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Extending Dirac and Faddeev-Jackiw Formalisms to Fractal First α-Order Lagrangian Systems

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Abstract

This paper presents the foundational concepts of fractal calculus before generalizing the Dirac Constraint Formalism and the Faddeev-Jackiw Formalism for first α-order Lagrangian systems in fractal spaces with non-integer dimensions. We provide a detailed analysis of the generalization process, highlighting the theoretical framework and key results, including the extended structure of the constraint systems in these Lagrangian formulations. Specific examples are discussed to demonstrate the practical application of the generalized formalism and to validate the consistency of our results. Moreover, graphical visualizations are included to enhance clarity, offering a visual interpretation of the findings and illustrating the relationship between the theory and its real-world implications.

Keywords Fractal Dirac Constraint Formalism · Fractal Faddeev-Jackiw Formalism · α-order Lagrangian systems

Mathematics Subject Classification 28A80 · 70H50 · 53A35

1 Introduction

Fractal geometry has emerged as a powerful tool for analyzing and understanding recurring patterns in various natural phenomena, including blood vessels, mountains, clouds, and coastlines. This article explores the distinctive properties and metrics associated with these formations. Key features of fractals include fractional dimensions, self-similarity, and fractal dimensions that exceed their topological dimensions [1–4].

Loosely speaking, a fractal dimension is an index for characterizing fractal patterns, such as similarity up to scale, by quantifying their complexity as a ratio of the change in detail to the change in scale. In our results, we shall also make use of key notions of self-similarity to precisely describe various properties of fractals, such as how a

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particular fractal object is similar to itself at all scales, from coarse to fine. Visually, when you zoom in on a fractal, the finer patterns resemble the original shape. Precise forms of self-similarity are a key property of fractals.

A fractal, as a geometric object, has the property that its fractal dimension is larger than its topological dimension. This often raises questions about a range of scales of self-similarity, which may involve stochastic (rather than deterministic) features. Consequently, we shall use certain concepts from stochastic processes. A stochastic process is a mathematical object involving probability and statistics that describes how random variables or measurements change over a particular parameter range, such as time or scale. It is often referred to as a random field when indexed by a parameter.

Fractal analysis integrates techniques from fractional calculus [5], stochastic processes [6], fractional space [7], and harmonic analysis [8–12]. Through adaptations of the Fundamental Theorem of Calculus, various Riemann-type integrals, such as s-Riemann, s-HK, and s-first-return integrals, have been developed. Researchers have investigated the relationships between these integrals and the Lebesgue integral as it pertains to the Hausdorff measure [13–16].

While there is a wealth of literature on fractal analysis, our focus is on selected studies that propose innovative techniques for Fourier expansions and wavelets specifically designed for certain classes of fractals [11, 17]. The intricate geometry and cross-scale self-similarity of fractals often necessitate approaches that extend beyond traditional mathematics [18].

A novel framework for fractal calculus has been proposed, extending classical calculus and offering an algorithmic, geometric, and physically intuitive approach. This generalization allows for the examination of functions exhibiting fractal characteristics, such as Cantor sets and Koch curves [19–22]. Non-local fractal calculus has been developed to model incompressible viscous fluids within fractal media and to address processes with memory, similar to the extensions of local calculus provided by the Riemann-Liouville and Caputo generalizations [23–26].

In the investigation of sub-and super-diffusion using fractal local derivatives, researchers have preserved locality and adhered to the central limit theorem [27, 28]. Fractal calculus has also found applications in physics, facilitating models of fractal space and time [29–35] and enabling the exploration of power law behavior and self-similarity solutions in these frameworks. The application of fractal Laplace, Sumudu, and Fourier transforms to Cantor sets and fractal curves has been utilized to analyze systems characterized by fractal temporal behavior [36–41]. Additionally, fractal derivatives and integrals have been calculated for various functions, including the Weierstrass function [42].

The use of nonstandard analysis with hyperreal and hyperinteger numbers has enabled the definition of derivative structures and integrals on fractal curves, as well as the specification of fractal integral and differential forms [43]. Fractal calculus has also been extended to encompass unbounded functions through gauge function generalizations [44]. Stability analysis for fractal differential equations has established criteria for unique and stable solutions [45]. Furthermore, a new framework that expands mean square calculus has been introduced, incorporating concepts such as the mean square derivative, fractal mean square integral, fractal mean square continuity, and the fractal order of random variables defined on fractal curves [46].

The development of fractal analogues to Newtonian, Lagrangian, Hamiltonian, and Appell mechanics has been proposed. Definitions of fractal velocity and acceleration have been established to derive the Langevin equation on fractal curves [47, 48].

The notion of fractal variations in calculus has been introduced, leading to the derivation of the general form of the fractal Euler equation and an alternative expression. Applications of the fractal Euler equation have been explored, including scenarios such as optical fractal paths near the event horizon of black holes and the determination of the shortest distance in fractal space [49].

The simplification of the transition from the Lagrangian to the Hamiltonian form, particularly when momenta are not independent of velocities, has been achieved [50].

The study of constrained Hamiltonian systems and their quantization has been elaborated, focusing on gauge theory quantization, which is pivotal for understanding fundamental interactions in nature [51].

The Faddeev-Jackiw approach offers a streamlined process by defining a velocity-linearized Lagrangian and employing a symplectic structure to derive generalized brackets from the Faddeev-Jackiw symplectic matrix [52].

Classical mechanical systems with internal constraints have been analyzed using the extended symplectic formalism of Faddeev-Jackiw, leading to the derivation of generalized brackets and corresponding equations of motion [53].

The Dirac-Bergmann algorithm has been applied to convert theories with singular Lagrangians into constrained Hamiltonian systems, encompassing gauge theories such as general relativity, electromagnetism, and string theory [54].

An iterative process has been employed to construct a symplectic supermatrix and identify associated constraints, analyzing the constraint structure within the context of the system's phase space [55]. The two-flavour Wess-Zumino model coupled to electromagnetism has been approached as a constrained system utilizing the Faddeev-Jackiw method [56].

A fractional constraint Hamiltonian formulation, employing Dirac brackets, has produced equations consistent with the fractional Euler-Lagrange equations, while the fractional Faddeev-Jackiw formalism has also been developed [57, 58]. The KPP-Fisher equation with non-local competitive losses and a fractal time derivative has been considered using F^{α}-calculus on the Cantor set. A dynamic system with a fractal time derivative, involving moments up to the second order, has been derived in the semiclassical approximation with respect to a small diffusion parameter within the class of trajectory-concentrated functions [59, 60].

The primary aim of this paper is to establish a Dirac constraint formalism and the Faddeev-Jackiw formalism within the context of fractal sets.

The organization of this paper is as follows: Section 2 provides a brief overview of fractal calculus. Section 3 presents the first α-order Lagrangian systems, detailing both the fractal Dirac constraint formalism and the fractal Faddeev-Jackiw formalism. The conclusion is found in Section 4.

2 Fundamental Definitions in Fractal Calculus

In this section, we present an overview of fractal calculus on the Cantor set $F \subset [a, b] \subset \mathbb{R}$, drawing from foundational insights in [19, 22].

Definition 1 The indicator function $\zeta_F(I)$ for the set F is defined as

$$\zeta_F(I) = \begin{cases} 1, & \text{if } F \cap I \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where $I = [a, b] \subset \mathbb{R}$.

Definition 2 The coarse-grained mass $\varphi_\delta^\alpha(F, a, b)$ of $F \cap [a, b]$ is defined as

$$\varphi_\delta^\alpha(F, a, b) = \inf_{|E| \leq \delta} \sum_{j=0}^{n-1} \Gamma(\alpha + 1)(z_{j+1} - z_j)^\alpha \zeta_F([z_j, z_{j+1}]),$$

where $|E| = \max_{0 \leq j \leq n-1}(z_{j+1} - z_j)$, $E_{[a,b]} = \{z_0 = a, z_1, \ldots, z_n = b\}, 0 < \alpha \leq 1$, and $\Gamma(\cdot)$ is the Gamma function.

Definition 3 The mass function $\varphi^\alpha(F, a, b)$ for F is given by

$$\varphi^\alpha(F, a, b) = \lim_{\delta \to 0} \varphi_\delta^\alpha(F, a, b).$$

Definition 4 The fractal ν-dimension of $F \cap [a, b]$ is defined by

$$\dim_\nu(F \cap [a, b]) = \inf\{\alpha : \varphi^\alpha(F, a, b) = 0\} = \sup\{\alpha : \varphi^\alpha(F, a, b) = \infty\}. \quad (1)$$

Definition 5 The integral staircase function $S_F^\alpha(z)$ is defined as

$$S_F^\alpha(z) = \begin{cases} \varphi^\alpha(F, a_0, z), & \text{if } z \geq a_0, \\ -\varphi^\alpha(F, z, a_0), & \text{if } z < a_0, \end{cases}$$

where $a_0 \in \mathbb{R}$ is a fixed constant.

Definition 6 For a function $w : F \to \mathbb{R}$, the $F\_\lim$ of $w(z)$ at a point $z \in F$ is the value L such that for any $\epsilon > 0$, there exists $\delta > 0$ satisfying

$$y \in F \text{ and } |y - z| < \delta \implies |w(y) - L| < \epsilon.$$

If such an L exists, it is denoted by

$$L = F\_\lim_{y \to z} w(y).$$

Definition 7 A function $w : F \to \mathbb{R}$ is F-continuous at $z \in F$ if

$$w(z) = F\_\lim_{y \to z} w(y),$$

whenever the $F\_\lim$ exists.

Definition 8 For a function w defined on an α-perfect fractal set F, the D_F^α-derivative of w at z is defined as

$$D_F^\alpha w(z) = \begin{cases} F\_\lim\limits_{y \to z} \frac{w(y)-w(z)}{S_F^\alpha(y)-S_F^\alpha(z)}, & \text{if } z \in F, \\ 0, & \text{if } z \notin F, \end{cases}$$

provided the $F\_\lim$ exists.

Definition 9 The $\mathcal{I}_F^\alpha$-integral of a bounded function $w(z)$, where $w \in B(F)$ (i.e., w is bounded on F), is defined as

$$\begin{aligned} \int_a^b w(z)\, d_F^\alpha z &= \sup_{E_{[a,b]}} \sum_{j=0}^{n-1} \inf_{z \in F \cap I} w(z)\, (S_F^\alpha(z_{j+1}) - S_F^\alpha(z_j)) \\ &= \inf_{E_{[a,b]}} \sum_{j=0}^{n-1} \sup_{z \in F \cap I} w(z)\, (S_F^\alpha(z_{j+1}) - S_F^\alpha(z_j)), \end{aligned}$$

where $z \in F$, and the infimum or supremum is taken over all partitions $E_{[a,b]}$ [19, 22].

3 First α-Order Lagrangian Systems

In this section, we present two approaches: the Fractal Dirac Constraint Formalism and the Fractal Faddeev-Jackiw Formalism for first α-order Lagrangian systems [61, 62].

3.1 Fractal Dirac Constraint Formalism

Consider the fractal functional action given by

$$S = \int_a^b L\left(q_i(t), D_{F,t}^\alpha q_i(t)\right) d_F^\alpha t, \quad t \in F, \quad i = 1, \ldots, N_q,$$

where q_i are the generalized coordinates, and $D_{F,t}^\alpha q_i(t)$ represents the corresponding generalized fractal velocities. In this context, $D_{F,t}^\alpha$ is a fractal derivative operator, and the indices i represent the different generalized coordinates, while the parameter α determines the fractal nature of the system. Assuming that the Lagrangian does not explicitly depend on fractal time, the fractal Euler-Lagrange equations can be derived as:

$$\frac{\partial L}{\partial q_i} - D_F^{\alpha}\left(\frac{\partial L}{\partial (D_{F,t}^{\alpha} q_i)}\right) = 0.$$

Here, D_F^{α} acts as a generalized fractal differential operator applied to the functional, capturing the fractal nature of the time evolution. Specifically, the operation

$$D_F^{\alpha} f\left(q(t), D_{F,t}^{\alpha} q(t)\right) = \frac{\partial f}{\partial q} D_{F,t}^{\alpha} q + \frac{\partial L}{\partial (D_{F,t}^{\alpha} q)} D_{F,t}^{2\alpha} q$$

is included to account for the second-order effects introduced by fractal time. This formulation follows from requiring that the variation of the action S be stationary, ensuring that the system is governed by a least action principle within the context of fractal geometry.

In the framework of fractal Hamiltonian formalism, the fractal canonical momentum is defined as

$$p_{\alpha}^{i} = \frac{\partial L}{\partial (D_{F,t}^{\alpha} q_i)}.$$

If this relation can be inverted to express $D_{F,t}^{\alpha} q_i$ in terms of p_{α}^{i} and q_i, the system is regular. However, if the determinant of the Hessian matrix of the fractal Lagrangian with respect to the fractal velocities vanishes, i.e.,

$$\det\left(H_{ij}\right) = \det\left(\frac{\partial^2 L}{\partial (D_{F,t}^{\alpha} q_i)\partial (D_{F,t}^{\alpha} q_j)}\right) = 0,$$

then the fractal Lagrangian is singular, indicating the presence of constraints that must be handled through the constraint formalism.

The fractal Poisson (FP) bracket is defined as

$$\{u, v\}_{FP} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_{\alpha}^{i}} - \frac{\partial u}{\partial p_{\alpha}^{i}}\frac{\partial v}{\partial q_i},$$

which describes the fundamental structure of the phase space evolution. The fractal fundamental Poisson bracket relations are

$$\{q_i, p_{\alpha}^{j}\}_{FP} = \delta_i^j,$$

where δ_i^j is the Kronecker delta. This shows that q_i and p_{α}^{j} form conjugate pairs in the context of fractal mechanics, analogous to standard Hamiltonian mechanics.

The canonical fractal Hamiltonian is defined as

$$H_c(q_i, p_{\alpha}^{i}) = p_{\alpha}^{i} D_{F,t}^{\alpha} q_i - L(q_i, D_{F,t}^{\alpha} q_i), \tag{2}$$

where $D^{\alpha}_{F,t} q_i$ must be expressed in terms of p^i_α if possible. The fractal Hamiltonian equations of motion follow as

$$D^{\alpha}_{F,t} q_i = \{q_i, H_c\}_{FP} = \frac{\partial H_c}{\partial p^i_\alpha}, \quad D^{\alpha}_{F,t} p^i_\alpha = \{p^i_\alpha, H_c\}_{FP} = -\frac{\partial H_c}{\partial q_i}.$$

This captures the fractal evolution of the system, accounting for both the generalized coordinates and their conjugate momenta in the context of fractal geometry.

In the case where the fractal Lagrangian is singular, the relation between the fractal momenta and fractal velocities may not be invertible. This leads to the appearance of primary constraints:

$$\phi_a(q_i, p^i_\alpha) \approx 0,$$

where the weak equality $\approx$ indicates that the constraints hold on a specific submanifold of the phase space. To ensure a consistent dynamical system, the fractal time derivatives of the primary constraints must also vanish:

$$D^{\alpha}_{F} \phi_a = \{\phi_a, H_T\}_{FP} = \{\phi_a, H_c\}_{FP} + u_m \{\phi_a, \phi_b\}_{FP} \approx 0,$$

where $H_T = H_c + u_m \phi_m$ is the fractal total Hamiltonian, and u_m are fractal Lagrange multipliers. This equation leads to two outcomes: one gives the values of the Lagrange multipliers u_m, and the other leads to new constraints, called secondary constraints.

By analyzing primary and secondary constraints, one can classify them as first-class or second-class, which is crucial for understanding gauge theories and the quantization of constrained systems. A constraint $\phi_a(q_i, p^i_\alpha)$ is first-class if its fractal Poisson bracket with all other constraints, including itself, weakly vanishes:

$$\{\phi_a, \phi_b\}_{FP} \approx 0. \tag{3}$$

Constraints satisfying Eq. (3) are associated with gauge symmetries of the system and represent redundancies in the description of physical states.

A constraint $\psi_a(q_i, p^i_\alpha)$ is second-class if its fractal Poisson bracket with at least one other constraint does not vanish:

$$\det\{\psi_a, \psi_b\}_{FP} \neq 0.$$

These constraints eliminate degrees of freedom rather than introducing gauge redundancies. Second-class constraints require modifying the fractal Poisson brackets to fractal Dirac brackets (FD) as follows:

$$\{u, v\}_{FD} = \{u, v\}_{FP} - \{u, \psi_a\}_{FP} C^{ab} \{\psi_b, v\}_{FP}, \tag{4}$$

where

$$C_{ab} = \{\psi_a, \psi_b\}_{FP},$$

must be invertible, with its inverse denoted by C^{ab} (i.e., det $C_{ab} \neq 0$) for consistency.

Remark 1 he classification of constraints determines the number of physical degrees of freedom N_{physical} in the system. For a system with N_q generalized coordinates and N_{FC} first-class constraints and N_{SC} second-class constraints, the number of physical degrees of freedom is given by:

$$N_{\text{physical}} = N_q - \frac{1}{2}(2N_{\text{FC}} + N_{\text{SC}}).$$

3.2 Fractal Faddeev-Jackiw Formalism

The Fractal Faddeev-Jackiw approach is presented for Lagrangians involving first-order fractal time derivatives. The fractal Lagrangian is defined as

$$L = a_i(q(t))D^{\alpha}_{F,t}q_i(t) - V(q(t)), \quad t \in F,$$

where $a_i(q)$ are coefficients defining the symplectic structure, and $V(q)$ is the potential. The corresponding fractal action is given by

$$S = \int_a^b \big(a_i(q(t))D^{\alpha}_{F,t}q_i(t) - V(q(t))\big)\, d^{\alpha}_F t, \quad t \in F. \tag{5}$$

Minimizing the action S using the fractal Euler-Lagrange equation yields

$$D^{\alpha}_F \frac{\partial L}{\partial q_i} = D^{\alpha}_F a_i(q) = \frac{\partial a_i}{\partial q_j} D^{\alpha}_{F,t} q_j = \frac{\partial L}{\partial q_i} = \frac{\partial a_j}{\partial q_i} D^{\alpha}_{F,t} q_j - \frac{\partial V}{\partial q_i}.$$

Equating both sides results in

$$\frac{\partial a_j}{\partial q_i} D^{\alpha}_{F,t} q_j = \frac{\partial a_i}{\partial q_j} D^{\alpha}_{F,t} q_j - \frac{\partial V}{\partial q_i}.$$

Rearranging terms gives

$$\left(\frac{\partial a_j}{\partial q_i} - \frac{\partial a_i}{\partial q_j}\right) D^{\alpha}_{F,t} q_j = \frac{\partial V}{\partial q_i}.$$

This can be compactly expressed as

$$\omega_{ij} D^{\alpha}_{F,t} q_j = \frac{\partial V}{\partial q_i},$$

where

$$\omega_{ij} = \frac{\partial a_j}{\partial q_i} - \frac{\partial a_i}{\partial q_j}$$

is the symplectic matrix. If the symplectic matrix ω_{ij} is invertible, the system has no constraints, and the dynamics follow directly from the symplectic structure. If ω_{ij} is singular, i.e., $\det(\omega_{ij}) = 0$, constraints exist, requiring the introduction of Lagrange multipliers to ensure an invertible symplectic matrix. Once invertibility is achieved, the symplectic equations of motion can be solved.

Example 1 Consider the fractal Lagrangian for a relativistic free particle:

$$L = -mc\sqrt{D_F^\alpha x^\mu D_F^\alpha x_\mu}, \quad \mu = 1, 2, 3, 4,$$

where $D_F^\alpha x^\mu$ and $D_F^\alpha x_\mu$ are the components of the fractal four-velocity, m is the particle mass, and c is the speed of light. This Lagrangian describes the motion of a relativistic particle in fractal spacetime, with D_F^α denoting the fractal derivative.

The Hessian of the system is computed using Eq. (2):

$$\begin{aligned} H_{\mu\nu} &= -mc\frac{\partial}{\partial D_F^\alpha x^\nu}\left(\frac{D_F^\alpha x_\mu}{\sqrt{D_F^\alpha x^\lambda D_F^\alpha x_\lambda}}\right) \\ &= -mc\left(\frac{\delta_{\mu\nu}}{\sqrt{D_F^\alpha x^\lambda D_F^\alpha x_\lambda}} - \frac{D_F^\alpha x_\mu D_F^\alpha x_\nu}{(D_F^\alpha x^\lambda D_F^\alpha x_\lambda)^{3/2}}\right). \end{aligned}$$

This Hessian is singular because its determinant is zero, indicating that the system has constraints.

The fractal canonical momenta are defined as

$$p_\mu^\alpha = \frac{\partial L}{\partial D_F^\alpha x^\mu} = -mc\frac{D_F^\alpha x_\mu}{\sqrt{D_F^\alpha x^\nu D_F^\alpha x_\nu}}.$$

Given the norm condition $D_F^\alpha x^\mu D_F^\alpha x_\mu = c^2$, we have

$$p_\mu^\alpha = -mc\frac{D_F^\alpha x_\mu}{c} = -mD_F^\alpha x_\mu.$$

Taking the norm of both sides yields

$$p_\mu^\alpha p^{\alpha\mu} = m^2 D_F^\alpha x_\mu D_F^\alpha x^\mu = m^2c^2.$$

This simplifies to

$$p_\mu^\alpha p^{\alpha\mu} - m^2c^2 = 0,$$

which is a constraint. To classify this constraint, we use the Dirac method, which involves checking whether the constraint is first-class. First-class constraints generate gauge symmetries in the system.

The constraint equation can be expressed as

$$\phi = p_\mu^\alpha p^{\mu\alpha} - m^2c^2 = 0.$$

To check if the constraint is first-class, we compute its Poisson bracket with itself:

$$\{\phi, \phi\} = 0.$$

Since the Poisson bracket vanishes, this confirms that the constraint is first-class. A first-class constraint indicates that the system possesses gauge freedom, which can be interpreted as the invariance of the system under reparameterizations of the fractal spacetime.

Thus, the constraint

$$p^{\alpha}_{\mu} p^{\alpha\mu} - m^2c^2 = 0$$

is a first-class constraint, implying that the system has a gauge symmetry. This gauge symmetry can be exploited when quantizing the system or analyzing its dynamics in phase space.

Example 2 Consider the fractal Lagrangian:

$$L = p_i D^{\alpha}_{F} q_i(t) - \left(\frac{(p^{\alpha}_i)^2}{2m} + eA_0(q) - eD^{\alpha}_{F} q_i A_i(q) \right),$$

where $q_i(t) : F \to \mathbb{R}$ are particle positions, p^{α}_i are generalized momenta, e is the electric charge, m is the particle mass, $A_0(q)$ is the scalar potential, and $A_i(q)$ is the vector potential. This Lagrangian is already in first-order form:

$$L = a_i(\xi)\xi^{\alpha}_i - V(\xi),$$

where $\xi_i = (q_i, p^{\alpha}_i)$, $a_i = (p^{\alpha}_i, 0)$, and

$$V(\xi) = \frac{(p^{\alpha}_i)^2}{2m} + eA_0(q) - eD^{\alpha}_{F} q_i A_i(q).$$

The symplectic matrix is

$$\omega_{ij} = \frac{\partial a_j}{\partial \xi_i} - \frac{\partial a_i}{\partial \xi_j} = \begin{pmatrix} 0 & -\delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix}.$$

The equations of motion follow from

$$\omega_{ij}\xi^{\alpha}_j = \frac{\partial V}{\partial \xi_i}.$$

This yields the same Hamiltonian equations as the Dirac formalism:

$$D^{\alpha}_{F} q_i = \frac{p^{\alpha}_i}{m}, \quad D^{\alpha}_{F} p^{\alpha}_i = -e\frac{\partial A_0}{\partial q_i}.$$

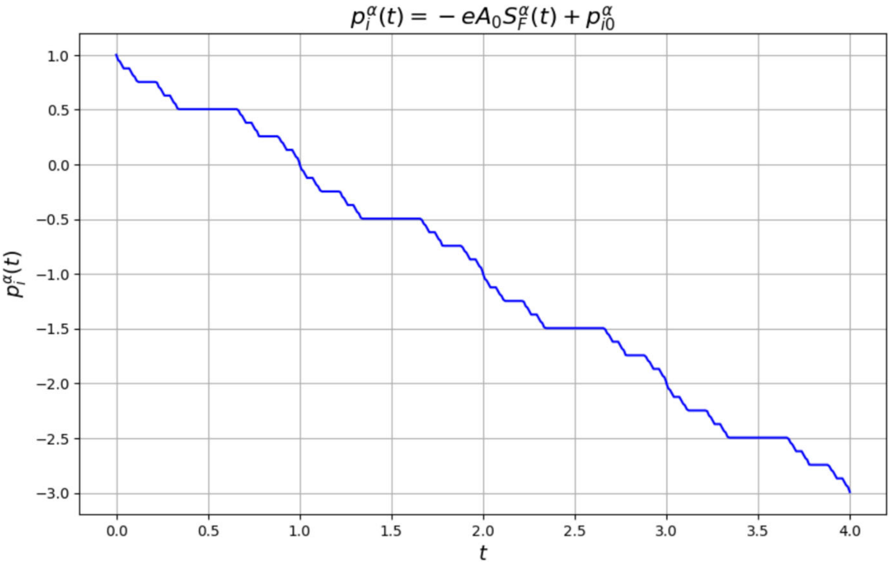

Fig. 1 Time evolution plot of Eq. (6)

For the special case where $A_0(q) = A_0 q_i$, the solutions for $q_i(t)$ is

$$p_i^\alpha(t) = -eA_0 S_F^\alpha(t) + p_{i0}^\alpha \tag{6}$$
$$\approx -eA_0 t^\alpha + p_{i0}^\alpha. \tag{7}$$

Figure 1 presents the time evolution described by Eq. (6). The plot illustrates the behavior of $p_i^\alpha(t)$ over time for various values of the parameter t.

In Fig. 2, Eq. (7) is plotted for different fractal dimension values α. The graph highlights the evolution of the function for $\alpha = 0.5$, $\alpha = 0.8$, and $\alpha = 1.0$.

The function $q_i(t)$ is defined as:

$$q_i(t) = -\frac{eA_0}{2m} S_F^\alpha(t)^2 + \frac{p_{i0}^\alpha}{m} S_F^\alpha(t) + q_{i0} \tag{8}$$
$$\approx -\frac{eA_0}{2m} t^{2\alpha} + \frac{p_{i0}^\alpha}{m} t^\alpha + q_{i0} \tag{9}$$

Figure 3 shows the plot of Eq. (8), illustrating the influence of the fractal structure on the time-dependent behavior. The plot demonstrates how the function evolves with changes in the time parameter t, considering the effects of the physical parameters A_0, p_{i0}^α, and q_{i0}.

Figure 4 presents the graph of Eq. (9) for various fractal dimension values α. The plot demonstrates how the function evolves with respect to time for $\alpha = 0.5$, $\alpha = 0.8$, and $\alpha = 1.0$.

These solutions describe the position and momentum of a particle under the influence of a linear scalar potential in fractal time.

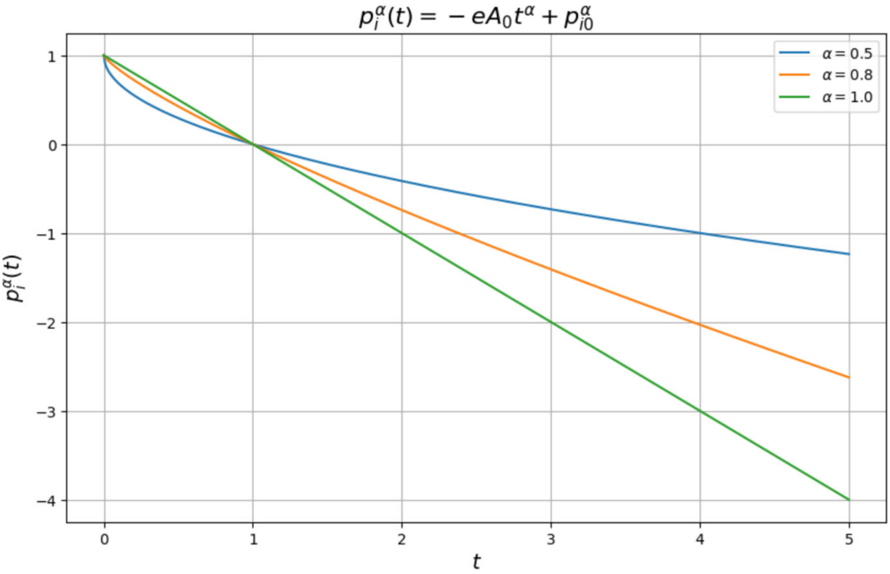

Fig. 2 Plot of Eq. (7) for various values of α

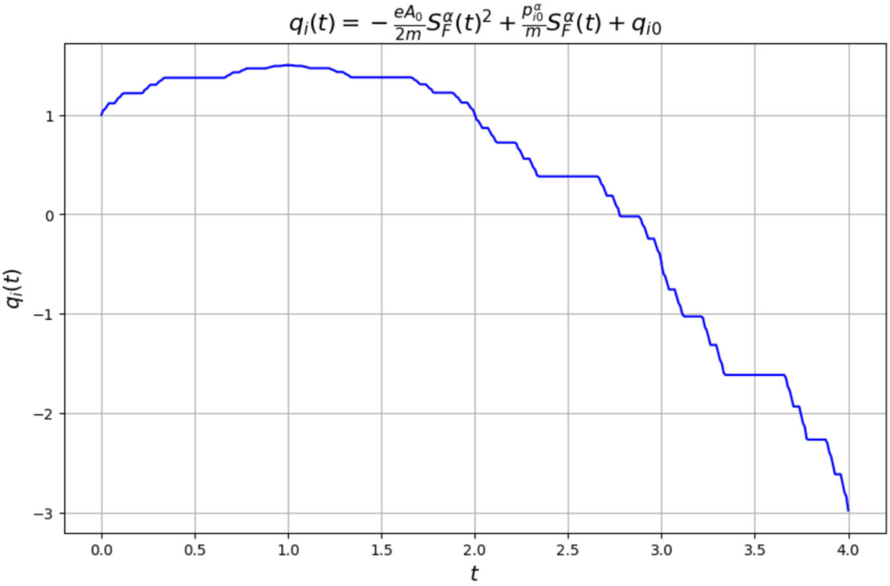

Fig. 3 Plot of Eq. (8), illustrating the fractal influence on the system's behavior

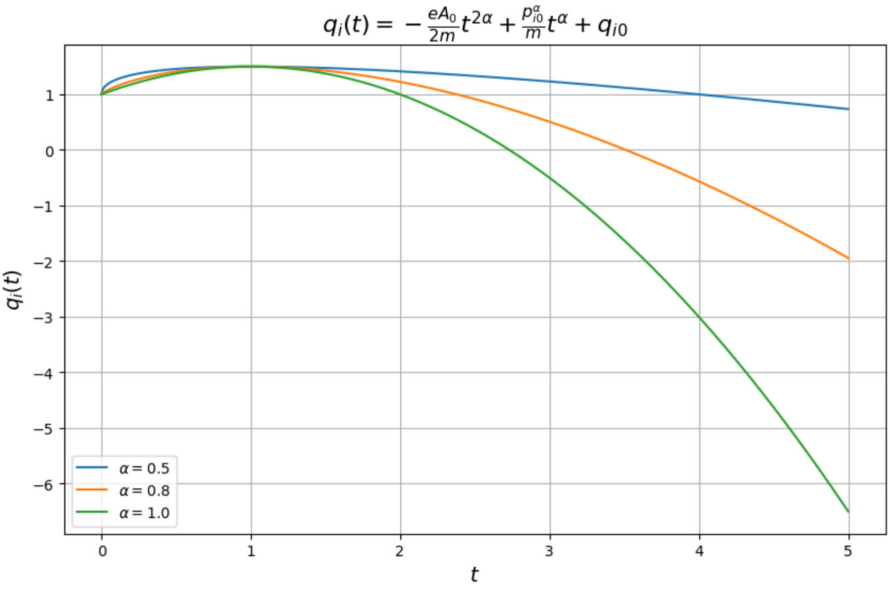

Fig. 4 Graph of Eq. (9) for different values of α

Example 3 Consider the fractal Lagrangian defined as

$$L(x, y, D_{F,t}^{\alpha}x, D_{F,t}^{\alpha}y) = xD_{F,t}^{\alpha}y - yD_{F,t}^{\alpha}x - V(x, y), \tag{10}$$

where $x(t)$ and $y(t)$ are functions mapping the fractal space F to $\mathbb{R}$. Here, $V(x, y)$ is a potential function.

The fractal Euler-Lagrange equations yield the equations of motion as follows:

For the x-component:

$$\frac{\partial L}{\partial x} - D_F^{\alpha}\frac{\partial L}{\partial(D_{F,t}^{\alpha}x)} = 0, \tag{11}$$

which simplifies to:

$$-\frac{\partial V}{\partial x} + D_{F,t}^{\alpha}y = 0. \tag{12}$$

For the y-component:

$$\frac{\partial L}{\partial y} - D_F^{\alpha}\frac{\partial L}{\partial(D_{F,t}^{\alpha}y)} = 0, \tag{13}$$

which simplifies to:

$$-\frac{\partial V}{\partial y} - D_{F,t}^{\alpha}x = 0. \tag{14}$$

The conjugate fractal momenta are defined as:

$$p_x^{\alpha} = \frac{\partial L}{\partial(D_{F,t}^{\alpha}x)} = -y, \tag{15}$$

$$p_y^\alpha = \frac{\partial L}{\partial (D_{F,t}^\alpha y)} = x. \tag{16}$$

The total fractal Hamiltonian is defined as:

$$H_T = p_x^\alpha D_{F,t}^\alpha x + p_y^\alpha D_{F,t}^\alpha y - L + u_1\phi_1 + u_2\phi_2, \tag{17}$$

where the constraints are given by:

$$\phi_1 = y + p_x^\alpha = 0, \tag{18}$$
$$\phi_2 = x - p_y^\alpha = 0. \tag{19}$$

Here, u_1 and u_2 are Lagrange multipliers enforcing the constraints.
The constraint matrix is given by:

$$C_{ij} = \{\phi_i, \phi_j\}_{FP} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}. \tag{20}$$

The inverse of this matrix is:

$$C^{ij} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{21}$$

Using the fractal Dirac bracket, the equations of motion become:
For the x-component:

$$D_{F,t}^\alpha x = \{x, H_T\}_{FD} = \frac{\partial V}{\partial y}. \tag{22}$$

For the y-component:

$$D_{F,t}^\alpha y = \{y, H_T\}_{FD} = -\frac{\partial V}{\partial x}. \tag{23}$$

For the fractal momenta:

$$D_{F,t}^\alpha p_x^\alpha = \{p_x^\alpha, H_T\}_{FD} = -\frac{\partial V}{\partial x}, \tag{24}$$
$$D_{F,t}^\alpha p_y^\alpha = \{p_y^\alpha, H_T\}_{FD} = -\frac{\partial V}{\partial y}. \tag{25}$$

These equations recover the fractal Euler-Lagrange equations (12) and (14), confirming the consistency of the Dirac formulation.
For the potential $V(x, y) = x^2 + y^2$, the equations of motion become:

$$D_{F,t}^\alpha x = 2y, \quad D_{F,t}^\alpha y = -2x, \quad D_{F,t}^\alpha p_x^\alpha = -2x, \quad D_{F,t}^\alpha p_y^\alpha = -2y. \tag{26}$$

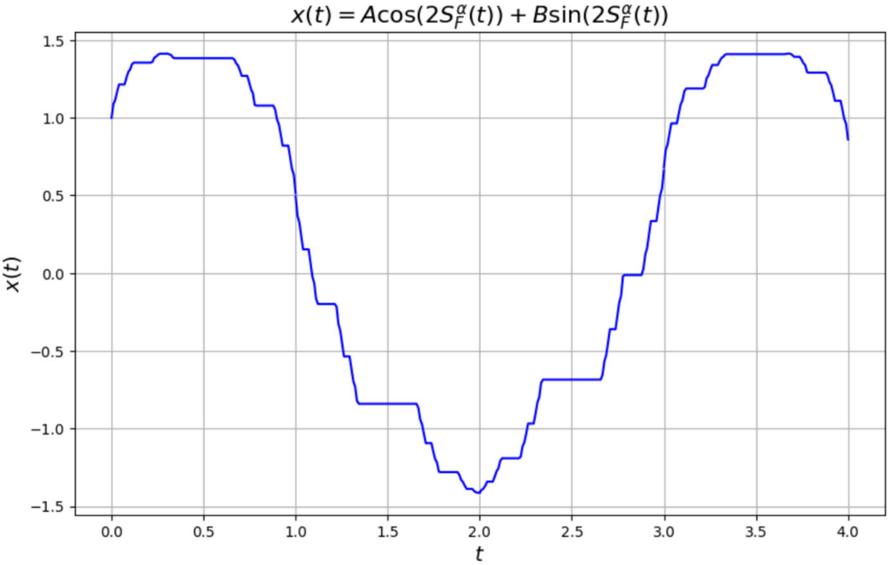

Fig. 5 Graph of Eq. (29) showing the solution in fractal time

This is a coupled fractal system of first-order linear differential equations. Taking the second F^α-derivative of $x(t)$:

$$D_{F,t}^{2\alpha} x = 2D_{F,t}^{\alpha} y = -4x. \tag{27}$$

This yields the fractal simple harmonic oscillator equation:

$$D_{F,t}^{2\alpha} x + 4x = 0. \tag{28}$$

The general solution is given by:

$$x(t) = A\cos(2S_F^\alpha(t)) + B\sin(2S_F^\alpha(t)) \tag{29}$$
$$\approx A\cos(2t^\alpha) + B\sin(2t^\alpha), \tag{30}$$

where A and B are constants determined by the initial conditions.

Figure 5 presents the plot of Eq. (29) for various values of t, demonstrating the behavior of the solution in fractal time.

In Fig. 6, Eq. (30) is plotted for different values of the fractal dimension α. The plot highlights the impact of the parameter α on the oscillatory behavior of $x(t)$, with results shown for $\alpha = 0.5$, $\alpha = 0.8$, and $\alpha = 1.0$. The variations in α demonstrate how the fractional exponent influences the frequency and amplitude patterns.

Using the equation $D_{F,t}^{\alpha} x = 2y$, we obtain:

$$y(t) = \frac{1}{2} D_{F,t}^{\alpha} x(t) = \frac{1}{2}\left(-A\sin(2S_F^\alpha(t)) + B\cos(2S_F^\alpha(t))\right). \tag{31}$$

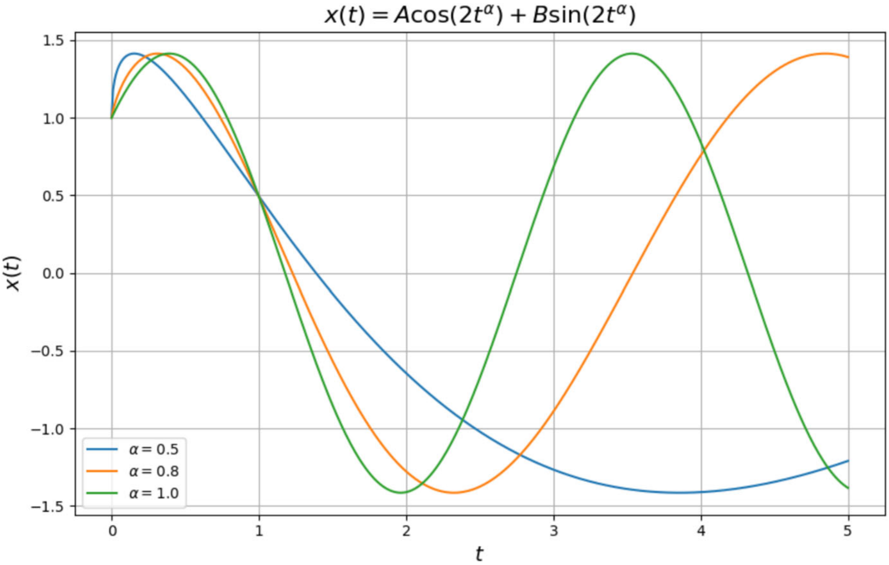

Fig. 6 Plot of Eq. (30) for different values of the fractal dimension α

Thus, we have

$$y(t) = \frac{1}{2}\left(-A\sin(2S_F^{\alpha}(t)) + B\cos(2S_F^{\alpha}(t))\right) \tag{32}$$
$$\approx \frac{1}{2}\left(-A\sin(2t^{\alpha}) + B\cos(2t^{\alpha})\right). \tag{33}$$

Figure 7 depicts the plot of Eq. (32) for different time shifts. This solution captures the oscillatory dynamics influenced by the fractal geometry of the Cantor set. The coefficients A and B regulate the amplitudes of the sine and cosine terms, respectively.

In Fig. 8, Eq. (33) is plotted for various values of the fractal dimension α. The graph highlights the oscillatory characteristics of the function, with the coefficients A and B controlling the amplitudes of the sine and cosine components. The parameter α determines the scaling of the time variable, illustrating how the fractional exponent affects the system's dynamic properties.

Using the equations $D_{F,t}^{\alpha} p_x^{\alpha} = -2x$ and $D_{F,t}^{\alpha} p_y^{\alpha} = -2y$:

$$p_x(t) = -A\sin(2S_F^{\alpha}(t)) - B\cos(2S_F^{\alpha}(t)) + C_1, \tag{34}$$
$$\approx -A\sin(2t^{\alpha}) - B\cos(2t^{\alpha}) + C_1. \tag{35}$$

Figure 9 illustrates the plot of Eq. (34) for a fixed value of $\alpha = 0.63$ and parameter values $A = B = C_1 = 1$. The graph reveals the oscillatory behavior of the solution, where the fractal staircase structure introduces complex dynamics. This plot visually demonstrates how the fractal geometry modifies the harmonic components of the solution.

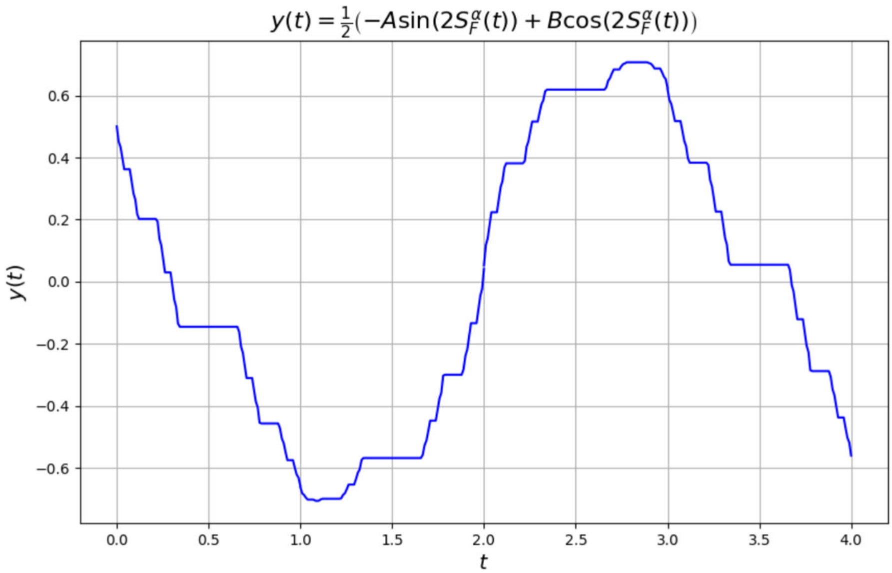

Fig. 7 Graph of Eq. (32) illustrating the system's oscillatory behavior under fractal geometry

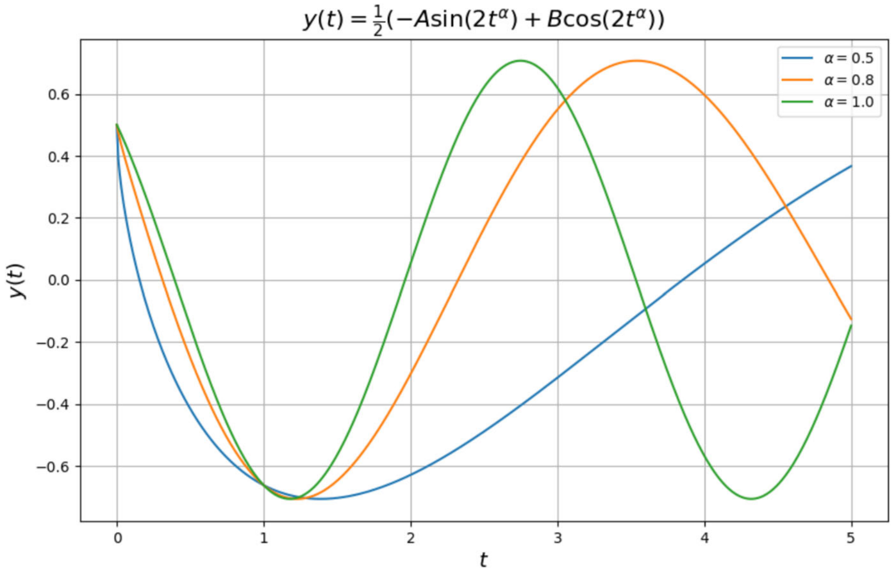

Fig. 8 Plot of Eq. (33) for different values of the fractal dimension α

Figure 10 shows the behavior of Eq. (35), an approximation of Eq. (34), for different values of the fractal dimension $\alpha = 0.5, 0.8, 1.0$. The plot illustrates that varying α significantly affects the frequency and phase of the oscillations. Lower values of α lead to more intricate waveforms, while classical harmonic oscillations emerge when $\alpha = 1.0$.

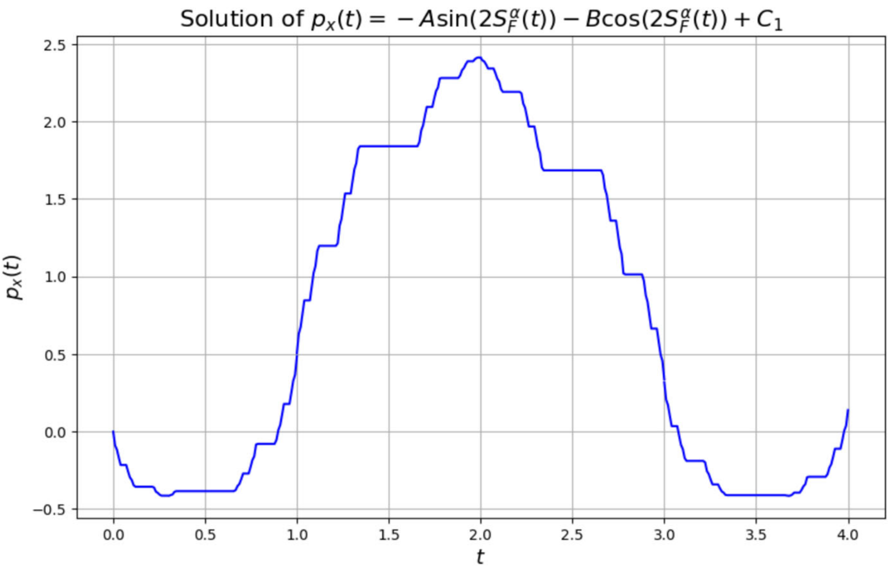

Fig. 9 Plot of Eq. (34) for $\alpha = 0.63$ and $A = B = C_1 = 1$

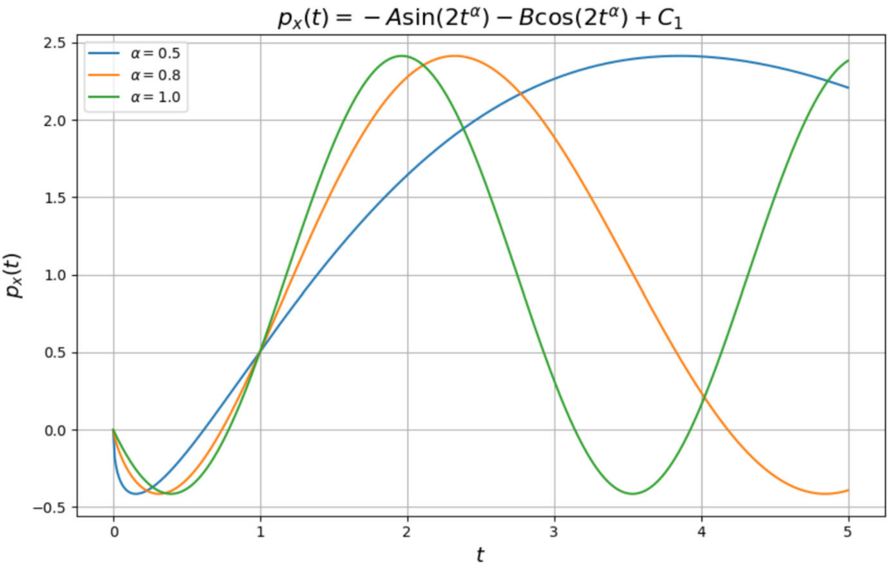

Fig. 10 Plot of Eq. (35) for different values of $\alpha = 0.5, 0.8, 1.0$

The corresponding expression for the generalized coordinate $p_y(t)$ is given by

$$p_y(t) = -\frac{A}{2}\cos(2S_F^\alpha(t)) + \frac{B}{2}\sin(2S_F^\alpha(t)) + C_2, \tag{36}$$

$$\approx -\frac{A}{2}\cos(2t^\alpha) + \frac{B}{2}\sin(2t^\alpha) + C_2, \tag{37}$$

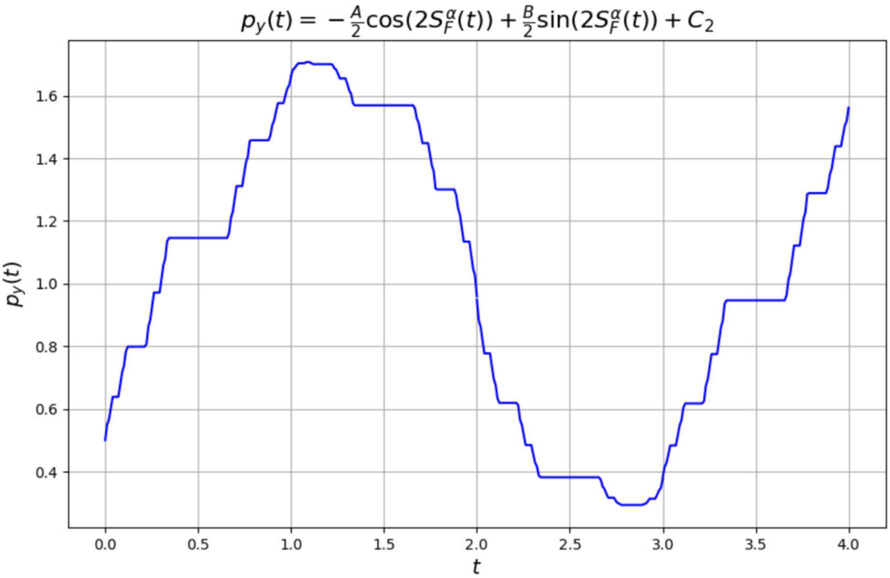

Fig. 11 Plot of Eq. (36) based on the Cantor set fractal structure

where C_1 and C_2 are constants determined by the initial conditions.

Figure 11 presents the plot of Eq. (36). The graph reveals the periodic and complex oscillatory behavior of $p_y(t)$ influenced by the fractal geometry of the Cantor set.

Figure 12 depicts the behavior of Eq. (37) for different values of α. The plot illustrates the sensitivity of the system to changes in the fractal dimension, with variations in α significantly altering the dynamic properties of $p_y(t)$.

These solutions describe the time evolution of the generalized coordinates $x(t)$ and $y(t)$ along with their corresponding fractal conjugate momenta $p_x(t)$ and $p_y(t)$ within the fractal system.

4 Conclusion

In this study, we have extended the Dirac Constraint Formalism and the Faddeev-Jackiw Formalism to first α-order Lagrangian systems, providing a novel framework for analyzing dynamical systems in fractal geometries. By presenting detailed examples, we have demonstrated the practical utility and theoretical consistency of these extended formalisms. Visual representations have highlighted the profound influence of the fractal dimension α on system dynamics, revealing intricate patterns and transitions that deviate from classical behavior. These insights contribute to a deeper understanding of the relationship between fractal geometry and constrained dynamical systems, offering a foundation for future explorations in mathematical physics and complex system analysis.

In simpler terms, our main result establishes a new and more efficient way to describe the dynamics of physical systems in fractal spaces, which are characterized

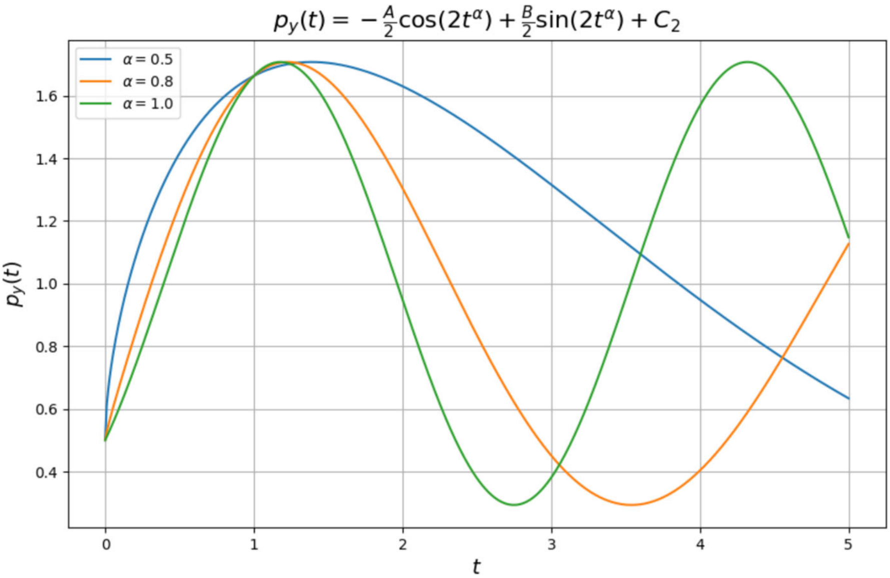

Fig. 12 Plot of Eq. (37) for different values of α

by non-integer dimensions. Interestingly, when the space has an integer dimension, the framework reduces to conventional results, making this approach both a generalization and an extension of traditional theories. This work bridges the gap between abstract theoretical models and practical applications by providing a clear mathematical foundation and visual representations of the generalized constraint structures.

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Data Availability It is not applicable; all the data have been provided in the paper.

Declarations

Conflicts of Interest The authors have no conflicts of interest to declare that are relevant to the content of this study.

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