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RESEARCH ARTICLE

The word problem for κ-terms over the pseudovariety of local groups

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Abstract
In this paper we study the κ-word problem for the pseudovariety **LG** of local groups, where κ is the canonical signature consisting of the multiplication and the pseudoinversion. We solve this problem by transforming each arbitrary κ-term α into another one α^* called the **LG**-canonical form of α and by showing that different canonical forms have different interpretations over **LG**. The procedure of construction of these canonical forms consists in applying reductions determined by a set Σ of κ-identities. As a consequence, Σ is a basis of κ-identities for the κ-variety generated by **LG**.

Keywords Local group · Pseudovariety · Finite semigroup · Implicit signature · Word problem · κ-term · Canonical form

1 Introduction

The notion of a *pseudovariety* has played a key role in the classification of finite semigroups. Recall that a pseudovariety of semigroups is a class of finite semigroups closed under taking subsemigroups, homomorphic images and finite direct products. The semidirect product operator on pseudovarieties of semigroups has received particular attention, as it allows to decompose complicated pseudovarieties into simpler

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ones, and which in turn is central to the applications of semigroup theory in computer science. Among the most studied semidirect products of pseudovarieties are those of the form $\mathbf{V} * \mathbf{D}$, where $\mathbf{V}$ is any pseudovariety and $\mathbf{D}$ is the pseudovariety of finite semigroups whose idempotents are right zeros [4,20,22]. If $\mathbf{V}$ is a pseudovariety, then $\mathbf{LV}$ denotes the pseudovariety of finite semigroups S whose local submonoids are in $\mathbf{V}$ (i.e., $eSe \in \mathbf{V}$ for all idempotents e of S). In general, $\mathbf{V} * \mathbf{D}$ is a subpseudovariety of $\mathbf{LV}$ but under certain conditions on the pseudovariety $\mathbf{V}$ the equality holds [20–22]. In particular, for the pseudovariety $\mathbf{G}$ of finite groups, $\mathbf{LG}$ is the class of finite local groups and it is well-known that $\mathbf{LG} = \mathbf{G} * \mathbf{D}$ [19].

Many applications involve solving the membership problem for specific pseudovarieties. A pseudovariety for which this is possible is said to be *decidable*. However, the semidirect product does not preserve decidability [11,17], and thus it is worth investigating stronger properties of the factors under which decidability of the semidirect product is guaranteed. This is the approach followed by Almeida and Steinberg that lead to the notion of *tameness* [6,7].

For a signature (or a type) σ of algebras and a class $\mathcal{C}$ of algebras of type σ (i.e., σ-algebras), the *σ-word problem for $\mathcal{C}$* consists in determining whether two given elements of the term algebra of type σ (i.e., σ-terms) over an alphabet have the same interpretation over every σ-algebra of $\mathcal{C}$. In the context of the study of tameness of pseudovarieties of semigroups, it is necessary to study the decidability of the σ-word problem over a pseudovariety $\mathbf{V}$, where σ is a set of implicit operations on semigroups containing the multiplication, called an *implicit signature*, since that is one of the properties required for $\mathbf{V}$ to be tame. For pseudovarieties of aperiodic semigroups it is common to use the signature ω consisting of the multiplication and the ω-power. For instance, the ω-word problem is already solved for the pseudovarieties $\mathbf{A}$ of finite aperiodic semigroups [16,23], $\mathbf{J}$ of $\mathcal{J}$-trivial semigroups [1], $\mathbf{LI}$ of locally trivial semigroups [9], $\mathbf{R}$ of $\mathcal{R}$-trivial semigroups [10] and $\mathbf{LSl}$ of local semilattices [12]. For non-aperiodic cases it is common to use the signature κ consisting of the multiplication and the $(\omega-1)$-power, usually called the *canonical signature*. We will use an extension of κ, denoted $\bar{\kappa}$ (and called the *completion* of κ in [5]), consisting of the multiplication and all the $(\omega+q)$-powers with q integer. It is easy to realize that the $\bar{\kappa}$-word problem is equivalent to the κ-word problem. As examples of pseudovarieties for which the κ-word problem is solved, we cite the pseudovarieties $\mathbf{S}$ of finite semigroups [13] and $\mathbf{CR}$ of completely regular semigroups [8].

This paper is a continuation of the work initiated in [14]. In that paper, the authors have shown that $\mathbf{LG}$ and $\mathbf{S}$ verify exactly the same identities involving $\bar{\kappa}$-terms of rank 0 or 1, and have given a proof (alternative to that contained in [13]) of the decidability of those $\bar{\kappa}$-identities. The present paper completes the proof of the decidability of the $\bar{\kappa}$-word problem (and, as a consequence, of the κ-word problem) over the pseudovariety $\mathbf{LG}$. We prove first that this problem can be reduced to consider only identities involving $\bar{\kappa}$-terms from a certain set $\mathcal{S}$ whose elements have rank at most 2. Next, a canonical form for rank 2 $\bar{\kappa}$-terms over $\mathbf{LG}$ is defined, thus extending the notion of canonical $\bar{\kappa}$-terms over $\mathbf{LG}$ given in [14] for rank 0 and 1. Finally, for canonical $\bar{\kappa}$-terms α and β, we show that the $\bar{\kappa}$-identity $\alpha = \beta$ holds over $\mathbf{LG}$ if and only if α and β are the same $\bar{\kappa}$-term. Since it is shown that each $\bar{\kappa}$-term can be algorithmically transformed into a unique canonical form with the same value over $\mathbf{LG}$, to test whether a $\bar{\kappa}$-identity $\alpha = \beta$ holds over **LG** it then suffices to verify if the canonical forms of the $\bar{\kappa}$-terms α and β are equal.

A fundamental tool in our work is that of $\mathbb{q}$-*root* of a $\bar{\kappa}$-term α from the set $\mathcal{S}$. We start by computing a certain parameter $\mathbb{q}_\alpha$, which is a positive integer and depends only on α. Then, for any given $\mathbb{q} \geq \mathbb{q}_\alpha$, the $\mathbb{q}$-*root* of α is an effectively computable word $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$, over a finite alphabet $\mathsf{V} \cup \mathsf{V}^{-1}$, which is reduced in the free group F_{V} generated by V. A pertinent property is that, if $\alpha, \beta \in \mathcal{S}$ and $\mathbb{q}$ is large enough, then **LG** satisfies $\alpha = \beta$ if and only if $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta)$. This result provides an alternative criterion to decide the $\bar{\kappa}$-word problem for **LG**. Moreover, each word $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$ is obtained as the reduced form in the free group F_{V} of another word $\mathsf{w}_{\mathbb{q}}(\alpha)$, called the $\mathbb{q}$-*outline* of α. The reduction process of an outline $\mathsf{w}_{\mathbb{q}}(\alpha)$ into the root $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$ was fundamental to us in the definition of a canonical form for rank 2 $\bar{\kappa}$-terms over **LG** since it served as a guide to some of the simplifications that should be operated at the $\bar{\kappa}$-term level. Informally speaking, if **LG** satisfies $\alpha = \beta$ and the outline $\mathsf{w}_{\mathbb{q}}(\beta)$ is "closer" than the outline $\mathsf{w}_{\mathbb{q}}(\alpha)$ to their common reduced form $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)(= \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta))$, then β should be considered to be "simpler" than α. The notion of $\mathbb{q}$-outline, introduced here for $\bar{\kappa}$-terms over **LG**, plays a similar role as a more general notion of superposition homomorphism that was used by Almeida and Azevedo [3] to provide a representation of the free pro-($\mathbf{V} * \mathbf{D}$) semigroup over A (see [2, Theorem 10.6.12]).

2 Preliminaries

This section introduces some terminology and notation. We assume familiarity with basic results of the theory of pseudovarieties and implicit operations. For further details and general background see [2,18]. For the main definitions and basic results about combinatorics on words, the reader is referred to [15].

2.1 $\bar{\kappa}$-terms

In this paper, we consider a finite alphabet A provided with a total order. The free semigroup (resp. the free monoid) generated by A is denoted by A^+ (resp. A^*). An element w of A^* is called a (finite) word and the empty word is denoted by ϵ. A word is said to be *primitive* if it cannot be written in the form u^n with $n > 1$. Words u and v are *conjugate* if there are words $w_1, w_2 \in A^*$ such that $u = w_1 w_2$ and $v = w_2 w_1$. A *Lyndon word* is a primitive word which is minimal in its conjugacy class for the lexicographic order.

Given an element s of a compact semigroup, the closed subsemigroup generated by s contains a unique idempotent, denoted s^ω or $s^{\omega+0}$. For $q \in \mathbb{N}$, $s^{\omega+q} = s^\omega s^q$ belongs to the maximal closed subgroup containing s^ω, and its group inverse is denoted by $s^{\omega-q}$. The following examples of implicit operations play an important role in the next sections: the binary implicit operation *multiplication* interpreted as the semigroup multiplication and, for each $q \in \mathbb{Z}$, the unary implicit operation $(\omega+q)$-*power* which, for a finite semigroup S, sends $s \in S$ to $s^{\omega+q}$.

We denote by $\bar{\kappa}$ the implicit signature consisting of the multiplication and the $(\omega+q)$-powers with $q \in \mathbb{Z}$. The free $\bar{\kappa}$-algebra generated by A in the variety defined by the identity $x(yz) = (xy)z$ will be denoted by $T_A^{\bar{\kappa}}$ and its elements are called *$\bar{\kappa}$-terms*. Every finite semigroup has a natural structure of an associative $\bar{\kappa}$-algebra (also known as a $\bar{\kappa}$-semigroup), via the interpretation of implicit operations as operations on finite semigroups. When referring to a *term* we will mean either a *$\bar{\kappa}$-term* or the empty word ϵ. A $\bar{\kappa}$-term of the form $\pi^{\omega+q}$ is called a *limit term*, and π and $\omega+q$ are called, respectively, its *base* and its *exponent*. Notice that $\pi^{\omega+0}$ is usually written as π^{ω} to make the notation more compact. If a term α can be written in the form $\alpha = \alpha_1\alpha_2$, then the terms α_1 and α_2 are said to be, respectively, a *prefix* and a *suffix* of α.

2.2 Portions of a $\bar{\kappa}$-term

The *rank* of a term α is the maximum number $\operatorname{rank}(\alpha)$ of nested exponents in it. So, the terms of rank 0 are the words from A^* and, for $i \geq 0$, a $\bar{\kappa}$-term of rank $i+1$ is an expression α of the form

$$\alpha = \rho_0\pi_1^{\omega+q_1}\rho_1 \cdots \pi_n^{\omega+q_n}\rho_n,$$

where $n \geq 1$, ρ_j is a term with rank at most i, π_ℓ is a rank i $\bar{\kappa}$-term and $q_\ell \in \mathbb{Z}$. This expression is uniquely determined and we call it the *rank configuration* of α. The number n is said to be the $(i+1)$-*length* of α. The subterms $\rho_0\pi_1^{\omega+q_1}$, $\pi_n^{\omega+q_n}\rho_n$ and $\pi_j^{\omega+q_j}\rho_j\pi_{j+1}^{\omega+q_{j+1}}$ are called, respectively, the *initial portion*, the *final portion* and the *crucial portions* of α. For a positive integer p, the *p-expansion* of α is the rank i $\bar{\kappa}$-term

$$\alpha^{(p)} = \rho_0\pi_1^p\rho_1 \cdots \pi_n^p\rho_n.$$

Suppose that $i = 0$, whence $\operatorname{rank}(\alpha) = 1$. The ω-terms $\rho_0\pi_1^{\omega}$, $\pi_n^{\omega}\rho_n$ and $\pi_j^{\omega}\rho_j\pi_{j+1}^{\omega}$ are said to be, respectively, the *initial ω-portion*, the *final ω-portion* and the *crucial ω-portions* of α. In case $i = 1$, so that $\operatorname{rank}(\alpha) = 2$, the (rank 1) *initial ω-portion*, *final ω-portion* and *crucial ω-portions* of α are, respectively, the *initial ω-portion*, *final ω-portion* and *crucial ω-portions* of the 2-expansion $\alpha^{(2)}$ of α. For example, if $\alpha = b(ab^{\omega}a)^{\omega-1}bc(c^{\omega-1}aa(bc)^{\omega-2})^{\omega-1}a^{\omega+1}$, then bab^{ω} and a^{ω} are the initial and the final ω-portions, respectively, and $b^{\omega}aab^{\omega}$, $b^{\omega}abcc^{\omega}$, $c^{\omega}aa(bc)^{\omega}$, $(bc)^{\omega}c^{\omega}$ and $(bc)^{\omega}a^{\omega}$ are the crucial ω-portions of α.

2.3 $\bar{\kappa}$-identities

A *$\bar{\kappa}$-identity* over A is a formal equality $\pi = \rho$ with $\pi, \rho \in T_A^{\bar{\kappa}}$. For a pseudovariety $\mathbf{V}$, the *$\bar{\kappa}$-word problem for* $\mathbf{V}$ consists in determining, for each given $\bar{\kappa}$-identity $\pi = \rho$, whether π and ρ have the same interpretation over every semigroup of $\mathbf{V}$. If so, we write $\mathbf{V} \models \pi = \rho$, as usual. Analogous definitions can be formulated for the signature κ.

Note that the following $\bar{\kappa}$-identities hold over every finite semigroup: $x^{\omega+q} = x^{\omega-1}x^{q+1}$ ($q \in \mathbb{N}_0$) and $x^{\omega-q} = (x^q)^{\omega-1} = (x^{\omega-1})^q$ ($q \in \mathbb{N}$). This means that the signatures κ and $\bar{\kappa}$ have the same expressive power and, consequently, the $\bar{\kappa}$-word problem is equivalent to the κ-word problem.

2.4 Rewriting rules for $\bar{\kappa}$-terms over S

The following set $\Sigma_{\mathbf{S}}$ of $\bar{\kappa}$-identities

$$\begin{cases} (x^{\omega+p})^{\omega+q} = x^{\omega+pq}, & (2.1)\\ (x^n)^{\omega+q} = x^{\omega+nq}, & (2.2)\\ x^{\omega+p}x^{\omega+q} = x^{\omega+p+q}, & (2.3)\\ x^n x^{\omega+q} = x^{\omega+q+n}, \quad x^{\omega+q}x^n = x^{\omega+q+n}, & (2.4)\\ (xy)^{\omega+q}x = x(yx)^{\omega+q}, & (2.5) \end{cases}$$

holds in the pseudovariety $\mathbf{S}$, where x and y represent arbitrary $\bar{\kappa}$-terms, $n \in \mathbb{N}$ and $p, q \in \mathbb{Z}$. Notice that, using (2.3)–(2.5), it is easy to deduce the $\bar{\kappa}$-identities

$$\begin{aligned} x^{\omega}(x^{\omega+p}y)^{\omega+q} &= (x^{\omega+p}y)^{\omega+q}, & (x^{\omega+p}y)^{\omega+q}x^{\omega} &= (x^{\omega+p}yx^{\omega})^{\omega+q},\\ (yx^{\omega+p})^{\omega+q}x^{\omega} &= (yx^{\omega+p})^{\omega+q}, & x^{\omega}(yx^{\omega+p})^{\omega+q} &= (x^{\omega}yx^{\omega+p})^{\omega+q}. \end{aligned} \tag{2.6}$$

Each $\bar{\kappa}$-identity $r = (u = v)$ can be seen as two rewriting rules $\overrightarrow{r} : u \to v$ and $\overleftarrow{r} : v \to u$. If we rewrite a $\bar{\kappa}$-term π interpreting a $\bar{\kappa}$-identity (2.i), with $i \in \{1, 2, 3, 4\}$, as a rewriting rule from left to right, we say that we make a (2.i)-*contraction*. The transformations resulting from interpreting the $\bar{\kappa}$-identities as rewriting rules on the opposite direction are called *expansions*. We will distinguish between left and right contractions/expansions of type (2.4) depending on whether the left or right identity (2.4) is used. An application of the identity (2.5) from left to right or from right to left is called a *shift right* and a *shift left*, respectively.

We will talk about the rank of a transformation of $\bar{\kappa}$-terms using a $\bar{\kappa}$-identity $\alpha = \beta$ as the number $\max\{\text{rank}(\alpha), \text{rank}(\beta)\}$. For example, if we rewrite $ab^{\omega+1}b(ca^{\omega+1})^{\omega-1}ca^{\omega+1}$ as $ab^{\omega+1}b(ca^{\omega+1})^{\omega}$, or as $ab^{\omega+2}(ca^{\omega+1})^{\omega-1}ca^{\omega+1}$, making right (2.4)-contractions, we say that it was made a rank 2 contraction in the first case, and a rank 1 contraction in the second one.

In what follows, we assume that the alphabet A is not a singleton set since, otherwise, every $\bar{\kappa}$-term with not null rank is equivalent to a rank 1 limit term with base the only letter of A, and the $\bar{\kappa}$-word problem is trivial in that case.

2.5 Local groups

A local group S is a semigroup such that eSe is a group for each idempotent e of S. Equivalently, we may say that S is a local group if and only if S has no idempotents or S has a completely simple minimal ideal containing all its idempotents [14, Proposition 2.1]. Groups and completely simple, locally trivial and nilpotent semigroups are examples of local groups.

Recall that **LI** is the join of **D** with its dual **K**, the pseudovariety of finite semigroups whose idempotents are left zeros. Therefore, a $\bar{\kappa}$-identity $\alpha = \beta$ holds in **LI** if and only if it holds in both **K** and **D**. In particular, when α and β are rank 1 or rank 2 $\bar{\kappa}$-terms, $\alpha = \beta$ holds in **LI** if and only if α and β have the same initial and final ω-portions. We also recall that **G** and **LI** are subpseudovarieties of **LG**, but **LG** is not the join of **G** with **LI**. Hence, if a $\bar{\kappa}$-identity $\alpha = \beta$ holds in **LG**, then it holds in both **G** and **LI** but the converse implication is not valid. It is well known that, if a pseudovariety **V** contains **LI** and $\mathbf{V} \models \alpha = \beta$, then either α and β are the same word or they both are $\bar{\kappa}$-terms of rank at least 1.

In [14] the authors defined a class of local groups denoted by $\mathcal{S}(G, L, \mathsf{f})$ in which G is a group, $L \subseteq A^+$ is a factorial language (i.e., a language that is closed under taking non-empty factors) and $\mathsf{f} : L \cup \ddot{L} \to G$ is a map that serves to define the semigroup operation, where $\ddot{L}$ is the subset of $A^+ \setminus L$ formed by the words whose proper factors belong to L. We have also constructed a finite local group $S_{\pi,\rho}$ of the form $\mathcal{S}(G, L, \mathsf{f})$, associated to each pair (π, ρ) of rank 1 canonical $\bar{\kappa}$-terms, such that $\mathbf{LG} \models \pi = \rho$ if and only if $S_{\pi,\rho} \models \pi = \rho$.

So, by the above considerations, it remains to deal with $\bar{\kappa}$-identities $\alpha = \beta$ such that $\mathrm{rank}(\alpha) \geq 1$ and $\mathrm{rank}(\beta) \geq 1$ where at least one of these inequalities is strict.

3 Some properties of $\bar{\kappa}$-terms over LG

In this section, we show some features of $\bar{\kappa}$-terms interpreted on finite local groups. Notice that **LG** is the pseudovariety of finite semigroups that satisfy the $\bar{\kappa}$-identity

$$(x^\omega y x^\omega)^\omega = x^\omega. \tag{3.1}$$

Let us consider the set of $\bar{\kappa}$-identities $\Sigma = \Sigma_{\mathbf{S}} \cup \{(x^\omega y x^\omega)^\omega = x^\omega\}$. Observe that the left side of the $\bar{\kappa}$-identity (3.1) is a rank 2 $\bar{\kappa}$-term while the $\bar{\kappa}$-term in the right side has rank 1. This is the key $\bar{\kappa}$-identity for the transformation of $\bar{\kappa}$-terms into other ones of rank at most 2 in Sect. 5.1. In Sect. 5.2, using the set Σ, we will further reduce any $\bar{\kappa}$-term to a canonical form over **LG**.

Two $\bar{\kappa}$-terms α and β are Σ-*equivalent* when $\Sigma \vdash \alpha = \beta$, that is, when the $\bar{\kappa}$-identity $\alpha = \beta$ is a syntactic consequence of Σ. Obviously, if α and β are Σ-equivalent, then $\mathbf{LG} \models \alpha = \beta$. One of the main goals is to prove that the converse implication also holds.

Let π be a $\bar{\kappa}$-term of rank at least 1. Then π is of the form $\pi = ux^{\omega+q}w$ for some integer q and some terms u, x and w. By (2.3), it follows that π may be transformed into $ux^\omega x^{\omega+q}w$. Therefore π is Σ-equivalent (it is $\Sigma_{\mathbf{S}}$-equivalent to be more precise) to some $\bar{\kappa}$-term of the form $ux^\omega v$ and we will often use this fact without further reference. In particular, using notably (2.6) and (3.1), we may derive

$$\pi^{\omega+1} = u(x^\omega vu)^\omega x^\omega v = u(x^\omega vux^\omega)^\omega v = ux^\omega v = \pi. \tag{3.2}$$

Notice that the $\bar{\kappa}$-identities $(x^\omega y x^\omega)^\omega = x^\omega (y x^\omega)^\omega = (x^\omega y)^\omega x^\omega$ are derived from $\Sigma_{\mathbf{S}}$ and that, for arbitrary integers p and q, $(x^{\omega+p} y x^{\omega+q})^\omega = x^\omega$ is a consequence of Σ. It is useful to point out the following consequences of this $\bar{\kappa}$-identity and (2.6),

$$x^{\omega+p}(yx^{\omega+q})^\omega = x^{\omega+p} = (x^{\omega+q}y)^\omega x^{\omega+p}. \tag{3.3}$$

Now, from these ones we deduce, as explained below, the following property of exponents, where r is an arbitrary integer,

$$x^{\omega+p}(yx^{\omega+q})^{\omega-1} = x^{\omega+p-r}(yx^{\omega+q-r})^{\omega-1}, \quad (x^{\omega+q}y)^{\omega-1}x^{\omega+p} = (x^{\omega+q-r}y)^{\omega-1}x^{\omega+p-r}. \tag{3.4}$$

Indeed, we deduce the first identity as follows (the second one being proved by symmetry)

$$\begin{aligned} x^{\omega+p}(yx^{\omega+q})^{\omega-1} &= x^{\omega+p-r}(x^{\omega+r}yx^{\omega+q-r})^{\omega-1}x^{\omega+r} \\ &= x^{\omega+p-r}(x^{\omega+r}yx^{\omega+q-r})^{\omega-1}x^{\omega+r}(yx^{\omega+q-r})^\omega \\ &= x^{\omega+p-r}(x^{\omega+r}yx^{\omega+q-r})^{\omega}(yx^{\omega+q-r})^{\omega-1} \\ &= x^{\omega+p-r}(yx^{\omega+q-r})^{\omega-1}. \end{aligned}$$

We gather in the following proposition a few $\bar{\kappa}$-identities exhibiting cancelation properties that are important in the reduction process.

Proposition 3.1 *The following $\bar{\kappa}$-identities are consequences of Σ, for all $p, q, r, s \in \mathbb{Z}$,*

$$x^{\omega+p}y(z^{\omega+q}wx^{\omega+r}y)^{\omega-1}z^{\omega+s} = x^{\omega+p}(z^{\omega+q}wx^{\omega+r})^{\omega-1}z^{\omega+s}, \tag{3.5}$$

$$x^{\omega+p}y(x^{\omega+q}y)^{\omega-1}x^{\omega+s} = x^{\omega+p-q+s}, \tag{3.6}$$

$$(x^{\omega+p}y)^{\omega-1}x^{\omega+q}(zx^{\omega+r})^{\omega-1} = x^\omega(zx^{\omega+p-q+r}yx^\omega)^{\omega-1}. \tag{3.7}$$

Proof The deduction of (3.5) can be made using $\Sigma_{\mathbf{S}}$ and (3.3) as follows

$$\begin{aligned} x^{\omega+p}y(z^{\omega+q}wx^{\omega+r}y)^{\omega-1}z^{\omega+s} &= x^{\omega+p}(z^{\omega+q}wx^{\omega+r})^{\omega}y(z^{\omega+q}wx^{\omega+r}y)^{\omega-1}z^{\omega+s} \\ &= x^{\omega+p}(z^{\omega+q}wx^{\omega+r})^{\omega-1}(z^{\omega+q}wx^{\omega+r}y)^{\omega}z^{\omega+s} \\ &= x^{\omega+p}(z^{\omega+q}wx^{\omega+r})^{\omega-1}z^{\omega+s}. \end{aligned}$$

The identity (3.6) is an immediate consequence of (3.5). For the identity (3.7), we prove $(x^\omega y)^{\omega-1}x^{\omega+q}(zx^\omega)^{\omega-1} = x^\omega(zx^{\omega-q}yx^\omega)^{\omega-1}$ which is a simpler and, clearly, equivalent condition. Using (3.4) in the first identity below, we have

$$\begin{aligned}(x^\omega y)^{\omega-1}x^{\omega+q}(zx^\omega)^{\omega-1} &= (x^\omega y)^{\omega-1}x^\omega(zx^{\omega-q})^{\omega-1}\\ &= (x^\omega y)^{\omega-1}(x^\omega zx^{\omega-q})^{\omega-1}x^\omega\\ &= (x^\omega y)^{\omega-1}(x^\omega zx^{\omega-q})^{\omega-1}(x^\omega zx^{\omega-q}yx^\omega)^\omega\\ &= (x^\omega y)^{\omega-1}(x^\omega zx^{\omega-q})^{\omega}yx^\omega(x^\omega zx^{\omega-q}yx^\omega)^{\omega-1}\\ &= (x^\omega y)^{\omega-1}x^\omega yx^\omega(x^\omega zx^{\omega-q}yx^\omega)^{\omega-1}\\ &= x^\omega(x^\omega zx^{\omega-q}yx^\omega)^{\omega-1}\\ &= x^\omega(zx^{\omega-q}yx^\omega)^{\omega-1}.\end{aligned}$$

This proves the proposition. □

It is also useful to emphasize the following properties.

Corollary 3.2 *Let τ and σ be $\bar{\kappa}$-terms.*

(a) If $\mathbf{LI} \models \tau = \sigma$, *then* $\Sigma \vdash \sigma(\tau\sigma)^{\omega-1} = \tau^{\omega-1}$.
(b) If $\mathbf{K} \models \tau = \sigma$, *then* $\Sigma \vdash \sigma^{\omega-1}\tau^{\omega-1} = (\tau^2\sigma)^{\omega-1}\tau$.
(c) If $\mathbf{D} \models \tau = \sigma$, *then* $\Sigma \vdash \sigma^{\omega-1}\tau^{\omega-1} = \sigma(\tau\sigma^2)^{\omega-1}$.

Proof Suppose that $\mathbf{LI} \models \tau = \sigma$. Then τ and σ are the same word (and the result is trivial), or they both have rank at least 1. In this case, τ and σ are $\Sigma_{\mathbf{S}}$-equivalent, respectively, to $\bar{\kappa}$-terms of the form $ux^\omega\tau'y^\omega v$ and $ux^\omega\sigma'y^\omega v$ with u, x, y, v words. Therefore, using $\Sigma_{\mathbf{S}}$ and (3.5), one derives

$$\sigma(\tau\sigma)^{\omega-1} = ux^\omega\sigma'(y^\omega vux^\omega\tau'y^\omega vux^\omega\sigma')^{\omega-1}y^\omega v = ux^\omega\tau'(y^\omega vux^\omega\tau'y^\omega vux^\omega\tau')^{\omega-1}y^\omega v = \tau^{\omega-1},$$

thus showing (a).

Now suppose that $\mathbf{K} \models \tau = \sigma$. Then, as above, τ and σ are the same word (in which case the result is immediate), or both τ and σ have rank at least 1. In this case, τ and σ are $\Sigma_{\mathbf{S}}$-equivalent, respectively, to $\bar{\kappa}$-terms of the form $ux^\omega\tau'$ and $ux^\omega\sigma'$ with u, x words. So, the deduction of (b) can be done, using $\Sigma_{\mathbf{S}}$ and (3.7), as follows

$$\begin{aligned}\sigma^{\omega-1}\tau^{\omega-1} &= (ux^\omega\sigma')^{\omega-1}(ux^\omega\tau'ux^\omega\tau')^{\omega-1}ux^\omega\tau'\\ &= u(x^\omega\sigma'u)^{\omega-1}x^\omega(\tau'ux^\omega\tau'ux^\omega)^{\omega-1}\tau'\\ &= ux^\omega(\tau'ux^\omega\tau'ux^\omega\sigma'ux^\omega)^{\omega-1}\tau'\\ &= (\tau^2\sigma)^{\omega-1}\tau.\end{aligned}$$

The proof of (c) can be made analogously. □

4 Canonical forms for $\bar{\kappa}$-terms over LG

In this section, we present the definitions of canonical forms for $\bar{\kappa}$-terms over $\mathbf{LG}$. The rank 0 and rank 1 canonical $\bar{\kappa}$-terms over $\mathbf{LG}$ were already introduced in [14], coincide with, respectively, rank 0 and rank 1 canonical $\bar{\kappa}$-terms over **S** defined in [13]. According to Proposition 5.1 below, in order to complete the definition of the canonical forms over **LG** it remains to introduce rank 2 **LG**-canonical forms.

Let α be a $\bar{\kappa}$-term and, if $\text{rank}(\alpha) \geq 1$, let

$$\alpha = \rho_0 \pi_1^{\omega+q_1} \rho_1 \cdots \pi_n^{\omega+q_n} \rho_n$$

be its rank configuration.

(C_0) If $\text{rank}(\alpha) = 0$, then α is said to be in **LG**-*canonical form*.

(C_1) If $\text{rank}(\alpha) = 1$ and, for each $j \in \{1, \ldots, n\}$,

(a) π_j is a Lyndon word;
(b) π_j is not a suffix of ρ_{j-1};
(c) π_j is not a prefix of any word $\rho_j \pi_{j+1}^{\ell}$ with $\ell \geq 0$, where π_{n+1} is the empty word;

then α is said to be in **LG**-*canonical form*. Notice that every rank 1 $\bar{\kappa}$-term can be effectively converted into a rank 1 canonical form by the reduction algorithm for rank 1 $\bar{\kappa}$-terms, defined in [14, Section 4] as follows:

(1) apply all possible (2.2)-contractions;
(2) turn the base of each limit term in the $\bar{\kappa}$-term into a Lyndon word, by means of a (2.4)-expansion (with $n = 1$) and a shift;
(3) apply all possible (2.4)-contractions;
(4) apply all possible (2.3)-contractions;
(5) replace each crucial portion $x^{\omega+p} u y^{\omega+q}$ not in canonical form by $x^{\omega+p+m} v y^{\omega+q-\ell}$, where ℓ is the minimum integer such that $|uy^{\ell}| \geq |x|$, m is the maximum integer such that x^m is a prefix of uy^{ℓ} and $x^m v = uy^{\ell}$, by means of applying a left (2.4)-expansion with $n = \ell$ and a right (2.4)-contraction with $n = m$.

(C_2) If $\text{rank}(\alpha) \in \{1, 2\}$, then α is said to be in *semi-canonical form (over* **S**) whenever the 2-expansion $\alpha^{(2)} = \rho_0 \pi_1^2 \rho_1 \cdots \pi_n^2 \rho_n$ is in canonical form. Notice that every rank 1 $\bar{\kappa}$-term is in semi-canonical form. We refer the reader to [13, Section 4.3] for the algorithm of calculation of the semi-canonical form of any rank 2 $\bar{\kappa}$-term. We will be particularly interested in rank 2 semi-canonical forms α such that $q_j = -1$ for all j, and denote by $\mathcal{S}_2$ the set of those $\bar{\kappa}$-terms.

(S_2) If $\alpha \in \mathcal{S}_2$ and α is irreducible for the rewrite system $\mathcal{R}$ defined in Sect. 5.2 below, then α is said to be in **LG**-*canonical form*.

The set of **LG**-canonical forms of rank i (with $i \in \{0, 1, 2\}$) is denoted $\mathcal{C}_i$. By [13] and Sect. 5.2, the following conditions are equivalent for a $\bar{\kappa}$-term α:

- α is in semi-canonical/**LG**-canonical form;
- every subterm of α is in semi-canonical/**LG**-canonical form;
- the initial, final and crucial portions of α are in semi-canonical/**LG**-canonical form;
- the initial, final and crucial ω-portions of α are in semi-canonical/**LG**-canonical form.

5 Canonical form algorithm

In this section, we describe an algorithm to compute a canonical form α^* of any given $\bar{\kappa}$-term α with $\mathrm{rank}(\alpha) \geq 1$. This algorithm consists in two major steps, presented in Sects. 5.1 and 5.2 . In step 1, we reduce α to a Σ-equivalent $\bar{\kappa}$-term α° in the set $\mathcal{S}$, mentioned in the Introduction. This set $\mathcal{S}$ is now identified as being $\mathcal{C}_1 \cup \mathcal{S}_2$. If $\alpha^\circ \in \mathcal{C}_1$, then α° is in rank 1 canonical form and so $\alpha^* = \alpha^\circ$. If $\alpha^\circ \in \mathcal{S}_2$, then step 2 turns α° into an element $\alpha^\odot$ of $\mathcal{C}_1 \cup \mathcal{C}_2$ and we let $\alpha^* = \alpha^\odot$. By Theorem 7.1 below, it follows that the $\bar{\kappa}$-term α^* is unique and so we call it the **LG**-canonical form of α.

5.1 Step 1: reduce to an element of $\mathcal{S}$

The first step consists in three sequential substeps.
Step 1.1. If $\mathrm{rank}(\alpha) \leq 2$, let $\alpha' = \alpha$. Otherwise, let α' be a rank 2 $\bar{\kappa}$-term obtained by recursively applying the procedure described in the proof of the following proposition.

Proposition 5.1 *Let γ be an arbitrary $\bar{\kappa}$-term such that $\mathrm{rank}(\gamma) = i + 1$ with $i \geq 2$. It is possible to effectively compute a $\bar{\kappa}$-term γ' such that γ' is Σ-equivalent to γ and $\mathrm{rank}(\gamma') = i$.*

Proof We begin by assuming that γ is of the form $\gamma = \pi^{\omega-1}$. The proof of this case is made by induction on the i-length m of π. Since π has rank i, it is of the form $\pi = w_0\sigma^{\omega+p}w_1$ with $\mathrm{rank}(\sigma) = i - 1$ and w_0 and w_1 with rank at most i. Using (3.4) and (3.2), one deduces

$$\begin{aligned}\gamma &= \pi^{\omega-1}\pi\pi^{\omega-1}\\ &= w_0(\sigma^{\omega+p}w_1w_0)^{\omega-1}\sigma^{\omega+p}(w_1w_0\sigma^{\omega+p})^{\omega-1}w_1\\ &= w_0(\sigma^{\omega+1}w_1w_0)^{\omega-1}\sigma^{\omega+2-p}(w_1w_0\sigma^{\omega+1})^{\omega-1}w_1\\ &= (w_0\sigma w_1)^{\omega-1}w_0\sigma^{\omega+2-p}w_1(w_0\sigma w_1)^{\omega-1}.\end{aligned}$$

If $m = 1$, this last $\bar{\kappa}$-term has rank i and, so, we take it to be γ'. Suppose now that $m > 1$. The $\bar{\kappa}$-term $\rho = w_0\sigma w_1$ is rank i and has i-length $m - 1$. So, by induction hypothesis, the $\bar{\kappa}$-term $\delta = \rho^{\omega-1}$ is Σ-equivalent to some rank i $\bar{\kappa}$-term δ'. Therefore, γ is Σ-equivalent to the rank i $\bar{\kappa}$-term $\gamma' = \delta' w_0\sigma^{\omega+2-p}w_1\delta'$. The proof of the case $\gamma = \pi^{\omega-1}$ is complete.

In general, by means of expansions of rank $i+1$ of types (2.2) and (2.4), if necessary, γ can be reduced to a $\bar{\kappa}$-term with rank configuration $\rho_0\pi_1^{\omega-1}\rho_1\cdots\pi_n^{\omega-1}\rho_n$. The $\bar{\kappa}$-term γ' is obtained from this by applying the above procedure to each subterm $\pi_j^{\omega-1}$. □

Step 1.2. If $\mathrm{rank}(\alpha') = 1$, let $\alpha'' = \alpha'$. Otherwise, let α'' be a $\bar{\kappa}$-term obtained from α' by the application of the first step of the **S** canonical form reduction algorithm described in [13, Section 4.3], and observe that α'' is a semi-canonical $\bar{\kappa}$-term such that $\mathrm{rank}(\alpha'') \in \{1, 2\}$.

Step 1.3. If rank$(\alpha'') = 1$, then we apply the rank 1 canonical form reduction algorithm [13,14], described in Sect. 4, to compute the canonical form of α''. This is an element of $\mathcal{C}_1$ and so it is chosen to be α°.

If rank$(\alpha'') = 2$, then, by means of expansions of rank 2 of types (2.2) and (2.4) if necessary, we obtain from α'' a $\bar{\kappa}$-term whose exponents of rank 2 limit subterms are equal to $\omega - 1$. This $\bar{\kappa}$-term is taken to be α°, since it is a semi-canonical form with rank configuration $\rho_0\pi_1^{\omega-1}\rho_1 \cdots \pi_n^{\omega-1}\rho_n$ meaning that it is an element of $\mathcal{S}_2$.

5.2 Step 2: compute the canonical form

Now, we complete the computation of the canonical form of α. If α° is rank 1, then it is in **LG**-canonical form and so let $\alpha^* = \alpha^\circ$.

To treat the remaining case, we define a rewriting system $\mathcal{R}$ with set of objects $\mathcal{S}$ and whose rules are described below. By Propositions 5.2 and 5.3, starting with the $\bar{\kappa}$-term α°, $\mathcal{R}$ produces, after a finite number of reductions, an irreducible (meaning that no rewriting rule can be applied to it) $\bar{\kappa}$-term $\alpha^\odot$ of $\mathcal{S}$. Then $\alpha^\odot \in \mathcal{C}_1 \cup \mathcal{C}_2$ and we let $\alpha^* = \alpha^\odot$.

The system $\mathcal{R}$ consists of rewriting rules of four types, called "shifts right", "eliminations", "agglutinations" and "shortenings". We do not include shifts left in $\mathcal{R}$ but they are used implicitly in the last three types of rules. The justification for this option is for the system to be terminating and for the canonical form to be unique. We list below the rewriting rules and justify that they transform $\bar{\kappa}$-terms into Σ-equivalent $\bar{\kappa}$-terms. The rank of terms x, y, z, u, v and w in every rule is bounded by assuming that the left side of each rule is a rank 2 $\bar{\kappa}$-term. The shift identity (2.5) is often used without reference.

Shifts right:

(sr.1) $(uv)^{\omega-1}u \to u(vu)^{\omega-1}$, where rank$(uv) = 1$ and $u \neq \epsilon$;

(sr.2) $(uv)^{\omega-1}(uw)^{\omega-1} \to u(vu)^{\omega-1}w(uwuw)^{\omega-1}$, where $u \in A^+$, rank$(v) =$ rank$(w) = 1$, $\mathbf{K} \not\models v = w$ and v and w do not have a common non-empty prefix.

Rule (sr.1) is a rank 2 shift right and rule (sr.2) is a result of applying the $\bar{\kappa}$-identity $\pi^{\omega-1} = \pi(\pi^2)^{\omega-1}$, which is a consequence of Σ, followed by a rank 2 shift right.

Eliminations:

(e.1) $x^{\omega+p}u(x^{\omega+q}u)^{\omega-1}x^{\omega+r} \to x^{\omega+p-q+r}$;

(e.2) $x^{\omega+p}uvx^{\omega+q}u(vx^{\omega+q}u)^{\omega-1} \to x^{\omega+p}u$;

(e.3) $(ux^{\omega+p}v)^{\omega-1}yzx^{\omega+q}vy(zx^{\omega+q}vy)^{\omega-1} \to (ux^{\omega+p}v)^{\omega-1}y$;

(e.4) $(ux^{\omega+p}vy)^{\omega-1}zx^{\omega+q}v(yzx^{\omega+q}v)^{\omega-1} \to ux^{\omega+p}v(yux^{\omega+p}vyux^{\omega+p}v)^{\omega-1}$ with $y \neq \epsilon$.

Rule (e.1) is a direct application of identity (3.6), while rule (e.2) also results from this identity but previously applying a rank 2 shift left. In its turn, rule (e.3) results from making a right (2.4)-expansion, followed by an application of (e.2) and ending with a right (2.4)-contraction. At last, rule (e.4) is obtained by applying the $\bar{\kappa}$-identity $\pi^{\omega-1} = (\pi^2)^{\omega-1}\pi$, followed by an application of (e.2) and ending with a rank 2 shift right.

Agglutinations:

$(a.1)$ $(x^{\omega+p}u)^{\omega-1}x^{\omega+q}v(yx^{\omega+r}v)^{\omega-1} \to x^{\omega}v(yx^{\omega+p-q+r}ux^{\omega}v)^{\omega-1}$;
$(a.2)$ $(ux^{\omega+p}v)^{\omega-1}(ux^{\omega+q}y)^{\omega-1} \to ux^{\omega+q}y(ux^{\omega+q}yux^{\omega+p}vux^{\omega+q}y)^{\omega-1}$;
$(a.3)$ $(ux^{\omega+p}v)^{\omega-1}y(zx^{\omega+q}vy)^{\omega-1} \to ux^{\omega+p}vy(zx^{\omega+q}vux^{\omega+p}vux^{\omega+p}vy)^{\omega-1}$.

Rule $(a.1)$ is derived from identity (3.7), whereas $(a.2)$ and $(a.3)$ follow from Corollary 3.2 (b) and (c) respectively.

Shortenings:

$(s.1)$ $\sigma(\tau\sigma)^{\omega-1} \to \tau^{\omega-1}$, where $\mathrm{rank}(\sigma) = \mathrm{rank}(\tau) = 1$ and $\mathbf{LI} \models \sigma = \tau$;
$(s.2)$ $x^{\omega+p}u(vx^{\omega+q}u)^{\omega-1} \to x^{\omega+p-q}u(vx^{\omega}u)^{\omega-1}$ with $q \neq 0$;
$(s.3)$ $(x^{\omega+p}u)^{\omega-1}x^{\omega+q} \to (x^{\omega}u)^{\omega-1}x^{\omega+q-p}$ with $p \neq 0$;
$(s.4)$ $x^{\omega+p}u(z^{\omega+q}vyx^{\omega+r}u)^{\omega-1}z^{\omega+s}v \to \delta(x, z, p, q, r, s)$;
$(s.5)$ $x^{\omega+p}uz^{\omega+q}v(yx^{\omega+r}uz^{\omega+q}v)^{\omega-1} \to \delta(x, z, p, q, r, q)$;

where

- $\delta(x, z, p, q, r, s)$ is the following $\bar{\kappa}$-term

$$\begin{cases} x^{\omega+p}(z^{\omega+q}vyx^{\omega+r})^{\omega-1}z^{\omega+s}v & \text{if } x^{\omega}z^{\omega} \text{ is in canonical form} \quad (5.1)\\ x^{\omega+p}v(yx^{\omega+r}v)^{\omega-1} & \text{if } x = z \text{ and } q = s\\ (x^{\omega+q}vy)^{\omega-1}x^{\omega+s}v & \text{if } x = z,\, q \neq s \text{ and } p = r\\ x^{\omega+p}a_{x,z}(z^{\omega+q}vyx^{\omega+r}a_{x,z})^{\omega-1}z^{\omega+s}v & \text{otherwise} \quad (5.2) \end{cases}$$

  with $a_{x,z}$ the least letter of the alphabet A such that $x^{\omega}a_{x,z}z^{\omega}$ is in canonical form (note that such letter exists since we are assuming A not singular);
- $u \neq \epsilon$ in rules $(s.4)$ and $(s.5)$;
- rules $(s.4)$ and $(s.5)$ apply in case (5.2) only if $u \neq a_{x,z}$.

Rule $(s.1)$ is a consequence of Corollary 3.2 (a). Rules $(s.2)$ and $(s.3)$ are derived from identities (3.4). In rules $(s.4)$ and $(s.5)$, applying identity (3.5) and shifts eventually, one gets from the left side of the rule the term

$$\delta_0 = x^{\omega+p}(z^{\omega+q}vyx^{\omega+r})^{\omega-1}z^{\omega+s}v.$$

The, possibly new, crucial ω-portion $\theta = x^{\omega}z^{\omega}$ of δ_0 may be not in canonical form and so δ_0 may be not in semi-canonical form. If θ is in canonical form, then $\delta(x, z, p, q, r, s) = \delta_0$.

Suppose now that θ is not a canonical term. Hence, as conditions (a) and (b) of the rank 1 canonical form definition hold, x must be a prefix of z^{ℓ} for some $\ell > 0$. So $z = z_1z_2$ and $x = (z_1z_2)^{\ell-1}z_1$ for some words z_1, z_2 with $z_1 \neq \epsilon$. Since x is a Lyndon word (and, so, it cannot have a proper prefix which is also a suffix), it follows that $\ell = 1$. We conclude that x is a prefix of z. Note that, conversely, if x is a prefix of z then θ is not in canonical form. This case is split into three subcases. If either $x = z$ and $q = s$, or $x = z$, $q \neq s$ and $p = r$, then δ_0 is Σ-equivalent to the semi-canonical terms $x^{\omega+p}(vyx^{\omega+r})^{\omega-1}v$ and $(x^{\omega+q}vy)^{\omega-1}x^{\omega+s}v$ respectively. Otherwise, δ_0 is Σ-equivalent to the semi-canonical term $x^{\omega+p}a_{x,z}(z^{\omega+q}vyx^{\omega+r}a_{x,z})^{\omega-1}z^{\omega+s}v$. In this case, we impose that $u \neq a_{x,z}$ to guarantee that the application of the rule does not return as a result the original $\bar{\kappa}$-term.

Proposition 5.2 *Let $\gamma \in \mathcal{S}_2$ and let γ' be a $\bar{\kappa}$-term obtained from γ by applying a rule of $\mathcal{R}$. Then $\gamma' \in \mathcal{S}$.*

Proof By the hypothesis of the proposition, $\gamma = \gamma_1\gamma_2\gamma_3$ with $\mathrm{rank}(\gamma_2) = 2$, $\gamma' = \gamma_1\gamma_2'\gamma_3$ and $\gamma_2 \to \gamma_2'$ is a rule of $\mathcal{R}$, since the rules apply only to rank 2 $\bar{\kappa}$-terms and $\mathrm{rank}(\gamma) = 2$. Moreover $\gamma_2 \in \mathcal{S}_2$ since $\gamma \in \mathcal{S}_2$.

Each ω-portion σ of γ_2' is an ω-portion of γ_2 for every rewriting rules with the only possible exceptions where $\sigma = x^\omega z^\omega$ or $\sigma = x^\omega a_{x,z} z^\omega$ and rule $\gamma_2 \to \gamma_2'$ is one of $(s.4)$ and $(s.5)$, with $\delta(x, z, p, q, r, s)$ given by (5.1) and (5.2) respectively. However, σ is in canonical form in both cases. Therefore $\gamma_2' \in \mathcal{S}$ in all cases, since $\gamma_2 \in \mathcal{S}_2$ by hypothesis. As γ_2 and γ_2' always have the same initial and final ω-portions, it follows that $\gamma' \in \mathcal{S}$. □

For a rank 1 $\bar{\kappa}$-term σ, with rank configuration $\sigma = u_0 x_1^{\omega+q_1} u_1 \cdots x_\ell^{\omega+q_\ell} u_\ell$, we define the *size* of σ, denoted $\mathrm{s}(\sigma)$, as the 4-tuple of non-negative integers

$$\mathrm{s}(\sigma) = \big(\ell, |q_1| + |q_2| + \cdots + |q_\ell|, |u_0 u_1 \cdots u_\ell|, \Sigma_{u_0 u_1 \cdots u_\ell}\big)$$

where $\Sigma_\epsilon = 0$ and, if $u_0 u_1 \cdots u_\ell = a_1 a_2 \cdots a_r$ and $a_1, a_2, \ldots, a_r \in A$, $\Sigma_{u_0 u_1 \cdots u_\ell}$ is the sum of the order of each letter a_i in the ordered alphabet A. We consider the image of the function size ordered by the lexicographic order. With this definition it can be seen that in a shortening $t \to t'$, the size of the base of the rank 2 limit term which occurs in t' is always strictly less than the size of that which occurs in t.

Now, the *size* of a rank 2 $\bar{\kappa}$-term α, with rank configuration $\alpha = \rho_0 \pi_1^{\omega-1} \rho_1 \cdots \pi_m^{\omega-1} \rho_m$, is introduced as the m-tuple

$$\mathrm{s}(\alpha) = \big(\mathrm{s}(\pi_1), \ldots, \mathrm{s}(\pi_m)\big)$$

consisting of the sizes of bases of the limit subterms of α. We consider sizes of rank 2 $\bar{\kappa}$-terms ordered by the shortlex order, that is, if α and β are rank 2 $\bar{\kappa}$-terms with 2-lengths m and n respectively, then $\mathrm{s}(\alpha) \le \mathrm{s}(\beta)$ if and only if $m < n$ or $m = n$ and $\mathrm{s}(\alpha) \le^{\mathrm{lex}} \mathrm{s}(\beta)$ for the lexicographic order $\le^{\mathrm{lex}}$. Notice that this ordering is a well-order on the set of sizes of rank 2 $\bar{\kappa}$-terms.

Let γ be a $\bar{\kappa}$-term from $\mathcal{S}_2$ with 2-length ℓ and let γ' be a $\bar{\kappa}$-term obtained from γ by applying a rewriting rule (r) of $\mathcal{R}$. Then $\gamma = \gamma_1\gamma_2\gamma_3$ where $\mathrm{rank}(\gamma_2) = 2$, $\gamma' = \gamma_1\gamma_2'\gamma_3$ and (r) is $\gamma_2 \to \gamma_2'$. We say that the rule is applied in position $j \in \{1, \ldots, \ell\}$ if the 2-length of γ_1 is $j - 1$ (where we assume the 2-length of γ_1 to be 0 in case its rank is lower than 2).

Proposition 5.3 *The rewriting system $\mathcal{R}$ is Noetherian.*

Proof Let $\gamma \in \mathcal{S}_2$ and let ℓ be the 2-length of γ. Suppose that

$$\gamma = \gamma_1 \to \gamma_2 \to \gamma_3 \cdots$$

is a chain of $\bar{\kappa}$-terms obtained from γ by the application of rewriting rules from $\mathcal{R}$. We want to show that this chain is finite. Suppose it is infinite. Since eliminations and agglutinations strictly decrease the rank or the 2-length of the $\bar{\kappa}$-term, and no rule increases rank or 2-length, they can be used at most ℓ times in the above chain. Without loss of generality, we may therefore assume that the chain uses only shifts right and shortenings. This means in particular that every $\bar{\kappa}$-term γ_j of the chain has the same 2-length ℓ.

Now, as shortenings strictly decrease the size of rank 2 $\bar{\kappa}$-terms, there must be an infinite number of steps where the sizes of the $\bar{\kappa}$-terms do not decrease, and so shifts right must be applied an infinite number of times. On the other hand, rule $(sr.1)$ can only be applied consecutively a finite number of times and preserves the size of rank 2 $\bar{\kappa}$-terms. It follows that shortenings and $(sr.1)$ can only be applied consecutively a finite number of times. Therefore, rule $(sr.2)$ must be applied an infinite number of times.

Let $j \in \{1, \ldots, \ell\}$ be the least position in which $(sr.2)$ is applied an infinite number of times. Whence, in positions less than j, $(sr.2)$ is applied only a finite number of times. Observe that shortenings and shifts right applied on a position i preserve the sizes of all the bases (of limit subterms) with the only exception of the base on position i (in case the rule is a shortening) and the base on position $i+1$ (in case the rule is $(sr.2)$). Consequently, shortenings and shifts right are used only a finite number of times in positions less than j. So, without loss of generality, we may assume that no rule is used in those positions. We may further assume that only rules $(sr.1)$ and $(sr.2)$ are used in position j. We claim that rule $(sr.2)$ may be used in position j only once. This contradicts the arguments that support the choice of j, so the proof of the claim concludes the proof of the proposition.

In order to prove the claim, suppose that $(sr.2)$ is used in some step, say k, in position j (of γ_k). So γ_k and γ_{k+1} are respectively of the forms $\rho_1(uv)^{\omega-1}(uw)^{\omega-1}\rho_2$ and $\rho_1 u(vu)^{\omega-1}w(uwuw)^{\omega-1}\rho_2$, where $u \in A^+$, $\text{rank}(v) = \text{rank}(w) = 1$, $\mathbf{K} \not\models v = w$ and v and w do not have a common non-empty prefix. Let k' be the first step after step k in which a rule is used in position j. Then, it is clear that $\gamma_{k'}$ is of the form $\rho_1 u(vu)^{\omega-1}\rho_3$ where ρ_3 and w have the same initial ω-portion, since shifts right and shortenings preserve such portions. Hence, from the assumption above on v and w, it is not possible to apply any shift right on position j of $\gamma_{k'}$. In particular, it is not possible to apply $(sr.2)$ again in position j, which means that in position j rule $(sr.2)$ could be applied only once. □

It is easy to verify that the following conditions are equivalent for any $\bar{\kappa}$-term α:

- α is in **LG**-canonical form;
- no intermediate step of the algorithm modifies α;
- $\alpha^* = \alpha$;
- every subterm of α is in **LG**-canonical form.

Example 5.4 Consider the following $\bar{\kappa}$-terms of $\mathcal{S}_2$,

$$\alpha = b(ab)^{\omega-5}cb(ab)^{\omega+2}c\Big(b(ab)^{\omega+2}c\Big)^{\omega-1}ac^{\omega-3}\Big(b^{\omega}a^{\omega-1}c\Big)^{\omega-1}b^{\omega}a^{\omega+1}c \Big(b^{\omega-2}aca^{\omega+4}c\Big)^{\omega-1}b^{\omega+1}$$
$$\beta = d^{\omega}b\Big(ad^{\omega-1}cd^{\omega+3}bad^{\omega}b\Big)^{\omega-1}\Big(ab(cd)^{\omega-2}a\Big)^{\omega-1}.$$

The **LG** canonical forms of α and β can be computed as follows:

$$\begin{aligned}
\alpha &\xrightarrow[(e.2)]{} b(ab)^{\omega-5}cac^{\omega-3}\Big(b^{\omega}a^{\omega-1}c\Big)^{\omega-1}b^{\omega}a^{\omega+1}c\Big(b^{\omega-2}aca^{\omega+4}c\Big)^{\omega-1}b^{\omega+1}\\
&\xrightarrow[(sr.1)]{} b(ab)^{\omega-5}cac^{\omega-3}b^{\omega}\Big(a^{\omega-1}cb^{\omega}\Big)^{\omega-1}a^{\omega+1}c\Big(b^{\omega-2}aca^{\omega+4}c\Big)^{\omega-1}b^{\omega+1}\\
&\xrightarrow[(a.1)]{} b(ab)^{\omega-5}cac^{\omega-3}b^{\omega}a^{\omega}c\Big(b^{\omega-2}aca^{\omega+2}cb^{\omega}a^{\omega}c\Big)^{\omega-1}b^{\omega+1}\\
&\xrightarrow[(s.4)]{} b(ab)^{\omega-5}cac^{\omega-3}\Big(b^{\omega-2}aca^{\omega+2}c\Big)^{\omega-1}b^{\omega+1}\\
&\xrightarrow[(s.3)]{} b(ab)^{\omega-5}cac^{\omega-3}\Big(b^{\omega}aca^{\omega+2}c\Big)^{\omega-1}b^{\omega+3}=\alpha^*;\\
\beta &\xrightarrow[(sr.2)]{} d^{\omega}ba\Big(d^{\omega-1}cd^{\omega+3}bad^{\omega}ba\Big)^{\omega-1}b(cd)^{\omega-2}a\Big(ab(cd)^{\omega-2}aab(cd)^{\omega-2}a\Big)^{\omega-1}\\
&\xrightarrow[(s.1)]{} \Big(d^{\omega-1}cd^{\omega+3}ba\Big)^{\omega-1}b(cd)^{\omega-2}a\Big(ab(cd)^{\omega-2}aab(cd)^{\omega-2}a\Big)^{\omega-1}\\
&\xrightarrow[(s.2)]{} \Big(d^{\omega-1}cd^{\omega+3}ba\Big)^{\omega-1}b(cd)^{\omega}a\Big(ab(cd)^{\omega-2}aab(cd)^{\omega}a\Big)^{\omega-1}=\beta^*.
\end{aligned}$$

6 Characterizing $\bar{\kappa}$-terms of $\mathcal{S}$ with finite words

In [14], the authors show that, for rank 1 canonical $\bar{\kappa}$-terms π and ρ, the $\bar{\kappa}$-identity $\pi = \rho$ holds over **LG** only when π and ρ are the same $\bar{\kappa}$-term. This is done by associating to the pair (π, ρ), when π and ρ are distinct rank 1 canonical $\bar{\kappa}$-terms, an alphabet V and a pair $(\mathsf{w}_\pi, \mathsf{w}_\rho)$ of distinct words over V. Afterwards, a finite local group $S_{\pi,\rho}$ is built from $(\mathsf{w}_\pi, \mathsf{w}_\rho)$ and it is shown that $S_{\pi,\rho}$ does not satisfy $\pi = \rho$.

In this section, we slightly improve the above construction and extend it to the elements of $\mathcal{S}_2$. To each element α of $\mathcal{S}$ is assigned a positive integer $\mathfrak{q}_\alpha$ defined by

$$\mathfrak{q}_\alpha = \begin{cases} 1 + \max\{|q| : \omega{+}q \text{ occurs in } \alpha\} & \text{when } \alpha \in \mathcal{C}_1 \\ 1 + \max\{|q| : \omega{+}q \text{ occurs in } \alpha^{(1)}\} & \text{when } \alpha \in \mathcal{S}_2 \end{cases}.$$

We will associate to α and any integer $\mathfrak{q} \geq \mathfrak{q}_\alpha$ a word over an alphabet of the form $\mathsf{V} \cup \mathsf{V}^{-1}$, denoted by $\mathsf{w}_\mathfrak{q}(\alpha)$ and called the $\mathfrak{q}$-*outline of* α. Its reduced form in the free group F_V is denoted by $\widetilde{\mathsf{w}}_\mathfrak{q}(\alpha)$ and named the $\mathfrak{q}$-*root of* α.

Outlines and roots

We begin by recalling the definition of a $\mathfrak{q}$-outline of a $\bar{\kappa}$-term $\alpha \in \mathcal{C}_1$, introduced (without a name) in [14]. We will make minor adjustments on that notion and on the notations. Let $\alpha = u_0 x_1^{\omega+q_1} u_1 \cdots x_n^{\omega+q_n} u_n$ be the rank configuration of α and notice that α is $\Sigma_\mathbf{S}$-equivalent to the $\bar{\kappa}$-term

$$(u_0x_1^{\omega})x_1^{\omega+q_1}(x_1^{\omega}u_1x_2^{\omega})x_2^{\omega+q_2}\cdots x_{n-1}^{\omega+q_{n-1}}(x_{n-1}^{\omega}u_{n-1}x_n^{\omega})x_n^{\omega+q_n}(x_n^{\omega}u_n).$$

The $\bar{\kappa}$-terms $u_0x_1^\omega$, $x_n^\omega u_n$, $x_i^\omega u_i x_{i+1}^\omega$ and x_j are the initial ω-portion, the final ω-portion, the crucial ω-portions and the bases of limit terms of α. We will represent them by symbols i_{u_0,x_1}, t_{x_n,u_n}, $\mathsf{c}_{x_i,u_i,x_{i+1}}$ and b_{x_j} of an alphabet V, called respectively an *initial*, a *final*, a *crucial* and a *base variable*. We associate to α and $\mathbb{q}$ the word $\mathsf{w}_{\mathbb{q}}(\alpha)$ over V, called the $\mathbb{q}$-*outline of* α, given by

$$\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{i}_{u_0,x_1}\mathsf{b}_{x_1}^{\mathbb{q}_1}\mathsf{c}_{x_1,u_1,x_2}\mathsf{b}_{x_2}^{\mathbb{q}_2}\cdots\mathsf{b}_{x_{n-1}}^{\mathbb{q}_{n-1}}\mathsf{c}_{x_{n-1},u_{n-1},x_n}\mathsf{b}_{x_n}^{\mathbb{q}_n}\mathsf{t}_{x_n,u_n},$$

where $\mathbb{q}_j = \mathbb{q}+q_j$. The condition $\mathbb{q} \geq \mathbb{q}_\alpha$ was introduced in [14] in order to avoid non-positive exponents in $\mathsf{w}_{\mathbb{q}}(\alpha)$. Let $\underline{\mathsf{w}}_{\mathbb{q}}(\alpha) = \mathsf{b}_{x_1}^{\mathbb{q}_1}\mathsf{c}_{x_1,u_1,x_2}\mathsf{b}_{x_2}^{\mathbb{q}_2}\cdots\mathsf{b}_{x_{n-1}}^{\mathbb{q}_{n-1}}\mathsf{c}_{x_{n-1},u_{n-1},x_n}\mathsf{b}_{x_n}^{\mathbb{q}_n}$, so that $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{i}_{u_0,x_1}\underline{\mathsf{w}}_{\mathbb{q}}(\alpha)\mathsf{t}_{x_n,u_n}$. We remark that the initial and final variables were not used in [14], where the initial and final ω-portions of α were taken into account by the introduction of two other variables. These two approaches are perfectly homologous but the (minor) changes introduced here seem to be more natural.

The $\mathbb{q}$-outline $\mathsf{w}_{\mathbb{q}}(\alpha)$, of any element α of $\mathcal{S}_2$, can be obtained by the application of the two following recursive steps.

1) Consider $\alpha = \pi^{\omega-1}$, with $\pi = u_0x_1^{\omega+q_1}u_1\cdots x_n^{\omega+q_n}u_n$. Notice that, for every positive integer k: the k-expansion $\alpha^{(k)}$ $(= \pi^k)$ belongs to $\mathcal{C}_1$; the initial and final ω-portions, $u_0x_1^\omega$ and $x_n^\omega u_n$, of π are the initial and final ω-portions of α and of $\alpha^{(k)}$; and

$$\mathsf{w}_{\mathbb{q}}(\alpha^{(k)}) = \mathsf{i}_{u_0,x_1}(\mathsf{b}_{x_1}^{\mathbb{q}_1}\mathsf{c}_{x_1,u_1,x_2}\mathsf{b}_{x_2}^{\mathbb{q}_2}\cdots\mathsf{b}_{x_n}^{\mathbb{q}_n}\mathsf{c}_{x_n,u_nu_0,x_1})^{k-1}\mathsf{b}_{x_1}^{\mathbb{q}_1}\mathsf{c}_{x_1,u_1,x_2}\mathsf{b}_{x_2}^{\mathbb{q}_2}\cdots\mathsf{b}_{x_n}^{\mathbb{q}_n}\mathsf{t}_{x_n,u_n}.$$

Furthermore, in the free group F_{V},

$$\mathsf{w}_{\mathbb{q}}(\alpha^{(k)}) = \mathsf{i}_{u_0,x_1}(\mathsf{b}_{x_1}^{\mathbb{q}_1}\mathsf{c}_{x_1,u_1,x_2}\mathsf{b}_{x_2}^{\mathbb{q}_2}\cdots\mathsf{b}_{x_n}^{\mathbb{q}_n}\mathsf{c}_{x_n,u_nu_0,x_1})^{k}\mathsf{c}_{x_n,u_nu_0,x_1}^{-1}\mathsf{t}_{x_n,u_n}.$$

Each finite group G verifies $g^\ell = 1_G$ for some positive integer $\ell > 2$. Therefore, over G,

$$\begin{aligned}\mathsf{w}_{\mathbb{q}}(\alpha^{(\ell-1)}) &= \mathsf{i}_{u_0,x_1}(\mathsf{b}_{x_1}^{\mathbb{q}_1}\mathsf{c}_{x_1,u_1,x_2}\mathsf{b}_{x_2}^{\mathbb{q}_2}\cdots\mathsf{b}_{x_n}^{\mathbb{q}_n}\mathsf{c}_{x_n,u_nu_0,x_1})^{\ell-1}\mathsf{c}_{x_n,u_nu_0,x_1}^{-1}\mathsf{t}_{x_n,u_n}\\ &= \mathsf{i}_{u_0,x_1}(\mathsf{b}_{x_1}^{\mathbb{q}_1}\mathsf{c}_{x_1,u_1,x_2}\mathsf{b}_{x_2}^{\mathbb{q}_2}\cdots\mathsf{b}_{x_n}^{\mathbb{q}_n}\mathsf{c}_{x_n,u_nu_0,x_1})^{-1}\mathsf{c}_{x_n,u_nu_0,x_1}^{-1}\mathsf{t}_{x_n,u_n}\\ &= \mathsf{i}_{u_0,x_1}\mathsf{c}_{x_n,u_nu_0,x_1}^{-1}\mathsf{b}_{x_n}^{-\mathbb{q}_n}\cdots\mathsf{b}_{x_2}^{-\mathbb{q}_2}\mathsf{c}_{x_1,u_1,x_2}^{-1}\mathsf{b}_{x_1}^{-\mathbb{q}_1}\mathsf{c}_{x_n,u_nu_0,x_1}^{-1}\mathsf{t}_{x_n,u_n}.\end{aligned}$$

In this case, we define the $\mathbb{q}$-*outline* of α as the following word over the alphabet $\mathsf{V}\cup\mathsf{V}^{-1}$,

$$\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{i}_{u_0,x_1}\mathsf{c}_{x_n,u_nu_0,x_1}^{-1}\mathsf{b}_{x_n}^{-\mathbb{q}_n}\mathsf{c}_{x_{n-1},u_{n-1},x_n}^{-1}\mathsf{b}_{x_{n-1}}^{-\mathbb{q}_{n-1}}\cdots\mathsf{b}_{x_2}^{-\mathbb{q}_2}\mathsf{c}_{x_1,u_1,x_2}^{-1}\mathsf{b}_{x_1}^{-\mathbb{q}_1}\mathsf{c}_{x_n,u_nu_0,x_1}^{-1}\mathsf{t}_{x_n,u_n}.$$

Denoting $\underline{\mathsf{w}}_{\mathbb{q}}(\alpha) = \mathsf{c}_{x_n,u_nu_0,x_1}^{-1}\mathsf{b}_{x_n}^{-\mathbb{q}_n}\mathsf{c}_{x_{n-1},u_{n-1},x_n}^{-1}\mathsf{b}_{x_{n-1}}^{-\mathbb{q}_{n-1}}\cdots\mathsf{b}_{x_2}^{-\mathbb{q}_2}\mathsf{c}_{x_1,u_1,x_2}^{-1}\mathsf{b}_{x_1}^{-\mathbb{q}_1}\mathsf{c}_{x_n,u_nu_0,x_1}^{-1}$,

$\mathsf{w}_{\mathbb{q}}(\alpha)$ may be written as $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{i}_{u_0,x_1}\underline{\mathsf{w}}_{\mathbb{q}}(\alpha)\mathsf{t}_{x_n,u_n}$ also in this case.

2) Suppose that $\alpha = \alpha_1\alpha_2$ and notice that, as observed in Sect. 4, each subterm α_j is a semi-canonical form. If α_j is rank 1 or rank 2, then $\alpha_j \in \mathcal{C}_1 \cup \mathcal{S}_2$ and we assume $\mathsf{w}_{\mathbb{q}}(\alpha_j)$ already defined and of the form $\mathsf{w}_{\mathbb{q}}(\alpha_j) = \mathsf{i}_{u_j,x_j}\underline{\mathsf{w}}_{\mathbb{q}}(\alpha_j)\mathsf{t}_{y_j,v_j}$.
If α_1 is rank 0, then we let $\mathsf{w}_{\mathbb{q}}(\alpha)$ be the word $\mathsf{i}_{\alpha_1 u_2,x_2}\underline{\mathsf{w}}_{\mathbb{q}}(\alpha_2)\mathsf{t}_{y_2,v_2}$. Symmetrically, if α_2 is rank 0, then we take $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{i}_{u_1,x_1}\underline{\mathsf{w}}_{\mathbb{q}}(\alpha_1)\mathsf{t}_{y_1,v_1\alpha_2}$. Finally, for $\mathrm{rank}(\alpha_j) \in \{1,2\}$, let $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{i}_{u_1,x_1}\underline{\mathsf{w}}_{\mathbb{q}}(\alpha_1)\mathsf{c}_{y_1,v_1u_2,x_2}\underline{\mathsf{w}}_{\mathbb{q}}(\alpha_2)\mathsf{t}_{y_2,v_2}$. In this case, the crucial variable $\mathsf{c}_{y_1,v_1u_2,x_2}$ are also denoted by $\mathsf{c}(\alpha_1,\alpha_2)$, whence $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{i}_{u_1,x_1}\underline{\mathsf{w}}_{\mathbb{q}}(\alpha_1)\mathsf{c}(\alpha_1,\alpha_2)\underline{\mathsf{w}}_{\mathbb{q}}(\alpha_2)\mathsf{t}_{y_2,v_2}$.
Observe that two different factorizations $(\alpha_1\alpha_2)\alpha_3$ and $\alpha_1(\alpha_2\alpha_3)$ of α determine the same word $\mathsf{w}_{\mathbb{q}}(\alpha)$, so the above definition is correct.

Let $\alpha \in \mathcal{S}$ and let ux^ω and $y^\omega v$ be, respectively, the initial and the final ω-portions of α. The variables $\mathsf{i}_{u,x}$ and $\mathsf{t}_{y,v}$ are also denoted respectively by $\mathsf{i}(\alpha)$ and $\mathsf{t}(\alpha)$. Then, by the above definition, it is clear that $\mathsf{w}_{\mathbb{q}}(\alpha)$ may be written as

$$\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{i}(\alpha)\underline{\mathsf{w}}_{\mathbb{q}}(\alpha)\mathsf{t}(\alpha) \tag{6.1}$$

for some word $\underline{\mathsf{w}}_{\mathbb{q}}(\alpha)$. Moreover each of $\mathsf{i}(\alpha)$ and $\mathsf{t}(\alpha)$ has exactly one occurrence in the word $\mathsf{w}_{\mathbb{q}}(\alpha)$. Now, let $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$ be the reduced form of $\mathsf{w}_{\mathbb{q}}(\alpha)$ in the free group F_{V} generated by V. The word $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$ is called the $\mathbb{q}$-*root* of α. By (6.1),

$$\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \mathsf{i}(\alpha)\widetilde{\underline{\mathsf{w}}}_{\mathbb{q}}(\alpha)\mathsf{t}(\alpha) \tag{6.2}$$

where $\widetilde{\underline{\mathsf{w}}}_{\mathbb{q}}(\alpha)$ is the reduced form of $\underline{\mathsf{w}}_{\mathbb{q}}(\alpha)$ in F_{V}. In particular, when $\alpha \in \mathcal{C}_1$ the outline $\mathsf{w}_{\mathbb{q}}(\alpha)$ is a word of V^+ and, so, $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\mathbb{q}}(\alpha)$.

Example 6.1 Consider the $\bar{\kappa}$-term α of Example 5.4. We have $\mathbb{q}_\alpha = 6$ and so, for any $\mathbb{q} \geq 6$, the $\mathbb{q}$-outline and the $\mathbb{q}$-root of α are the following

$$\begin{aligned}
\mathsf{w}_{\mathbb{q}}(\alpha) = {} & \mathsf{i}_{b,ab}\mathsf{b}_{ab}^{\mathbb{q}-5}\mathsf{c}_{ab,cb,ab}\mathsf{b}_{ab}^{\mathbb{q}+2}\mathsf{c}_{ab,cb,ab}\mathsf{c}_{ab,cb,ab}^{-1}\mathsf{b}_{ab}^{-(\mathbb{q}+2)}\mathsf{c}_{ab,cb,ab}^{-1}\mathsf{c}_{ab,ca,c}\mathsf{b}_c^{\mathbb{q}-3}\\
& \mathsf{c}_{c,\epsilon,b}\mathsf{c}_{a,c,b}^{-1}\mathsf{b}_a^{-(\mathbb{q}-1)}\mathsf{c}_{b,\epsilon,a}^{-1}\mathsf{b}_b^{-\mathbb{q}}\mathsf{c}_{a,c,b}^{-1}\mathsf{c}_{a,c,b}\mathsf{b}_b^{\mathbb{q}}\mathsf{c}_{b,\epsilon,a}\mathsf{b}_a^{\mathbb{q}+1}\mathsf{c}_{a,c,b}\\
& \mathsf{c}_{a,c,b}^{-1}\mathsf{b}_a^{-(\mathbb{q}+4)}\mathsf{c}_{b,ac,a}^{-1}\mathsf{b}_b^{-(\mathbb{q}-2)}\mathsf{c}_{a,c,b}^{-1}\mathsf{c}_{a,c,b}\mathsf{b}_b^{\mathbb{q}+1}\mathsf{t}_{b,\epsilon}\\
\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = {} & \mathsf{i}_{b,ab}\mathsf{b}_{ab}^{\mathbb{q}-5}\mathsf{c}_{ab,ca,c}\mathsf{b}_c^{\mathbb{q}-3}\mathsf{c}_{c,\epsilon,b}\mathsf{c}_{a,c,b}^{-1}\mathsf{b}_a^{-(\mathbb{q}+2)}\mathsf{c}_{b,ac,a}^{-1}\mathsf{b}_b^3\mathsf{t}_{b,\epsilon}.
\end{aligned}$$

The **LG** canonical form of α is $\alpha^* = b(ab)^{\omega-5}cac^{\omega-3}\big(b^\omega aca^{\omega+2}c\big)^{\omega-1}b^{\omega+3}$ and, so,

$$\begin{aligned}
&\mathsf{w}_{\mathbb{q}}(\alpha^*) = \mathsf{i}_{b,ab}\mathsf{b}_{ab}^{\mathbb{q}-5}\mathsf{c}_{ab,ca,c}\mathsf{b}_c^{\mathbb{q}-3}\mathsf{c}_{c,\epsilon,b}\mathsf{c}_{a,c,b}^{-1}\mathsf{b}_a^{-(\mathbb{q}+2)}\mathsf{c}_{b,ac,a}^{-1}\mathsf{b}_b^{-\mathbb{q}}\mathsf{c}_{a,c,b}^{-1}\mathsf{c}_{a,c,b}\mathsf{b}_b^{\mathbb{q}+3}\mathsf{t}_{b,\epsilon}\\
&\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha^*) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha).
\end{aligned}$$

Notice that the $\mathbb{q}$-outline of a $\bar{\kappa}$-term is a well-defined expression involving the parameter $\mathbb{q}$. Therefore, for $\alpha, \beta \in \mathcal{S}$ and $\mathbb{q}, \mathbb{q}' \geq \max\{\mathbb{q}_\alpha, \mathbb{q}_\beta\}$, $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\mathbb{q}}(\beta)$ if and only if $\mathsf{w}_{\mathbb{q}'}(\alpha) = \mathsf{w}_{\mathbb{q}'}(\beta)$. The condition $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\mathbb{q}}(\beta)$ implies that, either α and β are the same $\bar{\kappa}$-term, or one of them is obtained from the other applying a finite number of rank 2 shifts of the form $(uv)^{\omega-1}u = u(vu)^{\omega-1}$ with $u \in A^+$. In case α

and β are canonical forms, they are both irreducible for rule $(sr.1)$ and, so, $\alpha = \beta$ if and only if $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\mathbb{q}}(\beta)$.

A necessary condition for the identity of two $\bar{\kappa}$-terms over $\mathbf{LG}$

In this section we show that a necessary condition for the equality over $\mathbf{LG}$ of two $\bar{\kappa}$-terms of $\mathcal{S}$ is the equality of their roots.

Proposition 6.2 *Let $\alpha, \beta \in \mathcal{S}$ and let $\mathbb{q} \geq \max\{\mathbb{q}_\alpha, \mathbb{q}_\beta\}$. If $\mathbf{LG} \models \alpha = \beta$, then $\widetilde{w}_{\mathbb{q}}(\alpha) = \widetilde{w}_{\mathbb{q}}(\beta)$.*

Proof Assume that $\mathbf{LG} \models \alpha = \beta$. Then $\mathbf{LI} \models \alpha = \beta$, which means, by (6.2), that the $\mathbb{q}$-roots $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$ and $\widetilde{\mathsf{w}}_{\mathbb{q}}(\beta)$ have the same initial and final variables, say $\mathsf{i}_{u,x}$ and $\mathsf{t}_{y,v}$ respectively. Suppose, by way of contradiction, that $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) \neq \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta)$. The case in which $\alpha, \beta \in \mathcal{C}_1$ was already treated in [14, Theorem 5.1]. So, we assume without loss of generality that $\alpha \in \mathcal{S}_2$. We adapt the tools and results of [14] to manage the present situation by using expansions of α and of β in case $\beta \in \mathcal{S}_2$ (see Sect. 2.5 and [14] for more details and missing definitions).

We begin by building a finite local group $S_{\alpha,\beta}$ of the form $S_{\alpha,\beta} = \mathcal{S}(G, L, \mathsf{f})$ as follows. As $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) \neq \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta)$, there exists a finite group G that fails the identity $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\mathbb{q}}(\beta)$. Hence, there is a homomorphism $\eta : (\mathsf{V} \cup \mathsf{V}^{-1})^* \to G$ such that $\eta(\mathsf{w}_{\mathbb{q}}(\alpha)) \neq \eta(\mathsf{w}_{\mathbb{q}}(\beta))$ and $\eta(\mathsf{v}^{-1}) = \eta(\mathsf{v})^{-1}$ for each $\mathsf{v} \in \mathsf{V}$. For each variable v_* of V occurring in $\mathsf{w}_{\mathbb{q}}(\alpha)$ or $\mathsf{w}_{\mathbb{q}}(\beta)$, denote $\eta(\mathsf{v}_*)$ by $g_{\mathsf{v},*}$. By [14, Claim 1 of Section 5], the order of $g_{\mathsf{v},*}$ may be taken greater than $\max\{|\mathsf{w}_{\mathbb{q}}(\alpha)|, |\mathsf{w}_{\mathbb{q}}(\beta)|\}$. By (6.1) and the fact that η is a homomorphism,

$$\eta(\mathsf{w}_{\mathbb{q}}(\alpha)) = g_{\mathsf{i},u,x}\,\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\alpha))\,g_{\mathsf{t},y,v} \quad \text{and} \quad \eta(\mathsf{w}_{\mathbb{q}}(\beta)) = g_{\mathsf{i},u,x}\,\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\beta))\,g_{\mathsf{t},y,v}. \tag{6.3}$$

Next, let L and f be the ones that would be chosen by the process of [14, Theorem 5.1] for the rank 1 canonical forms α_1 and β_1 such that $\alpha_1 = \alpha^{(2)}$ and $\beta_1 = \beta^{(2)}$ when $\operatorname{rank}(\beta) = 2$ or $\beta_1 = \beta$ when $\operatorname{rank}(\beta) = 1$. This completes the definition of the semigroup $S_{\alpha,\beta} = \mathcal{S}(G, L, \mathsf{f})$.

Since $S_{\alpha,\beta}$ is a finite semigroup, there is a positive integer $\ell > 2$ such that $s^\omega = s^\ell$ for every $s \in S_{\alpha,\beta}$. In particular, as G is isomorphic to a subgroup of $S_{\alpha,\beta}$, $g^\ell = 1_G$ for all $g \in G$. Let $\widehat{\alpha} = \alpha^{(\ell-1)}$ and let $\widehat{\beta} = \beta^{(\ell-1)}$ in case $\operatorname{rank}(\beta) = 2$ and $\widehat{\beta} = \beta$ otherwise. Therefore, since $S_{\alpha,\beta} \in \mathbf{LG}$ and $\mathbf{LG} \models \alpha = \beta$, $S_{\alpha,\beta}$ satisfies $\widehat{\alpha} = \alpha = \beta = \widehat{\beta}$. On the other hand, $\mathbb{q}_{\widehat{\alpha}} = \mathbb{q}_\alpha$ and $\mathbb{q}_{\widehat{\beta}} = \mathbb{q}_\beta$, so that $\mathbb{q} \geq \max\{\mathbb{q}_{\widehat{\alpha}}, \mathbb{q}_{\widehat{\beta}}\}$. By the choice of ℓ, one can verify easily from the definition of $\mathbb{q}$-outline that the equalities $\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\widehat{\alpha})) = \eta(\underline{\mathsf{w}}_{\mathbb{q}}(\alpha))$ and $\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\widehat{\beta})) = \eta(\underline{\mathsf{w}}_{\mathbb{q}}(\beta))$ hold.

Now, let $\phi : T_A^{\bar{\kappa}} \to S_{\alpha,\beta}$ be the homomorphism of $\bar{\kappa}$-semigroups defined by $\phi(a) = a$ for $a \in A$. Since α_1 and $\widehat{\alpha}$ (resp. β_1 and $\widehat{\beta}$) have the same portions and the parameters L and f of the semigroup $S_{\alpha,\beta} = \mathcal{S}(G, L, \mathsf{f})$ depend only on those portions and on the homomorphism η, one can verify by the proof of [14, Theorem 5.1] that $\phi(\widehat{\alpha})$ and $\phi(\widehat{\beta})$ are triples of the form $(\_\,, h_0\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\widehat{\alpha}))h_1, \_)$ and $(\_\,, h_0\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\widehat{\beta}))h_1, \_)$ where h_0 is $g_{\mathsf{b},x}$ when $u \neq \epsilon$ and it is 1_G otherwise, and h_1 is $g_{\mathsf{b},y}$ when $v \neq \epsilon$ and it is 1_G otherwise. Since $S_{\alpha,\beta}$ satisfies $\widehat{\alpha} = \widehat{\beta}$, it follows that $\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\widehat{\alpha})) = \eta(\underline{\mathsf{w}}_{\mathbb{q}}(\widehat{\beta}))$. As $\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\widehat{\alpha})) = \eta(\underline{\mathsf{w}}_{\mathbb{q}}(\alpha))$ and $\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\widehat{\beta})) = \eta(\underline{\mathsf{w}}_{\mathbb{q}}(\beta))$, it follows that $\eta(\underline{\mathsf{w}}_{\mathbb{q}}(\alpha)) = \eta(\underline{\mathsf{w}}_{\mathbb{q}}(\beta))$, whence, by (6.3), $\eta(\mathsf{w}_{\mathbb{q}}(\alpha)) = \eta(\mathsf{w}_{\mathbb{q}}(\beta))$. However, we affirmed above that $\eta(\mathsf{w}_{\mathbb{q}}(\alpha)) \neq \eta(\mathsf{w}_{\mathbb{q}}(\beta))$ as a consequence of assuming that $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) \neq \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta)$. Hence, this condition does not hold, thus concluding the proof of the proposition. □

An immediate consequence of Proposition 6.2 is that, for any $\alpha \in \mathcal{S}_2$, $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha^*)$, where α^* is the canonical form of α and $\mathbb{q} \geq \max\{\mathbb{q}_\alpha, \mathbb{q}_{\alpha^*}\}$.

Properties of the $\mathbb{q}$-root of a $\bar{\kappa}$-term

In the remaining of the paper, for a given $\alpha \in \mathcal{S}_2$ with 2-length m, we will usually consider its rank configuration of the form

$$\alpha = \alpha_0\alpha_1^{\omega-1}\alpha_2\cdots\alpha_{2m-1}^{\omega-1}\alpha_{2m}. \tag{6.4}$$

Notice that the $\mathbb{q}$-outline $\mathsf{w}_{\mathbb{q}}(\alpha)$ may be written as

$$\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\alpha,0}\mathsf{w}_{\alpha,1}\mathsf{w}_{\alpha,2}\cdots\mathsf{w}_{\alpha,2m-1}\mathsf{w}_{\alpha,2m}$$

where: $\mathsf{w}_{\alpha,2i-1} = \underline{\mathsf{w}}_{\mathbb{q}}(\alpha_{2i-1}^{\omega-1})$ is a non-empty word on V^{-1} for each odd index $2i - 1 \in \{1, 3, \ldots, 2m-1\}$; $\mathsf{w}_{\alpha,2i'}$ is a non-empty word on V for each even index $2i' \in \{0, 2, \ldots, 2m\}$. We then call each $\mathsf{w}_{\alpha,2i-1}$ a *negative block* and each $\mathsf{w}_{\alpha,2i'}$ a *positive block* of $\mathsf{w}_{\mathbb{q}}(\alpha)$. Observe that, in each $\mathsf{w}_{\alpha,j}$ ($j \in \{0, 1, \ldots, 2m\}$), crucial variables alternate with powers of base variables. More precisely, for an odd j the alternation is of the form $\mathsf{c}_{x,\_,\_}^{-1}\mathsf{b}_x^{-r}\mathsf{c}_{\_,\_,x}^{-1}$, and for an even j it is of the form $\mathsf{c}_{\_,\_,x}\mathsf{b}_x^{r}\mathsf{c}_{x,\_,\_}$, where r is a positive integer. Moreover, $\mathsf{w}_{\alpha,j}$ begins and ends with a crucial variable except for $j = 0$, in which case it begins with the initial variable $\mathsf{i}(\alpha)$, and for $j = 2m$, in which case it ends with the final variable $\mathsf{t}(\alpha)$.

Although, for the calculation of the $\mathbb{q}$-root $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$, the occurrences of *spurs* (i.e., products of the form vv^{-1} or $v^{-1}v$ with $v \in \mathsf{V}$) in $\mathsf{w}_{\mathbb{q}}(\alpha)$ may be canceled in any order, we will assume that each cancelation step consists in deleting the leftmost occurrence of a spur. With this assumption, the process of cancelation of $\mathsf{w}_{\mathbb{q}}(\alpha)$ transforms each block $\mathsf{w}_{\alpha,j}$ into a unique and well-determined (possibly empty) word, called the *remainder* of $\mathsf{w}_{\alpha,j}$ and denoted $\mathsf{r}_{\alpha,j}$, so that

$$\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \mathsf{r}_{\alpha,0}\mathsf{r}_{\alpha,1}\mathsf{r}_{\alpha,2}\cdots\mathsf{r}_{\alpha,2m-1}\mathsf{r}_{\alpha,2m}.$$

In particular, the reduction process can, possibly, eliminate completely some of the negative blocks of $\mathsf{w}_{\mathbb{q}}(\alpha)$ or gather into a unique negative block of $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$ some factors occurring in distinct negative blocks of $\mathsf{w}_{\mathbb{q}}(\alpha)$, in which case the intermediate positive blocks are completely deleted.

For a finite word w over the alphabet $\mathsf{V} \cup \mathsf{V}^{-1}$, we define the *crucial length* of w as the number of occurrences of crucial variables in w, and denote it by $|\mathsf{w}|_{\mathsf{c}}$. For each $j \in \{0, 1, \ldots, 2m\}$, we denote by $\mathbb{c}_{\alpha,j}$ the number of occurrences of crucial variables in $\mathsf{w}_{\alpha,j}$ that are canceled in the computation of $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$, that is,

$$\mathbb{c}_{\alpha,j} = |\mathsf{w}_{\alpha,j}|_{\mathsf{c}} - |\mathsf{r}_{\alpha,j}|_{\mathsf{c}}.$$

Note that $|\mathsf{w}_{\alpha,j}|_{\mathsf{c}}$ is the 1-length of α_j in case $j \in \{0, 2m\}$ and it is equal to the 1-length of α_j plus one otherwise. Since the cancelations in $\mathsf{w}_{\alpha,j}$ are performed from the extremes, $\mathsf{w}_{\alpha,j} = \overleftarrow{\mathsf{w}}_{\alpha,j}\mathsf{r}_{\alpha,j}\overrightarrow{\mathsf{w}}_{\alpha,j}$ where $\overleftarrow{\mathsf{w}}_{\alpha,j}$ (resp. $\overrightarrow{\mathsf{w}}_{\alpha,j}$) is the longest prefix (resp. suffix) of $\mathsf{w}_{\alpha,j}$ that is canceled by variables occurring on its left side (resp. right side). The following lateral versions of $\mathbb{c}_{\alpha,j}$ will be convenient. We let

$$'\mathbb{c}_{\alpha,j} = |\overleftarrow{\mathsf{w}}_{\alpha,j}|_{\mathsf{c}}, \quad \mathbb{c}'_{\alpha,j} = |\overrightarrow{\mathsf{w}}_{\alpha,j}|_{\mathsf{c}},$$

and notice that $\mathbb{c}_{\alpha,j} = {}'\mathbb{c}_{\alpha,j} + \mathbb{c}'_{\alpha,j}$ and ${}'\mathbb{c}_{\alpha,j} = 0$ (resp. $\mathbb{c}'_{\alpha,j} = 0$) if and only if $\overleftarrow{\mathsf{w}}_{\alpha,j} = \epsilon$ (resp. $\overrightarrow{\mathsf{w}}_{\alpha,j} = \epsilon$) since each intermediate block begins and ends with a crucial variable.

The following lemma presents important properties of the $\mathbb{q}$-root of α in case $\alpha \in \mathcal{C}_2$.

Lemma 6.3 *Let α be a $\bar{\kappa}$-term of $\mathcal{C}_2$ with rank configuration of the form* (6.4) *and let* $j \in \{1, 2, \ldots, 2m-1\}$.

(a) If j is odd, then ${}'\mathbb{c}_{\alpha,j} \leq 2$ and $\mathbb{c}'_{\alpha,j} \leq 1$ with $\mathbb{c}_{\alpha,j} \leq 2$.
(b) $|r_{\alpha,j}|_c \neq 0$.

Remark 6.4 Note that, in the context of Lemma 6.3, for all $j \in \{1, 2, \ldots, 2m-1\}$, $\mathsf{r}_{\alpha,j}$ is non-empty by (b). Therefore, the number of negative blocks of $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha)$ is equal to the 2-length m of α. Moreover, the cancelation of the prefix $\overleftarrow{\mathsf{w}}_{\alpha,j}$ (resp. the suffix $\overrightarrow{\mathsf{w}}_{\alpha,j}$) of $\mathsf{w}_{\alpha,j}$ is caused only by the adjacent block $\mathsf{w}_{\alpha,j-1}$ (resp. $\mathsf{w}_{\alpha,j+1}$). That is, informally speaking, each block has only a "local influence". This means that, for each $j \in \{1, 2, \ldots, 2m\}$, $\overrightarrow{\mathsf{w}}_{\alpha,j-1}$ and $\overleftarrow{\mathsf{w}}_{\alpha,j}$ are mutually inverse words in F_{V} and, therefore, $\mathbb{c}'_{\alpha,j-1} = {}'\mathbb{c}_{\alpha,j}$.

Proof of Lemma 6.3 The proof is made by induction on m. Assume first that $m = 1$ and so $j = 1$, $\alpha = \alpha_0\alpha_1^{\omega-1}\alpha_2$ and $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\alpha,0}\mathsf{w}_{\alpha,1}\mathsf{w}_{\alpha,2}$. Let $\alpha_1 = u_0x_1^{\omega+q_1}u_1 \cdots x_n^{\omega+q_n}u_n$ be the rank configuration of α_1, whence

$$\mathsf{w}_{\alpha,1} = \mathsf{c}^{-1}_{x_n,u_nu_0,x_1}\mathsf{b}^{-\mathbb{q}_n}_{x_n}\mathsf{c}^{-1}_{x_{n-1},u_{n-1},x_n} \cdots \mathsf{b}^{-\mathbb{q}_2}_{x_2}\mathsf{c}^{-1}_{x_1,u_1,x_2}\mathsf{b}^{-\mathbb{q}_1}_{x_1}\mathsf{c}^{-1}_{x_n,u_nu_0,x_1}.$$

Supposing that α_1 is a generic rank 1 $\bar{\kappa}$-term with $n > 1$ and $q_n = 0$, we define the term $x^{\omega}_{n-1}u_{n-1}x^{\omega}_nu_n$ to be the *final ω2-portion of* α_1. To prove condition (a), we consider two cases.

Case 1. α_2 has not $u_0x_1^{\omega}$ as initial *ω-portion.*
Hence $\mathsf{c}_{x_n,u_nu_0,x_1}$ is not the initial variable of $\mathsf{w}_{\alpha,2}$ and, so, $\mathbb{c}'_{\alpha,1} = 0$. If α_0 has not final ω-portion $x_n^{\omega}u_n$, then $\mathsf{c}_{x_n,u_nu_0,x_1}$ is not the final variable of $\mathsf{w}_{\alpha,0}$, whence ${}'\mathbb{c}_{\alpha,1} = 0$ and $\mathbb{c}_{\alpha,1} = 0$.
Now, suppose that α_0 has final ω-portion $x_n^{\omega}u_n$. Since α is irreducible for shortenings $(s.2)$, $q_n = 0$ and α_0 is of the form $\alpha'_0x_n^{\omega+p}u_n$ with $p \in \mathbb{Z}$. On

the other hand, $\mathsf{w}_{\alpha,0} = \mathsf{i}(\alpha_0)\underline{\mathsf{w}}_{\mathbb{q}}(\alpha_0)\mathsf{c}(\alpha_0, \alpha_1)$, whence $\mathsf{w}_{\alpha,0}$ is of the form $\mathsf{w}'_{\alpha,0}\mathsf{b}_{x_n}^{\mathbb{p}}\mathsf{c}_{x_n,u_nu_0,x_1}$. Suppose $p \neq 0$. Hence, $\overleftarrow{\mathsf{w}}_{\alpha,1} = \mathsf{c}_{x_n,u_nu_0,x_1}^{-1}\mathsf{b}_{x_n}^{-p'}$ (and $\overrightarrow{\mathsf{w}}_{\alpha,0} = \mathsf{b}_{x_n}^{p'}\mathsf{c}_{x_n,u_nu_0,x_1}$) where p' is $\mathbb{q}$ when $p > 0$ and it is $\mathbb{q} + p$ when $p < 0$. Therefore ${}'\mathbb{c}_{\alpha,1} = 1$ and so $\mathbb{c}_{\alpha,1} = 1$.

Let now $p = 0$, so that $\alpha_0 = \alpha'_0 x_n^{\omega} u_n$. If $n = 1$, then $\alpha = \alpha'_0 x_1^{\omega} u_1 (u_0 x_1^{\omega} u_1)^{\omega-1} \alpha_2$ and $x_1^{\omega} u_1 u_0$ cannot be the final ω-portion of α'_0 since otherwise an elimination $(e.2)$ could be applied. So, arguing as above one deduces that $\mathbb{c}_{\alpha,1} = {}'\mathbb{c}_{\alpha,1} = 1$. These equalities hold also for $n > 1$ and α'_0 having not final ω-portion $x_{n-1}^{\omega} u_{n-1}$. It remains to treat the instance in which $n > 1$ and α'_0 has final ω-portion $x_{n-1}^{\omega} u_{n-1}$. In this case, $q_{n-1} = 0$, α_0 is of the form $\alpha''_0 x_{n-1}^{\omega+r} u_{n-1} x_n^{\omega} u_n$ and ${}'\mathbb{c}_{\alpha,1} \geq 2$. If $r \neq 0$, then $\mathbb{c}_{\alpha,1} = {}'\mathbb{c}_{\alpha,1} = 2$. Suppose now that $r = 0$ and notice that the $\bar{\kappa}$-term

$$\gamma = \begin{cases} x_{n-2}^{\omega} u_{n-2} & \text{if } n > 2 \\ x_2^{\omega} u_2 u_0 & \text{if } n = 2 \end{cases}$$

cannot be the final ω-portion of α''_0 since, otherwise, it would be possible to apply a shortening $(s.5)$, with $u = u_{n-2} x_{n-1}^{\omega} u_{n-1}$, and an elimination $(e.2)$ respectively. Whence $\mathbb{c}_{\alpha,1} = {}'\mathbb{c}_{\alpha,1} = 2$.

CASE 2. α_2 has initial *ω-portion* $u_0 x_1^{\omega}$.

Since α is irreducible for shifts right and shortenings $(s.3)$, $u_0 = \epsilon$, $q_1 = 0$ and α_2 is of the form $\alpha_2 = x_1^{\omega+s}\alpha'_2$ with $s \neq 0$. On the other hand, $\mathsf{w}_{\alpha,2} = \mathsf{c}(\alpha_1, \alpha_2)\underline{\mathsf{w}}_{\mathbb{q}}(\alpha_2)\mathsf{t}(\alpha_2)$, whence $\mathsf{w}_{\alpha,2}$ is of the form $\mathsf{w}_{\alpha,2} = \mathsf{c}_{x_n,u_n,x_1}\mathsf{b}_{x_1}^{\mathbb{q}+s}\mathsf{w}'_{\alpha,2}$. Therefore $\overrightarrow{\mathsf{w}}_{\alpha,1} = \mathsf{b}_{x_1}^{-s'}\mathsf{c}_{x_n,u_n,x_1}^{-1}$ (and $\overleftarrow{\mathsf{w}}_{\alpha,2} = \mathsf{c}_{x_n,u_n,x_1}\mathsf{b}_{x_1}^{s'}$) where s' is $\mathbb{q}$ when $s > 0$ and it is $\mathbb{q}+s$ when $s < 0$. It follows that $\mathbb{c}'_{\alpha,1} = 1$. If α_0 has not final ω-portion $x_n^{\omega} u_n$, then c_{x_n,u_n,x_1} is not the final variable of $\mathsf{w}_{\alpha,0}$ and, as a consequence, ${}'\mathbb{c}_{\alpha,1} = 0$ and $\mathbb{c}_{\alpha,1} = 1$. Suppose now that α_0 has final ω-portion $x_n^{\omega} u_n$. Hence $n > 1$ since α is irreducible for eliminations $(e.1)$. On the other hand, as α is irreducible for shortenings $(s.2)$, $q_n = 0$ and $\alpha_0 = \alpha'_0 x_n^{\omega+p} u_n$ with $p \in \mathbb{Z}$.

If $p \neq 0$, then one derives as above that ${}'\mathbb{c}_{\alpha,1} = 1$ and concludes that $\mathbb{c}_{\alpha,1} = 2$. Suppose now that $p = 0$ and notice that $x_{n-1}^{\omega} u_{n-1} x_n^{\omega} u_n$ can not be the final $\omega 2$-portion of α_0. Indeed, otherwise, it would be possible to apply an elimination $(e.1)$ if $n = 2$ and a shortening $(s.4)$ if $n > 2$, with $u = u_{n-1} x_n^{\omega} u_n$ in both cases. As a consequence, $\mathsf{c}_{x_{n-1},u_{n-1},x_n}\mathsf{b}_{x_n}^{\mathbb{q}}\mathsf{c}_{x_n,u_n,x_1}$ is not a suffix of $\mathsf{w}_{\alpha,0}$ and, so, the equalities ${}'\mathbb{c}_{\alpha,1} = 1$ and $\mathbb{c}_{\alpha,1} = 2$ also hold for $p = 0$.

The above analysis shows that, in all possible cases, ${}'\mathbb{c}_{\alpha,j} \leq 2$ and $\mathbb{c}'_{\alpha,j} \leq 1$ with $\mathbb{c}_{\alpha,j} \leq 2$, thus proving (a) for $m = 1$.

Condition (b) follows easily from (a). Indeed, $|\mathsf{w}_{\alpha,1}|_{\mathsf{c}} \geq 2$. So, by (a), $|\mathsf{r}_{\alpha,1}|_{\mathsf{c}} = 0$ if and only if $|\mathsf{w}_{\alpha,1}|_{\mathsf{c}} = \mathbb{c}_{\alpha,1} = 2$, in which case $n = 1$. Since, by the proof of (a), $\mathbb{c}_{\alpha,1} = 2$ only for $n > 1$, it follows that $|\mathsf{r}_{\alpha,1}|_{\mathsf{c}} > 0$, thus proving (b) for $m = 1$.

Let now $m > 1$ and suppose, by induction hypothesis, that the result holds for $\bar{\kappa}$-terms of $\mathcal{C}_2$ with 2-length at most $m - 1$. Let $\vec{\alpha} = \alpha_0 \alpha_1^{\omega-1} \alpha_2 \cdots \alpha_{2m-3}^{\omega-1} \alpha_{2m-2} u x^{\omega}$ and

$\overleftarrow{\alpha} = y^\omega v\alpha_{2m-2}\alpha_{2m-1}^{\omega-1}\alpha_{2m}$, where ux^ω and $y^\omega v$ are, respectively, the initial ω-portion of α_{2m-1} and the final ω-portion of α_{2m-3}. As $\mathbb{q} \geq \mathbb{q}_\alpha$ and $\mathbb{q}_\alpha = \max\{\mathbb{q}_{\vec{\alpha}}, \mathbb{q}_{\overleftarrow{\alpha}}\}$, we can write

$$\begin{aligned}\mathsf{w}_{\mathbb{q}}(\alpha) &= \mathsf{w}_{\alpha,0}\mathsf{w}_{\alpha,1}\mathsf{w}_{\alpha,2}\mathsf{w}_{\alpha,3}\cdots\mathsf{w}_{\alpha,2m}\\ \mathsf{w}_{\mathbb{q}}(\vec{\alpha}) &= \mathsf{w}_{\vec{\alpha},0}\mathsf{w}_{\vec{\alpha},1}\cdots\mathsf{w}_{\vec{\alpha},2m-2} = \mathsf{w}_{\alpha,0}\mathsf{w}_{\alpha,1}\cdots\mathsf{w}_{\alpha,2m-2}\mathsf{b}_x^{\mathbb{q}}\mathsf{t}_{x,\epsilon}\\ \mathsf{w}_{\mathbb{q}}(\overleftarrow{\alpha}) &= \mathsf{w}_{\overleftarrow{\alpha},0}\mathsf{w}_{\overleftarrow{\alpha},1}\mathsf{w}_{\overleftarrow{\alpha},2} = \mathsf{i}_{\epsilon,y}\mathsf{b}_y^{\mathbb{q}}\mathsf{w}_{\alpha,2m-2}\mathsf{w}_{\alpha,2m-1}\mathsf{w}_{\alpha,2m}.\end{aligned}$$

The $\bar{\kappa}$-terms $\vec{\alpha}$ and $\overleftarrow{\alpha}$ are clearly in $\mathcal{S}_2$. Moreover, as α is a canonical form, $\vec{\alpha}$ is necessarily in $\mathcal{C}_2$. Indeed, $\vec{\alpha}$ is irreducible for shifts right because α is irreducible for shifts right and agglutinations. Given the shape of the rewriting rules of $\mathcal{R}$, the only rules that could eventually be applied to $\vec{\alpha}$ are $(e.1)$, $(s.3)$ and $(s.4)$. However in these cases it would be possible to apply the same rule or an agglutination in α.

The $\bar{\kappa}$-term $\overleftarrow{\alpha}$ may not be in $\mathcal{C}_2$. Although, analyzing the possible reductions, as done for $\vec{\alpha}$, we conclude that the only rewriting rule that can be applied to $\overleftarrow{\alpha}$ is shortening $(s.1)$. This happens when $v = v'v''$ and $\overleftarrow{\alpha}$ is of the form $y^\omega v'\sigma(\tau\sigma)^{\omega-1}\alpha_{2m}$ with $v'' \in A^+, \sigma = v''\alpha_{2m-2}$ and $\mathbf{LI} \models \tau = \sigma$. In such case $y^\omega v'$ is not the final ω-portion of τ since agglutination $(a.3)$ does not apply on α. Moreover, the canonical form of $\overleftarrow{\alpha}$, obtained by applying the shortening $(s.1)$, is $\overleftarrow{\alpha}^* = y^\omega v'\tau^{\omega-1}\alpha_{2m}$. The respective $\mathbb{q}$-outline $\mathsf{w}_{\mathbb{q}}(\overleftarrow{\alpha}^*)$ is such that

$$\mathsf{w}_{\mathbb{q}}(\overleftarrow{\alpha}^*) = \mathsf{w}_{\overleftarrow{\alpha}^*,0}\mathsf{w}_{\overleftarrow{\alpha}^*,1}\mathsf{w}_{\overleftarrow{\alpha}^*,2} = \mathsf{r}_{\overleftarrow{\alpha}^*,0}\mathsf{r}_{\overleftarrow{\alpha}^*,1}\mathsf{w}_{\overleftarrow{\alpha}^*,1}\mathsf{w}_{\overleftarrow{\alpha}^*,2}\mathsf{r}_{\overleftarrow{\alpha}^*,2},$$

and $|\mathsf{r}_{\overleftarrow{\alpha}^*,0}|_{\mathsf{c}} = |\mathsf{w}_{\overleftarrow{\alpha}^*,0}|_{\mathsf{c}} = 1$. Since $\mathbb{q} \geq \mathbb{q}_{\overleftarrow{\alpha}} \geq \mathbb{q}_{\overleftarrow{\alpha}^*}$, $\widetilde{\mathsf{w}}_{\mathbb{q}}(\overleftarrow{\alpha}^*) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\overleftarrow{\alpha})$ by Proposition 6.2, and so $\mathsf{r}_{\overleftarrow{\alpha}^*,i} = \mathsf{r}_{\overleftarrow{\alpha},i}$ for $i = 0, 1, 2$.

By the induction hypothesis, the statement holds for both $\vec{\alpha}$ and $\overleftarrow{\alpha}^*$. In particular, the occurrences of crucial variables in $\mathsf{w}_{\vec{\alpha},2m-3}$ $(= \mathsf{w}_{\alpha,2m-3}$) are not all canceled in the computation of $\widetilde{\mathsf{w}}_{\mathbb{q}}(\vec{\alpha})$, and so $|\mathsf{r}_{\vec{\alpha},2m-3}|_{\mathsf{c}} \geq 1$. Analogously, there exist occurrences of crucial variables in $\mathsf{w}_{\overleftarrow{\alpha}^*,1}$ that are not canceled in the reduction of $\mathsf{w}_{\mathbb{q}}(\overleftarrow{\alpha}^*)$, which implies that $|\mathsf{r}_{\overleftarrow{\alpha},1}|_{\mathsf{c}} \geq 1$ since $|\mathsf{r}_{\overleftarrow{\alpha},1}|_{\mathsf{c}} = |\mathsf{r}_{\overleftarrow{\alpha}^*,1}|_{\mathsf{c}}$. Putting together these two facts, we deduce that $|\mathsf{r}_{\alpha,2m-3}|_{\mathsf{c}}$ and $|\mathsf{r}_{\alpha,2m-1}|_{\mathsf{c}}$ are both positive, thus showing, in particular, that each block has only a "local influence" in the reduction process. Furthermore, $\mathsf{r}_{\vec{\alpha},2m-3} = \mathsf{r}_{\alpha,2m-3}$, because we begin deleting the leftmost spurs, and ${}'\mathbb{c}_{\alpha,2m-1} \leq {}'\mathbb{c}_{\overleftarrow{\alpha},1}$. Therefore, statement (a) follows immediately from the induction hypothesis applied to $\vec{\alpha}$ and $\overleftarrow{\alpha}^*$.

To conclude the proof of statement (b), and of the lemma, it remains to show that $|\mathsf{r}_{\alpha,2m-2}|_{\mathsf{c}} \neq 0$. From $|r_{\alpha,2m-2}|_c \leq |r_{\alpha,2m-2}|$, we get ${}'\mathbb{c}_{\alpha,2m-1} = {}'\mathbb{c}_{\overleftarrow{\alpha},1}$ as an immediate consequence. We know already that the cancelations on $\mathsf{w}_{\alpha,2m-2}$ are determined only by the adjacent blocks $\mathsf{w}_{\alpha,2m-3}$ and $\mathsf{w}_{\alpha,2m-1}$. So, it suffices to consider the subterm $\alpha_{2m-3,2m-1} = \alpha_{2m-3}^{\omega-1}\alpha_{2m-2}\alpha_{2m-1}^{\omega-1}$ of α which, as one recalls, is a canonical form. To begin with, notice that $|\mathsf{w}_{\alpha,2m-2}|_{\mathsf{c}} = \ell + 1$ where ℓ is the 1-length of α_{2m-2}. On the other hand, by (a), ${}'\mathbb{c}_{\alpha,2m-2} = \mathbb{c}'_{\alpha,2m-3} \leq 1$ and $\mathbb{c}'_{\alpha,2m-2} = {}'\mathbb{c}_{\alpha,2m-1} \leq 2$ so that $\mathbb{c}_{\alpha,2m-2} \leq 3$. Suppose by way of contradiction that $|\mathsf{r}_{\alpha,2m-2}|_{\mathsf{c}} = 0$ and, so, that $\ell \leq 2$. Let us analyse, for each of the three possible values of ℓ, what could hypothetically be the forms of $\alpha_{2m-3,2m-1}$ and verify that, actually, those possibilities are not compatible with $\alpha_{2m-3,2m-1}$ being a canonical form.

1) $\ell = 0$, that is, $\alpha_{2m-2} = w_0 \in A^*$. In this case $|\mathsf{w}_{\alpha,2m-2}|_{\mathsf{c}} = 1$ and so, by hypothesis, $\mathbb{c}_{\alpha,2m-2} = 1$. Hence, either ${}^{\prime}\mathbb{c}_{\alpha,2m-2} = 1$ and $\mathbb{c}^{\prime}_{\alpha,2m-2} = 0$, or ${}^{\prime}\mathbb{c}_{\alpha,2m-2} = 0$ and $\mathbb{c}^{\prime}_{\alpha,2m-2} = 1$. Then $\alpha_{2m-3,2m-1}$ is of one of the forms $\alpha_{2m-3,2m-1} = (w_0 u x^{\omega+p} \rho_1)^{\omega-1} w_0 (u x^{\omega+q} \rho_3)^{\omega-1}$ or $\alpha_{2m-3,2m-1} = (\rho_1 y^{\omega+p} v)^{\omega-1} w_0 (\rho_3 y^{\omega+q} v w_0)^{\omega-1}$.
2) $\ell = 1$, say with $\alpha_{2m-2} = w_0 z_1^{\omega+q_1} w_1$. Then $|\mathsf{w}_{\alpha,2m-2}|_{\mathsf{c}} = \mathbb{c}_{\alpha,2m-2} = 2$ and either ${}^{\prime}\mathbb{c}_{\alpha,2m-2} = 1$ and $\mathbb{c}^{\prime}_{\alpha,2m-2} = 1$, or ${}^{\prime}\mathbb{c}_{\alpha,2m-2} = 0$ and $\mathbb{c}^{\prime}_{\alpha,2m-2} = 2$. In this circumstance, $\alpha_{2m-3,2m-1}$ is of one of the forms $\alpha_{2m-3,2m-1} = (z_1^{\omega} \rho_1)^{\omega-1} z_1^{\omega+q_1} w_1 (\rho_3 z_1^{\omega} w_1)^{\omega-1}$, in which case w_0 must be empty, or $\alpha_{2m-3,2m-1} = (\rho_1 y^{\omega+p} v)^{\omega-1} w_0 z_1^{\omega} w_1 (\rho_3 y^{\omega+r} v w_0 z_1^{\omega} w_1)^{\omega-1}$, in which case $q_1 = 0$.
3) $\ell = 2$, with $\alpha_{2m-2} = w_0 z_1^{\omega+q_1} w_1 z_2^{\omega+q_2} w_2$. Hence $|\mathsf{w}_{\alpha,2m-2}|_{\mathsf{c}} = \mathbb{c}_{\alpha,2m-2} = 3$ with ${}^{\prime}\mathbb{c}_{\alpha,2m-2} = 1$ and $\mathbb{c}^{\prime}_{\alpha,2m-2} = 2$. In this case $w_0 = \epsilon$, $q_2 = 0$ and $\alpha_{2m-3,2m-1}$ is of the form $\alpha_{2m-3,2m-1} = (z_1^{\omega} \rho_1)^{\omega-1} z_1^{\omega+q_1} w_1 z_2^{\omega} w_2 (\rho_3 z_1^{\omega} w_1 z_2^{\omega} w_2)^{\omega-1}$.

In all of the above situations it is possible to make a shift right or an agglutination on $\alpha_{2m-3,2m-1}$ and, so, this $\bar{\kappa}$-term is not a canonical form. Consequently, $|\mathsf{r}_{\alpha,2m-2}|_{\mathsf{c}} > 0$ and the proof is complete. □

It is useful, for later reference, to state the following facts shown in the proof of Lemma 6.3.

Remark 6.5 For an integer p let p' denote $\mathbb{q}$ when $p \geq 0$ and let it denote $\mathbb{q} + p$ otherwise. For a $\bar{\kappa}$-term α in the conditions of Lemma 6.3, let j be an odd position and let $\alpha_j = u_0 x_1^{\omega+q_1} u_1 \cdots x_n^{\omega+q_n} u_n$. Then,

(a) $\mathbb{c}^{\prime}_{\alpha,j} = 1$ if and only if $u_0 = \epsilon$, $q_1 = 0$ and α_{j+1} is of the form $\alpha_{j+1} = x_1^{\omega+p} \alpha'_{j+1}$ with $p \neq 0$. Moreover, in this case, $\overrightarrow{\mathsf{w}}_{\alpha,j} = \mathsf{b}_{x_1}^{-p'} \mathsf{c}_{x_n,u_n,x_1}^{-1}$.
(b) ${}^{\prime}\mathbb{c}_{\alpha,j} = 2$ if and only if $n > 1$, $q_{n-1} = q_n = 0$ and

$$\alpha_{j-1} = \begin{cases} \alpha'_{j-1} x_{n-1}^{\omega+p} x_n^{\omega} u_n & \text{if } x_{n-1}^{\omega} x_n^{\omega} \text{is in canonical form} \\ \alpha'_{j-1} x_{n-1}^{\omega+p} a_{x_{n-1},x_n} x_n^{\omega} u_n & \text{otherwise.} \end{cases}$$

Therefore, $\overleftarrow{\mathsf{w}}_{\alpha,j} = \mathsf{c}_{x_n,u_n u_0,x_1}^{-1} \mathsf{b}_{x_n}^{-\mathbb{q}} \mathsf{c}_{x_{n-1},\epsilon,x_n}^{-1} \mathsf{b}_{x_{n-1}}^{-p'}$ if $x_{n-1}^{\omega} x_n^{\omega}$ is in canonical form and $\overleftarrow{\mathsf{w}}_{\alpha,j} = \mathsf{c}_{x_n,u_n u_0,x_1}^{-1} \mathsf{b}_{x_n}^{-\mathbb{q}} \mathsf{c}_{x_{n-1},a_{x_{n-1},x_n},x_n}^{-1} \mathsf{b}_{x_{n-1}}^{-p'}$ otherwise.
(c) ${}^{\prime}\mathbb{c}_{\alpha,j} = 1$ if and only if $q_n = 0$, $\alpha_{j-1} = \alpha'_{j-1} x_n^{\omega+p} u_n$ and, when $n > 1$, $x_{n-1}^{\omega} u_{n-1} x_n^{\omega} u_n$ is not the final $\omega 2$-portion of α_{j-1}. In this case, $\overleftarrow{\mathsf{w}}_{\alpha,j} = \mathsf{c}_{x_n,u_n u_0,x_1}^{-1} \mathsf{b}_{x_n}^{-p'}$.
(d) for $\mathbb{c}^{\prime}_{\alpha,j} = {}^{\prime}\mathbb{c}_{\alpha,j} = 1$, $u_n = \epsilon$ if $x_n^{\omega} x_1^{\omega}$ is in canonical form and $u_n = a_{x_n,x_1}$ otherwise.

7 Uniqueness of the canonical forms

This section is dedicated to prove the following fundamental theorem, that shows the uniqueness of the canonical forms over **LG**.

Theorem 7.1 *Let α and β be canonical $\bar{\kappa}$-terms. If* $\mathbf{LG} \models \alpha = \beta$*, then* $\alpha = \beta$.

We begin by showing a preliminary result.

Proposition 7.2 *Let α and β be canonical forms such that* $\mathbf{LG} \models \alpha = \beta$.

(a) The $\bar{\kappa}$-terms α and β have the same rank.
(b) If rank$(\alpha) \leq 1$, then $\alpha = \beta$.

Proof By hypothesis $\mathbf{LG} \models \alpha = \beta$. Hence, as $\mathbf{LI}$ is a subpseudovariety of $\mathbf{LG}$ that separates different words and words from $\bar{\kappa}$-terms with rank at least 1, if one of α and β is a rank 0 $\bar{\kappa}$-term then they are the same $\bar{\kappa}$-term. We may therefore assume that α and β have at least rank 1. Then $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta)$ for $\mathbb{q} \geq \max\{\mathbb{q}_\alpha, \mathbb{q}_\beta\}$, by Proposition 6.2. Thus α and β must have the same rank, since the $\mathbb{q}$-root of a rank 1 $\bar{\kappa}$-term is a word from V^+ and, by Lemma 6.3, the $\mathbb{q}$-root of a rank 2 canonical form contains negative blocks. This proves (a). Statement (b) is a consequence of (a) and [14, Theorem 5.1]. □

To complete the proof of Theorem 7.1 it remains to treat the instance in which α and β are both rank 2 canonical forms.

Proposition 7.3 *Let $\alpha, \beta \in \mathcal{C}_2$. If* $\mathbf{LG} \models \alpha = \beta$*, then* $\alpha = \beta$.

This proposition is an immediate consequence of Proposition 6.2 and the following lemma.

Lemma 7.4 *Let $\alpha, \beta \in \mathcal{C}_2$ and let $\mathbb{q} \geq \max\{\mathbb{q}_\alpha, \mathbb{q}_\beta\}$. If $\widetilde{w}_{\mathbb{q}}(\alpha) = \widetilde{w}_{\mathbb{q}}(\beta)$, then $\alpha = \beta$.*

Proof Assume that $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta)$. By Lemma 6.3, the number of negative blocks in the $\mathbb{q}$-root of a rank 2 canonical form is precisely its 2-length. Then α and β have the same 2-length, say m. Consider the rank configurations $\alpha = \alpha_0\alpha_1^{\omega-1}\alpha_2 \cdots \alpha_{2m-1}^{\omega-1}\alpha_{2m}$ and $\beta = \beta_0\beta_1^{\omega-1}\beta_2 \cdots \beta_{2m-1}^{\omega-1}\beta_{2m}$ of α and β. As, for each $i \in \{0, 1, \ldots, 2m\}$, the remainders $\mathsf{r}_{\alpha,i}$ and $\mathsf{r}_{\beta,i}$ are non-empty by Lemma 6.3, the equality $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta)$ implies that $\mathsf{r}_{\alpha,i} = \mathsf{r}_{\beta,i}$. Since α and β are canonical forms, we observed already in the end of Sect. that $\alpha = \beta$ if and only if $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\mathbb{q}}(\beta)$. On the other hand, $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\mathbb{q}}(\beta)$ if and only if $\mathsf{w}_{\alpha,i} = \mathsf{w}_{\beta,i}$ for all i. Recall that, for $\gamma \in \{\alpha, \beta\}$: $\mathsf{w}_{\gamma,i} = \overleftarrow{\mathsf{w}}_{\gamma,i}\mathsf{r}_{\gamma,i}\overrightarrow{\mathsf{w}}_{\gamma,i}$; for $i \neq 0$, $\overrightarrow{\mathsf{w}}_{\gamma,i-1}$ and $\overleftarrow{\mathsf{w}}_{\gamma,i}$ are mutually inverse words in F_{V}; $\overleftarrow{\mathsf{w}}_{\gamma,0} = \overrightarrow{\mathsf{w}}_{\gamma,2m} = \epsilon$. Therefore, to deduce the equality $\alpha = \beta$ it suffices to prove that, for each odd position $j \in \{1, 3, \ldots, 2m-1\}$,

$$\overleftarrow{\mathsf{w}}_{\alpha,j} = \overleftarrow{\mathsf{w}}_{\beta,j} \text{ and } \overrightarrow{\mathsf{w}}_{\alpha,j} = \overrightarrow{\mathsf{w}}_{\beta,j}. \tag{7.1}$$

Throughout, let $j \in \{1, 3, \ldots, 2m-1\}$ be an odd integer and let $\alpha_j = u_0x_1^{\omega+q_1}u_1 \cdots x_n^{\omega+q_n}u_n$ and $\beta_j = v_0y_1^{\omega+p_1}v_1 \cdots y_k^{\omega+p_k}v_k$ be the rank configurations of α_j and β_j. To prove (7.1), let us show first that $\mathsf{w}_{\alpha,j}$ and $\mathsf{w}_{\beta,j}$ admit the same number of right cancelations of occurrences of crucial variables.

Claim 1 $\overleftarrow{\mathbb{c}}_{\alpha,j} = \overleftarrow{\mathbb{c}}_{\beta,j}$.

Proof We know from Lemma 6.3 that $\mathbb{c}'_{\alpha,j}, \mathbb{c}'_{\beta,j} \in \{0,1\}$. Suppose that $\mathbb{c}'_{\alpha,j} = 1$ and $\mathbb{c}'_{\beta,j} = 0$. As observed in Remark 6.5 (a), the equality $\mathbb{c}'_{\alpha,j} = 1$ gives $u_0 = \epsilon$, $q_1 = 0$ and $\alpha_{j+1} = x_1^{\omega+p}\alpha'_{j+1}$ for some integer $p \neq 0$. Hence $\mathsf{r}_{\alpha,j} = \mathsf{r}'_{\alpha,j}\mathsf{b}_{x_1}^{p}$ when $p < 0$, and $\mathsf{r}_{\alpha,j+1} = \mathsf{b}_{x_1}^{p}\mathsf{r}'_{\alpha,j+1}$ when $p > 0$. The equality $\mathbb{c}'_{\beta,j} = 0$ implies that $\mathsf{r}_{\beta,j}$ ends with a crucial variable and that $\mathsf{r}_{\beta,j+1}$ either begins with a crucial variable, or is equal to the final variable $\mathsf{t}(\beta)$ (in which case $j+1 = 2m$ and $\beta_{2m} \in A^*$). So, $\mathsf{r}_{\alpha,j} \neq \mathsf{r}_{\beta,j}$ or $\mathsf{r}_{\alpha,j+1} \neq \mathsf{r}_{\beta,j+1}$. This contradicts the fact that $\mathsf{r}_{\alpha,i} = \mathsf{r}_{\beta,i}$ for all i. Therefore $\mathbb{c}'_{\alpha,j} = 1$ and $\mathbb{c}'_{\beta,j} = 0$ does not apply, and neither does $\mathbb{c}'_{\alpha,j} = 0$ and $\mathbb{c}'_{\beta,j} = 1$ by symmetry, thus proving that $\mathbb{c}'_{\alpha,j} = \mathbb{c}'_{\beta,j}$. □

Let us now show the following:

Claim 2 *If* $'\mathbb{c}_{\alpha,j} = {}'\mathbb{c}_{\beta,j}$, *then* $\overleftarrow{\mathsf{w}}_{\alpha,j} = \overleftarrow{\mathsf{w}}_{\beta,j}$ *and* $\overrightarrow{\mathsf{w}}_{\alpha,j} = \overrightarrow{\mathsf{w}}_{\beta,j}$ *(and, so,* $\alpha_j = \beta_j$*).*

Proof Suppose that $'\mathbb{c}_{\alpha,j} = {}'\mathbb{c}_{\beta,j}$, whence $\mathbb{c}_{\alpha,j} = \mathbb{c}_{\beta,j}$ by Claim 1. Then, from $\mathsf{r}_{\alpha,j} = \mathsf{r}_{\beta,j}$ it follows that $n = k$ and that $\mathsf{w}_{\alpha,j}$ and $\mathsf{w}_{\beta,j}$ are of the form

$$\mathsf{w}_{\alpha,j} = \mathsf{c}^{-1}_{x_n,u_nu_0,x_1}\mathsf{b}_{x_n}^{-\mathbb{q}_n}\mathsf{c}^{-1}_{x_{n-1},u_{n-1},x_n}\cdots\mathsf{b}_{x_2}^{-\mathbb{q}_2}\mathsf{c}^{-1}_{x_1,u_1,x_2}\mathsf{b}_{x_1}^{-\mathbb{q}_1}\mathsf{c}^{-1}_{x_n,u_nu_0,x_1}$$
$$\mathsf{w}_{\beta,j} = \mathsf{c}^{-1}_{y_n,v_nv_0,y_1}\mathsf{b}_{y_n}^{-\mathbb{p}_n}\mathsf{c}^{-1}_{y_{n-1},v_{n-1},y_n}\cdots\mathsf{b}_{y_2}^{-\mathbb{p}_2}\mathsf{c}^{-1}_{y_1,v_1,y_2}\mathsf{b}_{y_1}^{-\mathbb{p}_1}\mathsf{c}^{-1}_{y_n,v_nv_0,y_1}.$$

We begin by showing the equality $\overrightarrow{\mathsf{w}}_{\alpha,j} = \overrightarrow{\mathsf{w}}_{\beta,j}$. If $\mathbb{c}'_{\alpha,j} = 0$ then $\overrightarrow{\mathsf{w}}_{\alpha,j} = \epsilon = \overrightarrow{\mathsf{w}}_{\beta,j}$. It remains to consider $\mathbb{c}'_{\alpha,j} = 1$. In this case $'\mathbb{c}_{\alpha,j} \leq 1$ by Lemma 6.3. Moreover, by Remark 6.5 (a), $u_0 = v_0 = \epsilon$, $q_1 = p_1 = 0$, $\alpha_{j+1} = x_1^{\omega+r}\alpha'_{j+1}$, $\beta_{j+1} = y_1^{\omega+s}\beta'_{j+1}$ for some non-zero integers r and s, $\overrightarrow{\mathsf{w}}_{\alpha,j} = \mathsf{b}_{x_1}^{-r'}\mathsf{c}^{-1}_{x_n,u_n,x_1}$ and $\overrightarrow{\mathsf{w}}_{\beta,j} = \mathsf{b}_{y_1}^{-s'}\mathsf{c}^{-1}_{y_n,v_n,y_1}$ where, for $t \in \{r,s\}$, $t' = \mathbb{q}$ when $t > 0$ and $t' = \mathbb{q} + t$ when $t < 0$. So, as $\mathsf{r}_{\alpha,j} = \mathsf{r}_{\beta,j}$, one deduces that $r = s$ and $x_1 = y_1$. To complete the proof of $\overrightarrow{\mathsf{w}}_{\alpha,j} = \overrightarrow{\mathsf{w}}_{\beta,j}$ it remains to show that $x_n = y_n$ and $u_n = v_n$. For $'\mathbb{c}_{\alpha,j} = 0$, this follows from the equalities $\mathsf{r}_{\alpha,j} = \mathsf{r}_{\beta,j}$ and $u_0 = v_0$. In case $'\mathbb{c}_{\alpha,j} = 1$, from the same arguments, we have also that $x_n = y_n$ and one deduces from Remark 6.5 (d) that $u_n = a_{x_n,x_1} = v_n$ or $u_n = \epsilon = v_n$.

Let us now show the equality $\overleftarrow{\mathsf{w}}_{\alpha,j} = \overleftarrow{\mathsf{w}}_{\beta,j}$. By Lemma 6.3, $'\mathbb{c}_{\alpha,j} \in \{0,1,2\}$. We have therefore to consider three cases.

1) $'\mathbb{c}_{\alpha,j} = 0$. In this case $\overleftarrow{\mathsf{w}}_{\alpha,j} = \epsilon = \overleftarrow{\mathsf{w}}_{\beta,j}$.
2) $'\mathbb{c}_{\alpha,j} = 1$. Then, by Remark 6.5 (c), $q_n = p_n = 0$, $\alpha_{j-1} = \alpha'_{j-1}x_n^{\omega+r}u_n$, $\beta_{j-1} = \beta'_{j-1}y_n^{\omega+s}v_n$ for some integers r and s, $\overleftarrow{\mathsf{w}}_{\alpha,j} = \mathsf{c}^{-1}_{x_n,u_nu_0,x_1}\mathsf{b}_{x_n}^{-r'}$ and $\overleftarrow{\mathsf{w}}_{\beta,j} = \mathsf{c}^{-1}_{y_n,v_nv_0,y_1}\mathsf{b}_{y_n}^{-s'}$ with r' and s' as above. The equality $\overleftarrow{\mathsf{w}}_{\alpha,j} = \overleftarrow{\mathsf{w}}_{\beta,j}$ is now a consequence of the fact that $\mathsf{r}_{\alpha,j}\overrightarrow{\mathsf{w}}_{\alpha,j} = \mathsf{r}_{\beta,j}\overrightarrow{\mathsf{w}}_{\beta,j}$.
3) $'\mathbb{c}_{\alpha,j} = 2$. In this case $\mathbb{c}'_{\alpha,j} = 0$ by Lemma 6.3. Moreover, by Remark 6.5 (b), $n > 1$, $q_n = q_{n-1} = p_n = p_{n-1} = 0$, $\alpha_{j-1} = \alpha'_{j-1}x_{n-1}^{\omega+r}u_{n-1}x_n^{\omega}u_n$ where $u_{n-1} = \epsilon$ if $x_{n-1}^{\omega}x_n^{\omega}$ is in canonical form and $u_{n-1} = a_{x_{n-1}x_n}$ otherwise, and $\beta_{j-1} = \beta'_{j-1}y_{n-1}^{\omega+s}v_{n-1}y_n^{\omega}v_n$ where $v_{n-1} = \epsilon$ if $y_{n-1}^{\omega}y_n^{\omega}$ is in canonical form and $v_{n-1} = a_{y_{n-1}y_n}$ otherwise. Whence, we have that $\overleftarrow{\mathsf{w}}_{\alpha,j} = \mathsf{c}^{-1}_{x_n,u_nu_0,x_1}\mathsf{b}_{x_n}^{-\mathbb{q}}\mathsf{c}^{-1}_{x_{n-1},u_{n-1},x_n}\mathsf{b}_{x_{n-1}}^{-r'}$ and $\overleftarrow{\mathsf{w}}_{\beta,j} = \mathsf{c}^{-1}_{y_n,v_nv_0,y_1}\mathsf{b}_{y_n}^{-\mathbb{q}}\mathsf{c}^{-1}_{y_{n-1},v_{n-1},y_n}\mathsf{b}_{y_{n-1}}^{-s'}$. As above, one deduces from $\mathsf{r}_{\alpha,j}\overrightarrow{\mathsf{w}}_{\alpha,j} = \mathsf{r}_{\beta,j}\overrightarrow{\mathsf{w}}_{\beta,j}$ that $\mathsf{c}_{x_n,u_nu_0,x_1} = \mathsf{c}_{y_n,v_nv_0,y_1}$ and $r' = s'$. So, as $x_n = y_n$, to prove $\overleftarrow{\mathsf{w}}_{\alpha,j} = \overleftarrow{\mathsf{w}}_{\beta,j}$ in this case, it remains to show that

$x_{n-1} = y_{n-1}$. Now, $\mathsf{r}_{\alpha,j-1}$ ends with one of the variables $\mathsf{b}_{x_{n-1}}$, $\mathsf{c}_{\_,\_,x_{n-1}}$ and $\mathsf{i}_{\_,x_{n-1}}$ and, similarly, $\mathsf{r}_{\beta,j-1}$ ends with one of the variables $\mathsf{b}_{y_{n-1}}$, $\mathsf{c}_{\_,\_,y_{n-1}}$ and $\mathsf{i}_{\_,y_{n-1}}$. Since $\mathsf{r}_{\alpha,j-1} = \mathsf{r}_{\beta,j-1}$ it follows that $x_{n-1} = y_{n-1}$.

We have proved that $\overleftarrow{\mathsf{w}}_{\alpha,j} = \overleftarrow{\mathsf{w}}_{\beta,j}$ in all cases. This concludes the proof of the claim. □

We now show that the number of left cancelations of occurrences of crucial variables coincides in $\mathsf{w}_{\alpha,j}$ and $\mathsf{w}_{\beta,j}$ which, in view of Claim 2, will be enough to conclude (7.1).

Claim 3 ${}'\mathbb{c}_{\alpha,j} = {}'\mathbb{c}_{\beta,j}$.

Proof The proof of this claim uses induction on j. By Lemma 6.3, both ${}'\mathbb{c}_{\alpha,j}$ and ${}'\mathbb{c}_{\beta,j}$ belong to $\{0, 1, 2\}$. There are, thus, three cases to look for regarding the value of ${}'\mathbb{c}_{\beta,j}$.
CASE 1. ${}'\mathbb{c}_{\beta,j} = 0$. By contradiction, suppose that ${}'\mathbb{c}_{\alpha,j} \neq 0$. Hence, there are two possibilities.
CASE 1.1. ${}'\mathbb{c}_{\alpha,j} = 2$. Then, by Remark 6.5 (b), $n > 1$, $q_{n-1} = q_n = 0$ and $\alpha_{j-1} = \alpha'_{j-1} x_{n-1}^{\omega+p} u_{n-1} x_n^{\omega} u_n$, with $u_{n-1} = \epsilon$ or $u_{n-1} = a_{x_{n-1},x_n}$. As above in the proof of Claim 1, for $p \neq 0$ this leads to a contradiction. Hence we assume that $p = 0$. We have $\mathbb{c}^{\backprime}_{\alpha,j} = 0$ by Lemma 6.3, whence $\mathbb{c}^{\backprime}_{\beta,j} = 0$ by Claim 1. So, $k = n - 2 \geq 1$ and

$$\mathsf{r}_{\alpha,j} = \mathsf{c}^{-1}_{x_{n-2},u_{n-2},x_{n-1}} \mathsf{b}^{-\mathbb{q}_{n-2}}_{x_{n-2}} \mathsf{c}^{-1}_{x_{n-3},u_{n-3},x_{n-2}} \mathsf{b}^{-\mathbb{q}_{n-3}}_{x_{n-3}} \cdots \mathsf{c}^{-1}_{x_1,u_1,x_2} \mathsf{b}^{-\mathbb{q}_1}_{x_1} \mathsf{c}^{-1}_{x_n,u_n u_0,x_1},$$
$$\mathsf{r}_{\beta,j} = \mathsf{c}^{-1}_{y_{n-2},v_{n-2}v_0,y_1} \mathsf{b}^{-\mathbb{p}_{n-2}}_{y_{n-2}} \mathsf{c}^{-1}_{y_{n-3},v_{n-3},y_{n-2}} \mathsf{b}^{-\mathbb{p}_{n-3}}_{y_{n-3}} \cdots \mathsf{c}^{-1}_{y_1,v_1,y_2} \mathsf{b}^{-\mathbb{p}_1}_{y_1} \mathsf{c}^{-1}_{y_{n-2},v_{n-2}v_0,y_1}.$$

As $\mathsf{r}_{\alpha,j} = \mathsf{r}_{\beta,j}$, we conclude that $x_n = y_{n-2}$, $x_{n-1} = y_1$, $u_{n-2} = v_{n-2}v_0 = u_n u_0$, and, for $i \in \{1, \ldots, n-2\}$, $x_i = y_i$, $q_i = p_i$ and, when $i \neq n-2$, $u_i = v_i$.
Furthermore, $\mathsf{r}_{\beta,j+1}$ begins with a crucial variable of the form $\mathsf{c}_{y_k,v_{k\_},\_}$ or it is equal to a terminal variable of the form $\mathsf{t}_{y_k,v_{k\_}}$. Moreover, either $\mathsf{r}_{\alpha,j+1}$ begins with a crucial variable of the form $\mathsf{c}_{x_n,u_{n\_},\_}$, or it is equal to a terminal variable of the form $\mathsf{t}_{x_n,u_{n\_}}$. As $u_n u_0 = v_{n-2}v_0$, $\mathsf{r}_{\alpha,j+1} = \mathsf{r}_{\beta,j+1}$ and it is not possible to make a rank 2 shift right at position j, neither in α nor in β, we must have $u_n = v_{n-2}$ and so $u_0 = v_0$. We have also that either $\mathsf{r}_{\beta,j-1}$ ends with a crucial variable of the form $\mathsf{c}_{\_,\_v_0,y_1}$ or it is equal to an initial variable of the form $\mathsf{i}_{\_v_0,y_1}$, and that either $\mathsf{r}_{\alpha,j-1}$ ends with a crucial variable of the form $\mathsf{c}_{\_,\_,x_{n-1}}$ or it is equal to an initial variable of the form $\mathsf{i}_{\_,x_{n-1}}$. Hence, $\alpha_j^{\omega-1} = (u_0 x_1^{\omega+q_1} \cdots u_{n-3} x_n^{\omega+q_{n-2}} u_n u_0 x_1^{\omega} u_{n-1} x_n^{\omega} u_n)^{\omega-1}$ and one of the two following situations happens:

(i) $\alpha_{j-1} = \alpha'_{j-1} u_0 x_1^{\omega} u_{n-1} x_n^{\omega} u_n$;
(ii) $\alpha_{j-1} = u''_0 x_1^{\omega} u_{n-1} x_n^{\omega} u_n$, $j > 1$ and u'_0 is a non-empty suffix of α_{j-2} with $u_0 = u'_0 u''_0$.

If situation (i) holds, α is not a canonical form as it allows the application of a shortening $(s.1)$ with $\sigma = u_0 x_1^{\omega} u_{n-1} x_n^{\omega} u_n$ and $\tau = u_0 x_1^{\omega+q_1} \cdots u_{n-3} x_n^{\omega+q_{n-2}} u_n$. In particular, this proves already the impossibility of Case 1.1. for $j = 1$.

Suppose now that situation (ii) holds. Then $j > 1$ and we will use the induction hypothesis to obtain a contradiction. Note that $x_n^{\omega} u_n u'_0$ can not be the final ω-portion of α_{j-2} (otherwise it would be possible to make an agglutination $(a.3)$). Consequently,

$\mathbb{c}'_{\beta,j-2} = \mathbb{c}'_{\alpha,j-2} = 0$ and $|\mathsf{r}_{\beta,j-1}|_\mathsf{c} = |\mathsf{r}_{\alpha,j-1}|_\mathsf{c} = 1$. Furthermore $\mathsf{r}_{\beta,j-1} = \mathsf{r}_{\alpha,j-1} = \mathsf{c}_{z,wu_0,x_1}$ where $z^\omega wu'_0$ is the final ω-portion of α_{j-2}. Hence, the final ω-portion of β_{j-2} is $z^\omega w'$ with w' a prefix of w. Assuming by induction hypothesis that ${}'\mathbb{c}_{\beta,j-2} = {}'\mathbb{c}_{\alpha,j-2}$, we have from Claim 2 that $\alpha_{j-2} = \beta_{j-2}$, and one deduces that $w = w'$ and $u'_0 = \epsilon$. So, actually, situation (ii) can not happen either.

Case 1.2. ${}'\mathbb{c}_{\alpha,j} = 1$. So, $k = n-1 \geq 1$ and, by Remark 6.5 (c), α_{j-1} is of the form $\alpha_{j-1} = \alpha'_{j-1}x_n^{\omega+p}u_n$ and $q_n = 0$. If $p \neq 0$, then we get a contradiction as above. So, we assume additionally that $p = 0$. Thereby, we get

$$\begin{aligned}
\mathsf{r}_{\alpha,j} &= \mathsf{c}^{-1}_{x_{n-1},u_{n-1},x_n}\mathsf{b}^{-\mathbb{q}_{n-1}}_{x_{n-1}}\mathsf{c}^{-1}_{x_{n-2},u_{n-2},x_{n-1}}\mathsf{b}^{-\mathbb{q}_{n-2}}_{x_{n-2}}\cdots\mathsf{c}^{-1}_{x_2,u_2,x_3}\mathsf{b}^{-\mathbb{q}_2}_{x_2}\mathsf{c}^{-1}_{x_1,u_1,x_2}\mathsf{r}'_{\alpha,j},\\
\mathsf{r}_{\beta,j} &= \mathsf{c}^{-1}_{y_{n-1},v_{n-1}v_0,y_1}\mathsf{b}^{-\mathbb{p}_{n-1}}_{y_{n-1}}\mathsf{c}^{-1}_{y_{n-2},v_{n-2},y_{n-1}}\mathsf{b}^{-\mathbb{p}_{n-2}}_{y_{n-2}}\cdots\mathsf{c}^{-1}_{y_2,v_2,y_3}\mathsf{b}^{-\mathbb{p}_2}_{y_2}\mathsf{c}^{-1}_{y_1,v_1,y_2}\mathsf{r}'_{\beta,j},
\end{aligned}$$

for some words $\mathsf{r}'_{\alpha,j}, \mathsf{r}'_{\beta,j} \in (\mathsf{V}^{-1})^*$. As $\mathsf{r}_{\alpha,j} = \mathsf{r}_{\beta,j}$, we conclude that, for $i \in \{1,\ldots,n-1\}$, $\mathsf{r}'_{\alpha,j} = \mathsf{r}'_{\beta,j}$, $u_{n-1} = v_{n-1}v_0$, $x_n = y_1$, $x_i = y_i$, $p_i = q_i$ if $i \neq 1$, and $u_i = v_i$ when $i \neq n-1$. Whence

$$\begin{aligned}
\alpha_{j-1}\alpha_j^{\omega-1} &= \alpha'_{j-1}x_n^\omega u_n(u_0x_n^{\omega+q_1}u_1x_2^{\omega+q_2}u_2\cdots x_{n-1}^{\omega+q_{n-1}}u_{n-1}x_n^\omega u_n)^{\omega-1},\\
\beta_{j-1}\beta_j^{\omega-1} &= \beta_{j-1}(v_0x_n^{\omega+p_1}u_1x_2^{\omega+q_2}u_2\cdots x_{n-1}^{\omega+q_{n-1}}v_{n-1})^{\omega-1}.
\end{aligned}$$

Suppose now that $\mathbb{c}'_{\alpha,j} = 1$. Hence $\mathbb{c}'_{\beta,j} = 1$, $q_1 = p_1 = 0$ and $u_0 = v_0 = \epsilon$. So, $u_{n-1} = v_{n-1}$, no crucial variables occur in either $\mathsf{r}'_{\alpha,j}$ or $\mathsf{r}'_{\beta,j}$ and $\alpha_{j-1}\alpha_j^{\omega-1}\alpha_{j+1}$ is of the form

$$\alpha_{j-1}\alpha_j^{\omega-1}\alpha_{j+1} = \alpha'_{j-1}x_n^\omega u_n(x_n^\omega u_1\cdots x_{n-1}^{\omega+q_{n-1}}u_{n-1}x_n^\omega u_n)^{\omega-1}x_n^{\omega+r}\alpha'_{j+1}$$

with $r \neq 0$ and $u_n \neq \epsilon$ since $\alpha \in \mathcal{C}_2$. Therefore, α is not a canonical form since it is possible to make a shortening $(s.4)$.

Suppose next that $\mathbb{c}'_{\alpha,j} = 0$ and so that $\mathbb{c}'_{\beta,j} = 0$. Then $\mathsf{r}'_{\alpha,j} = \mathsf{b}^{-\mathbb{q}_1}_{x_n}\mathsf{c}^{-1}_{x_n,u_nu_0,x_n}$ and $\mathsf{r}'_{\beta,j} = \mathsf{b}^{-\mathbb{p}_1}_{x_n}\mathsf{c}^{-1}_{x_{n-1},v_{n-1}v_0,x_n}$. Therefore $x_n = x_{n-1}$, $q_1 = p_1$ and $u_nu_0 = v_{n-1}v_0$ $(= u_{n-1})$. As in Case 1.1, analysing the first crucial variable of the remainder at position $j+1$ and the last crucial variable of the remainder at position $j-1$, we conclude that $u_n = v_{n-1}$ (whence $u_0 = v_0$) and $u_0x_n^\omega u_n$ is a suffix of $\alpha_{j-2}\alpha_{j-1}$. Consequently, $\alpha_j^{\omega-1} = (u_0x_n^{\omega+q_1}u_1x_2^{\omega+q_2}u_2\cdots x_n^{\omega+q_{n-1}}u_nu_0x_n^\omega u_n)^{\omega-1}$ and one of the two following situations happens:

(i) $\alpha_{j-1} = \alpha''_{j-1}u_0x_n^\omega u_n$;
(ii) $\alpha_{j-1} = u''_0x_n^\omega u_n$, $j > 1$ and u'_0 is a non-empty suffix of α_{j-2} where $u_0 = u'_0u''_0$.

If (i) holds, then α is not a canonical form since it admits the application of a shortening $(s.1)$ with $\sigma = u_0x_n^\omega u_n$ and $\tau = u_0x_n^{\omega+q_1}u_1x_2^{\omega+q_2}u_2\cdots x_n^{\omega+q_{n-1}}u_n$. If $j = 1$, this proves the impossibility of Case 1.2. If $j > 1$, it remains to consider situation (ii), in which case ${}'\mathbb{c}_{\beta,j-1} = {}'\mathbb{c}_{\alpha,j-1} = 0$ and $|\mathsf{r}_{\beta,j-1}|_\mathsf{c} = |\mathsf{r}_{\alpha,j-1}|_\mathsf{c} = 1$. Furthermore $\mathsf{r}_{\beta,j-1} = \mathsf{r}_{\alpha,j-1} = \mathsf{c}_{z,wu_0,x_n}$ where $z^\omega wu'_0$ is the final ω-portion of α_{j-2}. Consequently, the final ω-portion of β_{j-2} is $z^\omega w'$ with w' a prefix of w. Again assuming by induction hypothesis that ${}'\mathbb{c}_{\beta,j-2} = {}'\mathbb{c}_{\alpha,j-2}$, we have from Claim 2 that

$\alpha_{j-2} = \beta_{j-2}$, and this implies that $w = w'$ and $u'_0 = \epsilon$. Therefore, situation (ii) also does not occur.
In both cases, 1.1 and 1.2, we reached a contradiction. Therefore ${}'\mathbb{c}_{\alpha,j} = 0$ when ${}'\mathbb{c}_{\beta,j} = 0$. By symmetry it follows that ${}'\mathbb{c}_{\alpha,j} = 0$ if and only if ${}'\mathbb{c}_{\beta,j} = 0$.
CASE 2. ${}'\mathbb{c}_{\beta,j} = 2$. Then $\mathbb{c}`_{\alpha,j} = \mathbb{c}`_{\beta,j} = 0$, and ${}'\mathbb{c}_{\alpha,j} \neq 0$ by Case 1. Suppose that ${}'\mathbb{c}_{\alpha,j} = 1$. Hence $k = n+1$, $q_n = p_n = p_{n+1} = 0$, and α_{j-1} and β_{j-1} are of the forms, respectively, $\alpha_{j-1} = \alpha'_{j-1} x_n^{\omega+q} u_n$ and $\beta_{j-1} = \beta'_{j-1} y_n^{\omega+p} v_n y_{n+1}^{\omega} v_{n+1}$. Furthermore,

$$\mathsf{r}_{\alpha,j} = \mathsf{b}_{x_n}^{q'} \mathsf{c}_{x_{n-1},u_{n-1},x_n}^{-1} \cdots \mathsf{b}_{x_2}^{-\mathbb{q}_2} \mathsf{c}_{x_1,u_1,x_2}^{-1} \mathsf{b}_{x_1}^{-\mathbb{q}_1} \mathsf{c}_{x_n,u_n u_0,x_1}^{-1},$$
$$\mathsf{r}_{\beta,j} = \mathsf{b}_{y_n}^{p'} \mathsf{c}_{y_{n-1},v_{n-1},y_n}^{-1} \cdots \mathsf{b}_{y_2}^{-\mathbb{p}_2} \mathsf{c}_{y_1,v_1,y_2}^{-1} \mathsf{b}_{y_1}^{-\mathbb{p}_1} \mathsf{c}_{y_{n+1},v_{n+1} v_0,y_1}^{-1},$$

where, for $t \in \{p, q\}$, t' is 0 when $t \geq 0$ and it is t when $t < 0$. From the equality $\mathsf{r}_{\alpha,j} = \mathsf{r}_{\beta,j}$ it follows that, for $i \in \{1, \cdots, n-1\}$, $q' = p'$, $x_n = y_n = y_{n+1}$, $u_n u_0 = v_{n+1} v_0$, $x_i = y_i$, $u_i = v_i$ and $p_i = q_i$. Again, analysing the first crucial variables of $\mathsf{r}_{\alpha,j+1}$ and $\mathsf{r}_{\beta,j+1}$, we conclude that $u_n = v_{n+1}$, so that $u_0 = v_0$. Whence,

$$\beta_{j-1} \beta_j^{\omega-1} = \beta'_{j-1} x_n^{\omega+p} v_n x_n^{\omega} u_n (u_0 x_1^{\omega+q_1} u_1 \cdots x_{n-1}^{\omega+q_{n-1}} u_{n-1} x_n^{\omega} v_n x_n^{\omega} u_n)^{\omega-1}.$$

So, β is not a canonical $\bar{\kappa}$-term, either because $v_n = \epsilon$ or because $v_n \neq \epsilon$ and it allows the application of a shortening ($s.5$). This is in contradiction with the hypothesis and so ${}'\mathbb{c}_{\alpha,j} = 2 = {}'\mathbb{c}_{\beta,j}$.
CASE 3. ${}'\mathbb{c}_{\beta,j} = 1$. From the previous cases it is now immediate that ${}'\mathbb{c}_{\alpha,j} = {}'\mathbb{c}_{\beta,j} = 1$.

We have proved in all cases that ${}'\mathbb{c}_{\alpha,j} = {}'\mathbb{c}_{\beta,j}$ and, so, the proof of Claim 3 is complete. □

The ending of the proof of the proposition is now clear. By Claim 3, ${}'\mathbb{c}_{\alpha,j} = {}'\mathbb{c}_{\beta,j}$ and, so, by Claim 2 (which uses Claim 1) one deduces that $\overleftarrow{\mathsf{w}}_{\alpha,j} = \overleftarrow{\mathsf{w}}_{\beta,j}$ and $\overrightarrow{\mathsf{w}}_{\alpha,j} = \overrightarrow{\mathsf{w}}_{\beta,j}$ for every odd position j. As observed above this entails that $\mathsf{w}_{\mathbb{q}}(\alpha) = \mathsf{w}_{\mathbb{q}}(\beta)$ and, so, as α and β are canonical forms, that $\alpha = \beta$. □

The next result, which also follows from Lemma 7.4, is a weaker version of the reciprocal of Proposition 6.2.

Proposition 7.5 *Let $\alpha, \beta \in \mathcal{C}_1 \cup \mathcal{S}_2$ and let $\mathbb{q} \geq \max\{\mathbb{q}_\alpha, \mathbb{q}_{\alpha^*}, \mathbb{q}_\beta, \mathbb{q}_{\beta^*}\}$. If $\widetilde{w}_{\mathbb{q}}(\alpha) = \widetilde{w}_{\mathbb{q}}(\beta)$, then $\mathbf{LG} \models \alpha = \beta$.*

Proof Assume that $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta)$. By Proposition 6.2, $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha^*)$ and $\widetilde{\mathsf{w}}_{\mathbb{q}}(\beta) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta^*)$, where α^* and β^* are the canonical forms of α and β. Therefore, $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha^*) = \widetilde{\mathsf{w}}_{\mathbb{q}}(\beta^*)$ and, by Lemma 7.4, $\alpha^* = \beta^*$. Hence $\mathbf{LG} \models \alpha^* = \beta^*$ and so, as every $\bar{\kappa}$-term is Σ-equivalent to its canonical form, $\mathbf{LG} \models \alpha = \beta$. □

8 Main results

The main results of this paper may now be easily deduced.

Theorem 8.1 *The $\bar{\kappa}$-word problem for* **LG** *is decidable.*

Proof The solution of the $\bar{\kappa}$-word problem for **LG** consists in, given two $\bar{\kappa}$-terms α and β, to compute their respective canonical forms α^* and β^*. Then, by Theorem 7.1, $\mathbf{LG} \models \alpha = \beta$ if and only if $\alpha^* = \beta^*$. □

By the above proof, to test whether a $\bar{\kappa}$-identity $\alpha = \beta$ holds over **LG**, it is necessary to compute the canonical forms of the $\bar{\kappa}$-terms α and β and verify they are the same. An alternative test requests the calculation of $\mathbb{q}$-roots. If α and β are not finite words, then one computes $\bar{\kappa}$-terms α° and β° using the procedure described in Sect. 5.1. Their $\mathbb{q}$-outlines are well-defined expressions $\mathsf{w}_{\mathbb{q}}(\alpha^\circ)$ and $\mathsf{w}_{\mathbb{q}}(\beta^\circ)$ parameterized by $\mathbb{q}$. Making all possible cancellations, one obtains well-defined expressions, also parameterized by $\mathbb{q}$, that coincide with the $\mathbb{q}$-roots $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha^\circ)$ and $\widetilde{\mathsf{w}}_{\mathbb{q}}(\beta^\circ)$ for $\mathbb{q}$ large enough (see Example 6.1 as an instance). So, by Propositions 6.2 and 7.5 , $\mathbf{LG} \models \alpha = \beta$ if and only if $\widetilde{\mathsf{w}}_{\mathbb{q}}(\alpha^\circ)$ and $\widetilde{\mathsf{w}}_{\mathbb{q}}(\beta^\circ)$ are the same expression.

Theorem 8.2 *The set Σ is a basis of $\bar{\kappa}$-identities for $\mathbf{LG}^{\bar{\kappa}}$.*

Proof We have to prove that, for all $\bar{\kappa}$-terms α and β, $\mathbf{LG} \models \alpha = \beta$ if and only if $\Sigma \vdash \alpha = \beta$. The only if part follows from the fact that **LG** verifies all the $\bar{\kappa}$-identities of Σ. For the if part recall that, by Sect. 5, there exist canonical forms α^* and β^* that may be computed from α and β using the $\bar{\kappa}$-identities of Σ. Therefore, if $\mathbf{LG} \models \alpha = \beta$ then $\mathbf{LG} \models \alpha^* = \beta^*$ and so, by Theorem 7.1, $\alpha^* = \beta^*$. Since $\Sigma \vdash \{\alpha = \alpha^*, \beta = \beta^*\}$ it follows by transitivity that $\Sigma \vdash \alpha = \beta$. □

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