Global operator calculus on spin groups *

P. Cerejeiras [1†], M. Ferreira [1,2†], U. Kähler [1*†], J. Wirth [3†]

[1]CIDMA - Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Campus Universitário de Santiago, 3810-193, Aveiro, Portugal.
[2]School of Technology and Management, Polytechnic of Leiria, Campus 2, Morro do Lena - Alto do Vieiro, P-2411-901, Leiria, Portugal.
[3]Institute of Analysis, Dynamics and Modelling; Department of Mathematics, University of Stuttgart, Pfaffenwaldring 57, 70569, Stuttgart, Germany.

*Corresponding author(s). E-mail(s): ukaehler@ua.pt;
Contributing authors: pceres@ua.pt; milton.ferreira@ipleiria.pt; jens.wirth@iadm.uni-stuttgart.de;
†These authors contributed equally to this work.

Abstract

In this paper, we use the representation theory of the group $\mathbf{Spin}(\boldsymbol{m})$ to develop aspects of the global symbolic calculus of pseudo-differential operators on $\mathbf{Spin}(3)$ and $\mathbf{Spin}(4)$ in the sense of Ruzhansky-Turunen-Wirth. A detailed study of $\mathbf{Spin}(3)$ and $\mathbf{Spin}(4)$-representations is made including recurrence relations and natural differential operators acting on matrix coefficients. We establish the calculus of left-invariant differential operators and of difference operators on the group $\mathbf{Spin}(4)$ and apply this to give criteria for the subellipticity and the global hypoellipticity of pseudo-differential operators in terms of their matrix-valued full symbols. Several examples of first and second order globally hypoelliptic differential operators are given, including some that are locally neither invertible nor hypoelliptic.
The paper presents a particular case study for higher dimensional spin groups.

Keywords: Spin group, Spin representations, difference operators, pseudo-differential operators, Fourier transform, microlocal analysis, elliptic operators, global hypoellipticity

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1 Introduction

The classic principal calculus of L. Hörmander over manifolds, which is based on the notion of the symbol via localizations, has several limitations such as in the characterisation of global and local hypoellipticity. This is due to the fact that one uses local Euclidean Fourier analysis on manifolds which makes only the principal part of a symbol to be coordinate-invariant. But in the case of Lie groups one has another approach based on harmonic analysis over these groups which allows for a global approach. For instance, S. Zelditch [29] used the non-Euclidean harmonic analysis of Helgason to replace the local Euclidean Fourier analysis to obtain a pseudo-differential calculus on hyperbolic surfaces in the plane. Hereby, Helgason's non-Euclidean harmonic analysis is based on a Fourier transform given by eigenfunctions of the invariant Laplacian over a suitable homogeneous space which has its own drawbacks. For a detailed description on the historic development of calculi of pseudo-differential operators we refer to [21, 26].

During the last decade, a new and full symbol calculus over compact groups was developed by M. Ruzhansky, V. Turunen, and J. Wirth which represents a non-commutative extension of the classical Kohn-Nirenberg quantization. This calculus has several advantages over the classic principal calculus of L. Hörmander. Given a Lie group G one makes full use of its representation theory and the corresponding harmonic analysis to create a global Fourier transform which allows the study of matrix-valued symbols defined on $G \times \widehat{G}$ and their characterisation using results from harmonic analysis on phase space. This full symbol calculus has been extended to the case of type-1 groups [18] and, recently, to a subelliptic pseudo-differential calculus on compact Lie group [2] as well as to the case of nilpotent groups [9, 20].

As with all these abstract approaches there appears always the question of its realization in concrete cases. The full symbol calculus over compact groups has been explicitly worked out in the case of the n-dimensional torus and the case of SU(2). But it also raises the question of how this calculus would look like in one of the most important cases of compact groups, the case of Spin(m). The classic approach to a Fourier symbol calculus over Spin(m), i.e., the case of spinor-valued functions, consists in constructing Gelfand pairs and employing the spherical Dirac or Laplace operator, see the classic work by J. Dieudonné [7]. Naturally, these approaches restrict also the class of pseudo-differential operator symbols which can be considered and to overcome this problem a full symbol calculus in the sense of M. Ruzhansky, V. Turunen, and J. Wirth becomes even more important. In this paper, we are going to establish this calculus for the group Spin(4). This choice is based on two reasons. First of all the structure of group representations of Spin(m) makes it difficult to obtain explicit formulae in the general case so that the case of Spin(4) will provide further insight into the general case. Secondly, the case of Spin(4) is important by itself in applications. For instance, Spin(4) is the translation group on the three-sphere which appears in the study of diffraction tomography and the construction of wavelet and Gabor frames over the three-sphere [3]. While classically diffraction tomography is used to establish the so-called orientation density function of a fixed specimen, recent advances have also created new interests in time-dependent versions of diffraction tomography in structural analysis. Furthermore, discussion of perturbations of wavelet and Gabor frames over the three-sphere will require the study of pseudo-differential operators over Spin(4) in the same way as in the classic case. Additionally, Spin(4) also appears in quantum gravity (see, e.g., in Spin(4) BF models [19]). Among other things, these investigations require tools for the study of symbols of pseudo-differential operators and global hypoellipticity of differential operators defined over function spaces on Spin(4).

After recalling some necessary facts about the abstract case of the full symbol calculus by M. Ruzhansky, V. Turunen, and J. Wirth and Spin(3)-representations, we are going to study Spin(4)-representations and their connections with harmonic and spinor-valued monogenic polynomials. Hereby, we establish the necessary tools for a full symbol calculus like matrix-coefficients, recurrence relations and difference operators acting on them. This will allow us to work out details of the Fourier transform on Spin(4), which in turn gives rise to the full symbol calculus. Furthermore, we are going to obtain conditions on ellipticity and hypoellipticity.

In the end we provide some examples of differential operators to show the symbol calculus in action.

2 Preliminaries on the harmonic analysis for compact groups

We start with some basic notations and results about harmonic analysis of a compact Lie group. Let G be a compact Lie group of real dimension n with unit element e. A finite-dimensional unitary representation ξ of G is a continuous group homomorphism $\xi : G \to U(d_\xi)$ of G into the group of unitary matrices of a certain dimension d_ξ. The representation ξ is irreducible if $\xi(x)A = A\xi(x)$ for all $x \in G$ and some $A \in \mathbb{C}^{d_\xi \times d_\xi}$ implies $A = cI$ is a multiple of the identity. This is equivalent to the statement that $\mathbb{C}^{d_\xi}$ does not have non-trivial ξ-invariant subspaces $V \subset \mathbb{C}^{d_\xi}$ with $\xi(x)V \subset V$ for all $x \in G$.

Two representations ξ_1 and ξ_2 are equivalent if there exists an invertible matrix B with $\xi_1(x)B = B\xi_2(x)$ for all $x \in G$. Let $\widehat{G}$ denote the set of all equivalence classes of irreducible representations.

We further equip G by its normalized Haar measure. The group structure gives rise to left and right translations, $L_x : \phi \mapsto \phi(x^{-1}\,\cdot)$ and $R_x : \phi \mapsto \phi(\cdot\, x)$ of functions on the group. These left- and right-translations are unitary on the Hilbert space $\mathrm{L}^2(G)$ of square integrable functions and therefore the translations give rise to unitary representations $x \mapsto L_x$ and $x \mapsto R_x$ of the group G on the Hilbert space $\mathrm{L}^2(G)$. These representations split into irreducibles and give rise to the Peter–Weyl Theorem in the following form, see [10], [21], or [28].

Theorem 2.1 (Peter–Weyl)**.** *The space* $\mathrm{L}^2(G)$ *decomposes as the orthogonal direct sum of minimal bi-invariant subspaces parameterised by* $\widehat{G}$, *that is*

$$\mathrm{L}^2(G) = \bigoplus_{[\xi]\in\widehat{G}} \mathcal{H}^\xi, \qquad \mathcal{H}^\xi = \{x \mapsto \mathrm{tr}(A\xi(x)) \mid A \in \mathbb{C}^{d_\xi \times d_\xi}\}. \tag{2.1}$$

The Fourier transform of $f \in \mathrm{L}^2(G)$ is a matrix valued function on $\widehat{G}$ defined by

$$\widehat{f}(\xi) = \int_G f(x)\, \xi^*(x)\, \mathrm{d}x \tag{2.2}$$

with inverse given by

$$f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \ \mathrm{tr}(\xi(x)\, \widehat{f}(\xi)). \tag{2.3}$$

Furthermore, the following Parseval identity holds

$$\|f\|^2_{\mathrm{L}^2(G)} = \sum_{[\xi]\in\widehat{G}} d_\xi\, \|\widehat{f}(\xi)\|^2_{\mathrm{HS}}, \tag{2.4}$$

where $\|A\|^2_{\mathrm{HS}} = \mathrm{tr}(A^*A)$ is the Frobenius or Hilbert–Schmidt norm of a matrix A.

On the group G the convolution of two integrable functions $\phi, \psi \in \mathrm{L}^1(G)$ is defined by

$$(\phi * \psi)(x) = \int_G \phi(y)\, \psi(y^{-1}x)\, \mathrm{d}y. \tag{2.5}$$

The following convolution theorem on G is well-known. Note the change in the order of the factors.

Theorem 2.2. *Let $\phi, \psi \in \mathrm{L}^1(G)$. Then $\phi * \psi \in \mathrm{L}^1(G)$ and $(\widehat{\phi * \psi})(\xi) = \widehat{\psi}(\xi)\, \widehat{\phi}(\xi)$.*

The Laplace–Beltrami operator $\mathcal{L}_G \in \mathrm{Diff}^2(G)$ on the group G is bi-invariant, i.e., it commutes with all L_x and R_x. Therefore, all of its eigenspaces are bi-invariant subspaces of $\mathrm{L}^2(G)$. Since $\mathcal{H}_\xi$ are minimal bi-invariant subspaces, each of them has to be eigenspace of $\mathcal{L}_G$ and we denote the corresponding eigenvalue by $-\lambda_\xi^2$. Hence, we obtain the following decomposition

$$\mathcal{L}_G\, \phi = -\sum_{\xi\in\widehat{G}} d_\xi\, \lambda_\xi^2\ \mathrm{tr}(\xi(x)\widehat{\phi}(\xi)). \tag{2.6}$$

The notion of Fourier series extends naturally to $C^\infty(G)$ and the space of distributions $\mathcal{D}'(G)$ with convergence in the respective topologies. Now, any operator A on G mapping $C^\infty(G)$ to $\mathcal{D}'(G)$ gives rise to a matrix-valued full symbol $\sigma_A(x,\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$, $x \in G$ defined by

$$\sigma_A(x,\xi) := \xi(x)^*(A\xi)(x) \tag{2.7}$$

which can be understood either pointwise or distributionally, as the product of a smooth matrix-valued function $\xi^*(x)$ with the matrix-valued distribution $A\xi$, i.e. $\sigma_A(\cdot,\xi) = \xi^* A\xi$ as a distribution in the first variable, for all $[\xi] \in \widehat{G}$. Then it can be shown that

$$Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi\ \mathrm{tr}(\xi(x)\, \sigma_A(x,\xi)\, \widehat{f}(\xi)) \tag{2.8}$$

holds as $\mathcal{D}'$-convergent series. If it happens that the operator A maps $C^\infty(G)$ to itself, then (2.8) holds in the strong topology of $C^\infty(G)$. For A and σ_A related by (2.8) we write $A = \mathrm{Op}(\sigma_A)$. For a comprehensive treatment of this quantization we refer to [21] and [24].

We denote the right-convolution kernel of A by R_A, so that

$$Af(x) = \int_G K_A(x,y)\, f(y)\, \mathrm{d}y = \int_G R_A(x, y^{-1}x)\, f(y)\, \mathrm{d}y. \tag{2.9}$$

The symbol σ_A and the right-convolution kernel R_A are related by $\sigma_A(x,\xi) = \int_G R_A(x,y)\, \xi(y)^*\, \mathrm{d}y$.

The class $\Psi^m(G)$ of Hörmander's pseudo-differential operators on G was fully characterised in [21] and [22] using commutator properties with the vector fields in Sobolev spaces, and in [23] by the behaviour of their matrix symbols. Before we give a characterisation of the class $\Psi^m(G)$ we fix some notations.

We say that Q_ξ is a difference operator of order k if it is given by $Q_\xi \widehat{f}(\xi) = \widehat{\varphi_Q f}(\xi)$ for a function $\varphi = \varphi_Q \in C^\infty(G)$ vanishing of order k at the identity $e \in G$, i.e., $(P_x \varphi_Q)(e) = 0$ for all left-invariant differential operators $P_x \in \mathrm{Diff}^{k-1}(G)$ of order $k-1$. We denote the set of all difference operators of order k as $\mathrm{diff}^k(\widehat{G})$. In the sequel, for a function $\varphi \in C^\infty(G)$ it will be also convenient to denote the associated difference operator, acting on Fourier coefficients, by $\Delta_\varphi \widehat{f}(\xi) := \widehat{\varphi f}(\xi)$.

Definition 2.3 (cf. [23])**.** Let G be a compact Lie group of dimension n with unit element e. A collection of $k \geq n$ first order difference operators $\Delta_1, \ldots, \Delta_k \in \mathrm{diff}^1(\widehat{G})$ is called admissible, if the corresponding functions $\varphi_1, \ldots, \varphi_k \in C^\infty(G)$ satisfy $\mathrm{d}\varphi_j \neq 0, j = 1, \ldots, k$, and $\mathrm{rank}(\mathrm{d}\varphi_1(e), \ldots, \mathrm{d}\varphi_k(e)) = n$. It follows, in particular, that e is an isolated zero of the family $\{\varphi_j\}_{j=1}^k$. An admissible collection is called strongly admissible if $\cap_j \{x \in G : \varphi_j(x) = 0\} = \{e\}$.

The previous definition was adapted to a collection of $k \geq n$ first order difference operators since this happens in our case. For a given admissible selection of difference operators on a compact Lie Group G we use the multi-index notation

$$\Delta_\xi^\alpha := \Delta_1^{\alpha_1} \cdots \Delta_k^{\alpha_k} \quad \text{and} \quad \varphi^\alpha(x) := \varphi_1(x)^{\alpha_1} \cdots \varphi_k(x)^{\alpha_k} \tag{2.10}$$

Furthermore, there exist corresponding differential operators $\partial_x^{(\alpha)} \in \mathrm{Diff}^{|\alpha|}(G)$ such that the Taylor's formula

$$f(x) = \sum_{|\alpha| \leq N-1} \frac{1}{\alpha!} \varphi^\alpha(x^{-1})\, \partial_x^{(\alpha)} f(e) + \mathcal{O}(h(x)^N), \quad h(x) \to 0 \tag{2.11}$$

holds true for any smooth function $f \in C^\infty(G)$ and with $h(x)$ the geodesic distance from x to the identity element e (see [21, Sec. 10.6]). Additionally, we introduce operators ∂_x^α as follows. Let $\partial_{x_j} \in \mathrm{Diff}^1(G), 1 \leq j \leq n = \dim G$, be a collection of left-invariant first order differential operators corresponding to some linearly independent family of the left-invariant vector fields on G. We denote $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. The following theorem characterises Hörmander's class of pseudo-differential operators $\Psi^m(G)$ by the behaviour of their matrix symbols.

Theorem 2.4 (cf. [23, Thm. 2.2])**.** *Let A be a linear continuous operator from $C^\infty(G)$ to $\mathcal{D}'(G)$. Then the following statements are equivalent:*

(A) $A \in \Psi^m(G)$.

(B) *For every left-invariant differential operator* $P_x \in \mathrm{Diff}^k(G)$ *of order* k *and every difference operator* $Q_\xi \in \mathrm{diff}^l(\widehat{G})$ *of order* l *the symbol estimate*

$$\|Q_\xi\, P_x\, \sigma_A(x,\xi)\|_{\mathrm{op}} \leq C_{Q_\xi,P_x}\, \langle\xi\rangle^{m-l} \tag{2.12}$$

is valid, where $\langle\xi\rangle = (1+\lambda_\xi^2)^{1/2}$ *and* $-\lambda_\xi^2$ *are the eigenvalues of* $\mathcal{L}_G$.

(C) *For an admissible selection* $\Delta_1,\ldots,\Delta_m \in \mathrm{diff}^1(\widehat{G})$ *we have*

$$\|\Delta_\xi^\alpha\, \partial_x^\beta\, \sigma_A(x,\xi)\|_{\mathrm{op}} \leq C_{\alpha,\beta} \langle\xi\rangle^{m-|\alpha|} \tag{2.13}$$

for all multi-indices α,β. *Moreover, sing supp* $R_A(x,\cdot) \subseteq \{e\}$.

(D) *For a strongly admissible selection* $\Delta_1,\ldots,\Delta_m \in \mathrm{diff}^1(\widehat{G})$ *we have*

$$\|\Delta_\xi^\alpha\, \partial_x^\beta\, \sigma_A(x,\xi)\|_{\mathrm{op}} \leq C_{\alpha,\beta}\, \langle\xi\rangle^{m-|\alpha|} \tag{2.14}$$

for all multi-indices α,β.

The set of symbols σ_A satisfying either of equivalent conditions (B)-(D) is also denoted by $S^m(G)$, such that the operator quantization gives an isomorphism $\mathrm{Op} : S^m(G) \to \Psi^m(G)$. The composition of pseudodifferential operators gives again a pseudo-differential operator with a symbol, which can be expressed as an asymptotic expansion.

Theorem 2.5 (cf. [21, Thm. 10.7.9])**.** *If* $A \in \Psi^{m_1}(G)$ *and* $B \in \Psi^{m_2}(G)$ *then* $A\circ B \in \Psi^{m_1+m_2}(G)$ *satisfies*

$$\sigma_{A\circ B}(x\,\xi) \sim \sum_\alpha \frac{1}{\alpha!} \left(\Delta_\xi^\alpha \sigma_A(x,\xi)\right) \left(\partial_x^{(\alpha)} \sigma_B(x,\xi)\right) \tag{2.15}$$

in the sense that $\sigma_{A\circ B} - \sum_{|\alpha|<N} \frac{1}{\alpha!}\left(\Delta_\xi^\alpha \sigma_A\right)\left(\partial_x^{(\alpha)}\sigma_B\right) \in S^{m_1+m_2-N}(G)$ *holds true.*

Note that the proof in[21] omits the crucial remainder estimates for the underlying Taylor expansion. For a complete proof including the remainder estimates see e.g. [4, Sections 9.5 and 9.7].

3 Irreducible representations of Spin(m)

3.1 Notation

First we introduce some basic notation about Clifford algebras. We refer to [5] for a more detailed overview. Let $(\mathrm{e}_1,\ldots,\mathrm{e}_m)$ be the standard basis of the Euclidean space $\mathbb{R}^m$ and $\mathbb{R}_{0,m}$ be the real Clifford algebra generated by the vectors $\mathrm{e}_1,\ldots,\mathrm{e}_m$ such that $\mathrm{e}_j^2 = -1$ for $j = 1,\ldots,m$, and $\mathrm{e}_i\mathrm{e}_j = -\mathrm{e}_j\mathrm{e}_i$, for $i,j = 1,\ldots,m$, and $i \neq j$. An element $a \in \mathbb{R}_{0,m}$ is of the form $a = \sum_A a_A\, \mathrm{e}_A$, $a_A \in \mathbb{R}$ for ordered subsets $A \subseteq \{1,\ldots,m\}$ and with $\mathrm{e}_\varnothing = \mathrm{e}_0 = 1$. The k-vector part of a is given by $[a]_k = \sum_{|A|=k} a_A \mathrm{e}_A$ and $a = \sum_{k=0}^m [a]_k$. Vectors $\underline{x} \in \mathbb{R}^m$ are identified with 1-vectors $\underline{x} = \sum_{j=1}^m x_j \mathrm{e}_j \in \mathbb{R}_{0,m}$.

The Clifford product of two 1-vectors $\underline{x}$ and $\underline{y}$ in $\mathbb{R}^m$ splits in a scalar part given by minus the inner product in $\mathbb{R}^m$ and the wedge product:

$$\underline{x}\,\underline{y} = -\underline{x}\cdot\underline{y} + \underline{x}\wedge\underline{y}. \tag{3.1}$$

It holds $-\underline{x}\cdot\underline{y} = \frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\,\underline{x})$ and $\underline{x}\wedge\underline{y} = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\,\underline{x})$. These can be extended to the whole Clifford algebra by setting

$$-\mathrm{e}_j\cdot[a]_k \;\equiv\; [\mathrm{e}_j[a]_k]_{k-1} \;=\; \frac{1}{2}(\mathrm{e}_j[a]_k + (-1)^{k-1}[a]_k\mathrm{e}_j) \tag{3.2}$$

$$\mathrm{e}_j\wedge[a]_k \;\equiv\; [\mathrm{e}_j[a]_k]_{k+1} \;=\; \frac{1}{2}(\mathrm{e}_j[a]_k - (-1)^{k-1}[a]_k\mathrm{e}_j). \tag{3.3}$$

The Dirac operator on $\mathbb{R}^m$ is given by $\partial_{\underline{x}} = \mathrm{e}_1\partial_{x_1} + \ldots + \mathrm{e}_m\partial_{x_m}$ and its null solutions are called (left) monogenic functions. Right monogenic functions can also be defined considering the multiplication of the partial derivatives by the basis elements on the right.

The complex Clifford algebra $\mathbb{C}_m$ is the complexification of $\mathbb{R}_{0,m}$, i.e. $\mathbb{C}_m = \mathbb{C}\otimes\mathbb{R}_{0,m}$. The main anti-involution in $\mathbb{C}_m$ is defined by

$$\overline{a} = \sum_{A\subset M}\overline{a_A}\,\overline{\mathrm{e}_A}, \qquad \overline{\mathrm{e}_i\mathrm{e}_j} = \overline{\mathrm{e}_j}\,\overline{\mathrm{e}_i}, \qquad \overline{\mathrm{e}_i} = -\mathrm{e}_i, \qquad \overline{\mathrm{e}_0} = \mathrm{e}_0. \tag{3.4}$$

The complex Clifford algebra is equipped with the Clifford inner product defined by

$$\langle a,b\rangle_{\mathbb{C}_m} = [\overline{a}b]_0 = \sum_{|A|=0}^{m}(-1)^{|A|}\overline{a_A}\,b_A. \tag{3.5}$$

We will frequently use the Witt basis vectors

$$T_j = \frac{1}{2}(\mathrm{e}_{2j-1} - \mathbf{i}\,\mathrm{e}_{2j}), \qquad T_j^\dagger = -\frac{1}{2}(\mathrm{e}_{2j-1} + \mathbf{i}\,\mathrm{e}_{2j}), \qquad j = 1,\ldots M \tag{3.6}$$

for $M = \lfloor\frac{m}{2}\rfloor$. They satisfy $T_j^2 = 0 = (T_j^\dagger)^2$ together with $T_iT_j = -T_jT_i$ and $T_iT_j^\dagger = -T_j^\dagger T_i$ for $i\neq j$ and $T_iT_i^\dagger + T_i^\dagger T_i = 1$. For even m they generate all of $\mathbb{C}_m$.

Later on we will use spaces of $\mathbb{C}_m$-valued polynomials on $\mathbb{R}^m$. For this, we recall the Fischer inner product

$$\langle P,Q\rangle := [\overline{P}(\partial_{\underline{x}})Q(\underline{x})]_0\Big|_{\underline{x}=0} \;=\; [(\overline{P}(\partial_{\underline{x}})Q)(0)]_0 \tag{3.7}$$

defined for two such polynomials P and Q. This definition implies immediately that homogeneous polynomials of different degree are Fischer orthogonal. The multiplication by the variable x_i and the derivative ∂_{x_i} are Fischer-adjoint while the generators

e_i of the Clifford algebra $\mathbb{C}_m$ are skew-adjoint

$$\langle x_i P, Q\rangle = \langle P, \partial_{x_i} Q\rangle, \qquad \langle \mathrm{e}_i P, Q\rangle = -\langle P, \mathrm{e}_i Q\rangle. \tag{3.8}$$

3.2 The spin group and H- and L-representations

The spin group Spin(m) is realised as the set of even products of unit vectors, that is,

$$\mathrm{Spin(m)} = \{\prod_{j=1}^{2k} s_j \mid s_j \in \mathbb{R}^m,\ |s_j| = 1,\ k \in \mathbb{N}\} \subset \mathbb{R}^+_{0,m}, \tag{3.9}$$

where $\mathbb{R}^+_{0,m} = \mathrm{span}_{\mathbb{R}}\{\mathrm{e}_A \mid |A| \text{ even}\}$ denotes the even subalgebra of $\mathbb{R}_{0,m}$. The spin group is a double covering of SO(m) as seen by the group action $\mathbb{R}^m \ni \underline{x} \mapsto \overline{s}\underline{x}s \in \mathbb{R}^m$ on vectors. There are two distinguished representations of the spin group on $\mathbb{C}_m$-valued functions on $\mathbb{C}_m$ defined by

$$\mathrm{H}(s)f(x) = sf(\overline{s}xs)\overline{s} \qquad \text{and} \qquad \mathrm{L}(s)f(x) = sf(\overline{s}xs), \tag{3.10}$$

where $x \in \mathbb{C}_m$, $s \in \mathrm{Spin}(m)$, and $f : \mathbb{C}_m \to \mathbb{C}_m$. The H-representation corresponds to the standard representation of SO(m) on scalar-valued functions $f \in \mathrm{L}^2(\mathbb{S}^{m-1})$, while the L-representation corresponds to the half-spin representations. Models for all irreducible representations arise from decomposing H and L into irreducibles. Although spin representations are an old topic (see [1]), here we follow [27] which construct representation models based on simplicial harmonic and monogenic polynomials, i.e., harmonic and monogenic polynomials of simplicial variables. This is an extension of the work in [11], where the authors consider simplicial harmonic polynomials which provide only models for irreducible representations with integer weight of the SO(m) group.

The Lie algebra $\mathfrak{spin}(m)$ can be realised as the space

$$\mathfrak{spin}(m) \cong \mathbb{R}^{(2)}_{0,m} = \mathrm{span}\left\{\frac{1}{2}\mathrm{e}_{i,j} := \frac{1}{2}\mathrm{e}_i\mathrm{e}_j \,\middle|\, i < j, i, j = 1, \ldots, m\right\} \tag{3.11}$$

of bivectors in $\mathbb{R}_{0,m}$. The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{spin}(m)$ is given by

$$\mathfrak{h} = \mathrm{span}\left\{\frac{1}{2}\mathrm{e}_{2j-1,2j} \,\middle|\, j = 1, \ldots, M = \left\lfloor\frac{m}{2}\right\rfloor\right\}. \tag{3.12}$$

The exponential of $\mathfrak{h}$ yields the maximal torus $\mathbb{T} \subset \mathrm{Spin}(m)$

$$\mathbb{T} = \left\{s = \exp\left(\frac{1}{2}t_1\mathrm{e}_{1,2}\right)\cdots\exp\left(\frac{1}{2}t_M\,\mathrm{e}_{2M-1,2M}\right) \mid t_i \in [0, 2\pi[\right\}, \quad M = \left\lfloor\frac{m}{2}\right\rfloor, \tag{3.13}$$

from which we can label all the unitary irreducible spin representations. Any representation $R : \mathrm{Spin}(m) \to \mathrm{Aut}(V)$ of Spin(m) on some vector space V is determined by the restriction to the maximal torus and any representation of the maximal torus invariant under the adjoint action of the group comes from a representation of the group itself. The space V splits into subspaces generated by weight vectors $v \in V$ satisfying

$$\mathrm{d}R\Big(\frac{1}{2}t_1\mathrm{e}_{12} + \cdots + \frac{1}{2}t_M\,\mathrm{e}_{2M-1,2M}\Big)v = \Big(\mathbf{i}\sum_{j=1}^{M} l_j t_j\Big)v \tag{3.14}$$

for the derived representation $\mathrm{d}R : \mathfrak{spin}(m) \to \mathrm{Aut}(V)$ and with weights $l = (l_1, \ldots, l_M)$ consisting entirely of either integer or half integer numbers. Factoring out the action of the Weyl group, we obtain the highest weights given by the ordering

$$\begin{aligned} l = (l_1, \ldots, l_M) : l_1 \geq l_2 \geq \ldots \geq l_M \geq 0 \quad &\text{if } m = 2M+1, \\ l = (l_1, \ldots, l_M) : l_1 \geq l_2 \geq \ldots \geq |l_M| \quad &\text{if } m = 2M, \end{aligned} \tag{3.15}$$

where all $l_i \in \mathbb{Z}$ or all $l_i \in \mathbb{Z} + \frac{1}{2}$.

3.3 Explicit models

To construct explicit models for irreducible representations of $\mathrm{Spin}(m)$ we follow [27] and consider $k \leq m$ vector variables $\underline{x}_1, \ldots, \underline{x}_k$, where $\underline{x}_i = \sum_{j=1}^{m} x_{ij}\mathrm{e}_j$ and $\mathbb{C}_m$-valued polynomials in these k vector variables. A polynomial $P(\underline{x}_1, \underline{x}_1 \wedge \underline{x}_2, \ldots, \underline{x}_1 \wedge \ldots \wedge \underline{x}_k)$ depending on the simplicial variables $\underline{x}_1 \wedge \ldots \wedge \underline{x}_i$ is called a simplicial polynomial. A harmonic simplicial polynomial P is a simplicial polynomial satisfying

$$\Delta_{\underline{x}_i} P(\underline{x}_1, \underline{x}_1 \wedge \underline{x}_2, \ldots, \underline{x}_1 \wedge \ldots \wedge \underline{x}_k) = 0, \quad i = 1, \ldots, k. \tag{3.16}$$

The space of these polynomials is denoted by $\mathcal{H}(\underline{x}_1, \ldots, \underline{x}_k)$. It is invariant under the H-action. A monogenic simplicial polynomial P is characterised by the condition

$$\partial_{\underline{x}_i} P(\underline{x}_1, \underline{x}_1 \wedge \underline{x}_2, \ldots, \underline{x}_1 \wedge \ldots \wedge \underline{x}_k) = 0, \quad i = 1, \ldots, k. \tag{3.17}$$

The space of monogenic simplicial polynomials is denoted by $\mathcal{M}(\underline{x}_1, \ldots, \underline{x}_k)$ and is invariant under the L-action.

Different to the notation from [27] we will parameterise representations and representation spaces by their weights and not by degrees of homogeneity of the polynomials in the representation spaces.[1]

Case 1. For the highest weight $(l_1, \ldots, l_{M-1}, l_M)$, $l_i \in \mathbb{N}_0$, we take the highest weight vector

$$\begin{aligned} &\omega_{(l_1,\ldots,l_M)}(\underline{x}_1, \ldots, \underline{x}_M) \\ &\quad = \langle \underline{x}_1, T_1\rangle^{l_1-l_2}\langle \underline{x}_1 \wedge \underline{x}_2, T_1 \wedge T_2\rangle^{l_2-l_3} \cdots \langle \underline{x}_1 \wedge \cdots \wedge \underline{x}_M, T_1 \wedge \ldots \wedge T_M\rangle^{l_M} \end{aligned} \tag{3.18}$$

[1] Although both parameterisations of representations have their own merits, we will always use weights as parameters later on. This allows for direct comparison to abstract results on compact groups as used in [23], [25] or [17], [2].

and let Spin(m) act by the H-representation on it. We recall that for $s \in \mathrm{Spin}(m)$ the H-representation on simplicial functions is given by

$$\mathrm{H}(s)F(\underline{x}_1, \ldots, \underline{x}_1 \wedge \ldots \wedge \underline{x}_M) = s\, F(\overline{s}\, \underline{x}_1 s, \ldots, \overline{s}\, \underline{x}_1 \wedge \ldots \wedge \underline{x}_M\, s)\overline{s}. \tag{3.19}$$

We denote the resulting representation space by

$$\mathcal{H}_{(l_1,\ldots,l_M)}(\mathbb{R}^m) = \mathrm{span}\{\mathrm{H}(s)\omega_{(l_1,\ldots,l_M)} \mid s \in \mathrm{Spin}(m)\}. \tag{3.20}$$

Case 2. For the highest weight $(l_1, \ldots, l_{M-1}, -l_M)$, $l_i \in \mathbb{N}_0$, we have to consider the H-action on the highest weight vector

$$\begin{aligned}&\omega_{(l_1,\ldots,-l_M)}(\underline{x}_1, \ldots, \underline{x}_M)\\ &\quad = \langle \underline{x}_1, T_1\rangle^{l_1-l_2} \langle \underline{x}_1 \wedge \underline{x}_2, T_1 \wedge T_2\rangle^{l_2-l_3} \cdots \langle \underline{x}_1 \wedge \cdots \wedge \underline{x}_M, T_1 \wedge \ldots \wedge T_{M-1} \wedge T_M^\dagger\rangle^{l_M}.\end{aligned} \tag{3.21}$$

The resulting representation space will be denoted by $\mathcal{H}_{(l_1,\ldots,-l_M)}(\mathbb{R}^m)$.

For half-integer weights we have to realise the representations in the spinor space of the complex Clifford algebra $\mathbb{C}_m$. We distinguish between even and odd m. For even $m = 2M$ we use the pairwise commuting idempotents $\mathcal{I}_j = T_j T_j^\dagger = \frac{1}{2}(1 - \mathbf{i}\, \mathrm{e}_{2j-1}\mathrm{e}_{2j})$, $j = 1, \ldots, M$, together with $\mathcal{I}'_M = T_M^\dagger T_M = \frac{1}{2}(1 + \mathbf{i}\, \mathrm{e}_{2j-1}\mathrm{e}_{2j})$, to construct the primitive idempotents

$$\mathcal{I}_+ = \mathcal{I}_1 \cdots \mathcal{I}_M \qquad \text{and} \qquad \mathcal{I}_- = \mathcal{I}_1 \cdots \mathcal{I}_{M-1}\mathcal{I}'_M. \tag{3.22}$$

and define

$$\mathcal{S}_m^+ = \mathbb{C}_m^+ \mathcal{I}_+, \qquad \mathcal{S}_m^- = \mathbb{C}_m^+ \mathcal{I}_-. \tag{3.23}$$

They are both Spin(m)-invariant, minimal and inequivalent.

In the odd dimensional case $m = 2M + 1$ there is up to equivalence only one spinor space and we use

$$\mathcal{S} = \mathcal{S}_{m+1}^+ = \mathbb{C}_{m+1}^+ \mathcal{I}_+. \tag{3.24}$$

Case 3. For the highest weight $(l_1, \ldots, l_{M-1}, l_M)$, $l_i \in \mathbb{N}_0 + \frac{1}{2}$, we take the highest weight vector

$$\begin{aligned}&\omega_{(l_1,\ldots,l_M)}(\underline{x}_1, \ldots, \underline{x}_M)\\ &\quad = \langle \underline{x}_1, T_1\rangle^{l_1-l_2} \langle \underline{x}_1 \wedge \underline{x}_2, T_1 \wedge T_2\rangle^{l_2-l_3} \cdots \langle \underline{x}_1 \wedge \cdots \wedge \underline{x}_M, T_1 \wedge \ldots \wedge T_M\rangle^{l_M - \frac{1}{2}} \mathcal{I}_+,\end{aligned} \tag{3.25}$$

and let Spin(m) act by the L-representation on it. We recall that for $s \in \mathrm{Spin}(m)$ the L-representation on simplicial spinor functions is given by

$$\mathrm{L}(s)F(\underline{x}_1, \ldots, \underline{x}_1 \wedge \ldots \wedge \underline{x}_M) = s\, F(\overline{s}\, \underline{x}_1\, s, \ldots, \overline{s}\, \underline{x}_1 \wedge \ldots \wedge \underline{x}_M\, s). \tag{3.26}$$

We denote the resulting representation space by

$$\mathcal{M}_{(l_1,\dots,l_M)}(\mathbb{R}^m) = \operatorname{span}\{\mathrm{L}(s)\omega_{(l_1,\dots,l_M)} \mid s \in \mathrm{Spin}(m)\}. \tag{3.27}$$

Case 4. For the highest weight $(l_1, \dots, l_{M-1}, -l_M)$, $l_i \in \mathbb{N}_0 + \frac{1}{2}$, we have to consider the L-action on the highest weight vector

$$\omega_{(l_1,\dots,-l_M)}(\underline{x}_1, \dots, \underline{x}_M) = \\ \langle \underline{x}_1, T_1\rangle^{l_1-l_2} \langle \underline{x}_1 \wedge \underline{x}_2, T_1 \wedge T_2\rangle^{l_2-l_3} \cdots \langle \underline{x}_1 \wedge \cdots \wedge \underline{x}_M, T_1 \wedge \ldots \wedge T_{M-1} \wedge T_M^\dagger\rangle^{l_M-\frac{1}{2}} \mathcal{I}_-. \tag{3.28}$$

The resulting representation space will be denoted by $\mathcal{M}_{(l_1,\dots,-l_M)}(\mathbb{R}^m)$.

To summarize, in the odd dimensional case ($m = 2M + 1$) the irreducible representations of Spin(m) are obtained considering the H- and L-actions on the weight vectors (3.18) and (3.25), correspondingly. In the even dimensional case ($m = 2M$) all the irreducible representations of Spin(m) are obtained considering the H-action on the weight vectors (3.18) and (3.21), and the L-action on the weight vectors (3.25) and (3.28).

4 Spin(3) representations

In this section we collect results on Spin(3)-representations, in particular constructing the irreducible modules from the theory explained in Section 3. These turn out to be also important in the construction of Spin(4)-representations later on.

The group Spin(3) is the universal cover of SO(3) and can be realised inside the even Clifford algebra $\mathbb{R}_3^+$. Moreover, it is isomorphic to the special unitary group SU(2) and also isomorphic to the unit 3-sphere $\mathbb{S}^3$ understood as the group of unit length quaternions, i.e.

$$\mathrm{Spin}(3) = \{a \in \mathbb{R}_{0,3}^+ : |a| = 1\} \cong \mathbb{S}^3 \cong \mathrm{SU}(2). \tag{4.1}$$

We want to make these isomorphisms explicit for later use. An element of Spin(3) is of the form $s = a_0 + a_1 e_{12} + a_2 e_{13} + a_3 e_{23}$ such that $|s|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2 = 1$. It acts on vectors $\underline{x} = x_1 e_1 + x_2 e_2 + x_3 e_3 \in \mathbb{R}^3$ by $\underline{x} \mapsto \bar{s}\underline{x}s$ and this mapping is represented by the SO(3) rotation matrix

$$\begin{pmatrix} 1 - 2a_1^2 - 2a_2^2 & 2(a_0a_1 - a_2a_3) & 2(a_0a_2 + a_1a_3) \\ -2(a_0a_1 + a_2a_3) & 1 - 2a_1^2 - 2a_3^2 & 2(a_0a_3 - a_1a_2) \\ -2(a_0a_2 - a_1a_3) & -2(a_0a_3 + a_1a_2) & 1 - 2a_2^2 - 2a_3^2 \end{pmatrix} \tag{4.2}$$

as a straightforward calculation within $\mathbb{R}_{0,3}$ shows. Identifying $e_{12} = \mathbf{i}$, $e_{13} = \mathbf{j}$ und $e_{23} = \mathbf{k}$ with the quaternion units yields an isomorphism $\mathbb{R}_{0,3}^+ \cong \mathbb{H}$ and identifies the spin group with the group of unit length quaternions. We identify $\mathbb{H} \cong \mathbb{C}^2$ by writing $q = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = q_1 + q_2\mathbf{j}$ with $q_1 = a_0 + \mathbf{i}a_1 \in \mathbb{C}$ and $q_2 = a_2 + \mathbf{i}a_3 \in \mathbb{C}$. Then in particular $\overline{q} = \overline{q_1} - q_2\mathbf{j}$ is the quaternion conjugation and quaternion multiplication corresponds to matrix multiplication for the associated matrices

$$\begin{pmatrix} q_1 & q_2 \\ -\overline{q}_2 & \overline{q}_1 \end{pmatrix}, \tag{4.3}$$

which belong to SU(2) whenever $|q| = 1$. This completes the isomorphisms in (4.1). We rewrite the rotation matrix (4.2) in these complex coordinates for Spin(3). This yields

$$\frac{1}{2}\begin{pmatrix} (q_1^2 - q_2^2 + \overline{q}_1^2 - \overline{q}_2^2) & \mathbf{i}(-q_1^2 + q_2^2 + \overline{q}_1^2 - \overline{q}_2^2) & 2(q_1\overline{q}_2 + \overline{q}_1 q_2) \\ \mathbf{i}(q_1^2 + q_2^2 - \overline{q}_1^2 - \overline{q}_2^2) & (q_1^2 + q_2^2 + \overline{q}_1^2 + \overline{q}_2^2) & 2\mathbf{i}(q_1\overline{q}_2 - \overline{q}_1 q_2) \\ -2(\overline{q}_1\overline{q}_2 + q_1 q_2) & 2\mathbf{i}(q_1 q_2 - \overline{q}_1\overline{q}_2) & 2|q_1|^2 - 2|q_2|^2 \end{pmatrix}. \tag{4.4}$$

For later calculations we need the H-action on the polynomials $z_1 = x_1 + \mathbf{i}x_2$, $\overline{z}_1 = x_1 - \mathbf{i}x_2$ and x_3. Using the rotation matrix (4.4) applied to $\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ we obtain

$$\mathrm{H}(q)\, z_1 = \overline{q}_1^2\, z_1 + 2\overline{q}_1 q_2 x_3 - q_2^2\, \overline{z}_1, \tag{4.5a}$$

$$\mathrm{H}(q)\, \overline{z}_1 = -\overline{q}_2^2\, z_1 + 2q_1\overline{q}_2 x_3 + q_1^2\, \overline{z}_1, \tag{4.5b}$$

$$\mathrm{H}(q)\, x_3 = -\overline{q}_1\overline{q}_2 z_1 + (|q_1|^2 - |q_2|^2)x_3 - q_1 q_2 \overline{z}_1. \tag{4.5c}$$

4.1 Representations of $\mathbb{S}^3 \subset \mathbb{H}$

Before explicitly giving irreducible Spin(3)-representations based on the general theory of Section 3, we will recall the closely related irreducible $\mathbb{S}^3$-representations from [13, 21] (see also [10, 23, 28]). Since the quaternionic unit sphere $\mathbb{S}^3$ can be viewed as a subset of $\mathbb{C}^2$ through $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ we write the quaternion multiplication and conjugation in $\mathbb{S}^3$ as

$$(z_1, z_2) \bullet (w_1, w_2) = (z_1 w_1 - z_2\overline{w}_2, z_1 w_2 + z_2\overline{w}_1), \tag{4.6a}$$

$$(z_1, z_2)^* = (\overline{z}_1, -z_2). \tag{4.6b}$$

Let $\mathcal{P}$ be the space of all polynomials $P(z, w) = \sum c_{j,k} z^j w^k$ in two complex variables, and let $\mathcal{P}_m \subset \mathcal{P}$ be the space of homogeneous polynomials of degree m

$$\mathcal{P}_m = \Big\{ P(z, w) = \sum_{j=0}^{m} c_j z^j w^{m-j} \;\Big|\; c_j \in \mathbb{C},\, j = 0, \dots, m \Big\}. \tag{4.7}$$

An orthonormal basis of $\mathcal{P}_m$ with respect to the Fischer inner product (or equivalently, with respect to the normalised L^2 inner product) consists of the set of functions

$$P_k^m(z_1, z_2) = \frac{1}{\sqrt{(m-k)!}\sqrt{k!}} z_1^{m-k} z_2^k, \quad 0 \le k \le m,\; m \in \mathbb{N}. \tag{4.8}$$

The group $\mathbb{S}^3$ naturally acts on $\mathcal{P}_m$ by right translation $R_{(w_1,w_2)}f(z_1,z_2) = f((z_1,z_2)\bullet(w_1,w_2))$. Next, we follow [13] and express the linear map $R_{(w_1,w_2)} : \mathcal{P}_m \to \mathcal{P}_m$ with respect to the orthonormal basis P_k^m, $j = 0,\dots,m$. By straightforward computations we have

$$\begin{aligned}
&\sqrt{(m-j)!}\,\sqrt{j!}\,R_{(w_1,w_2)}P_j^m(z_1,z_2) \\
&= \sqrt{(m-j)!}\,\sqrt{j!}\,P_j^m((z_1,z_2)\bullet(w_1,w_2)) \\
&= (z_1w_1 - z_2\overline{w}_2)^{m-j}(z_1w_2 + z_2\overline{w}_1)^j \\
&= \left(\sum_{k=0}^{m-j}\binom{m-j}{k}(z_1w_1)^{m-j-k}(-z_2\overline{w}_2)^k\right)\left(\sum_{l=0}^{j}\binom{j}{l}(z_1w_2)^{j-l}(z_2\overline{w}_1)^l\right) \\
&= \sum_{i=0}^{m} z_1^{m-i}z_2^i \sum_{k+l=i}(-1)^k\binom{m-j}{k}\binom{j}{l}w_1^{m-j-k}\,\overline{w}_1^l w_2^{j-l}\,\overline{w}_2^k \\
&= \sum_{i=0}^{m}\frac{1}{\sqrt{(m-i)!}\,\sqrt{i!}}z_1^{m-i}z_2^i \\
&\quad\times \sum_{k=\max(0,i-j)}^{\min(i,m-j)}\sqrt{(m-i)!}\,\sqrt{i!}\binom{m-j}{k}\binom{j}{i-k}w_1^{m-j-k}\,\overline{w}_1^{i-k}w_2^{j-i+k}\,(-\overline{w}_2)^k\,.
\end{aligned} \tag{4.9}$$

Hence, for each $j = 0,\dots,m$, we obtain

$$R_{(w_1,w_2)}P_j^m(z_1,z_2) = \sum_{i=0}^{m}P_i^m(z_1,z_2)\sum_{k=\max(0,i-j)}^{\min(i,m-j)}C_{i,j,m}^k\,w_1^{m-j-k}\,\overline{w}_1^{i-k}w_2^{j-i+k}\,(-\overline{w}_2)^k \tag{4.10}$$

where

$$C_{i,j,m}^k = \sqrt{\binom{m}{i}^{-1}\binom{m}{j}}\binom{m-j}{k}\binom{j}{i-k}, \tag{4.11}$$

and, therefore, the matrix elements of the unitary representation

$$\boldsymbol{\xi}_m : \mathrm{SU}(2) \cong \mathbb{S}^3 \to \mathrm{U}(m+1) \tag{4.12}$$

are given by

$$\boldsymbol{\xi}_m(w_1,w_2)_{i,j} = \sum_{k=\max(0,i-j)}^{\min(i,m-j)}C_{i,j,m}^k\,w_1^{m-j-k}\,\overline{w}_1^{i-k}w_2^{j-i+k}\,(-\overline{w}_2)^k, \quad 0 \le i,j \le m. \tag{4.13}$$

In particular, the first row contains the holomorphic polynomials, i.e.

$$\boldsymbol{\xi}_m(w_1,w_2)_{0,j} = \sqrt{\binom{m}{j}}w_1^{m-j}w_2^j, \quad 0 \le j \le m, \tag{4.14}$$

while the first column contains polynomials of the form

$$\boldsymbol{\xi}_m(w_1,w_2)_{i,0} = \sqrt{\binom{m}{i}} w_1^{m-i}(-\overline{w}_2)^i, \quad 0 \le i \le m. \tag{4.15}$$

The following recurrence relations are taken from [13] and follow easily by direct computation.

Theorem 4.1 (cf. [13])**.** *For every $m \in \mathbb{N}_0$ and all $0 \le i,j \le m$ the following recurrence relations hold*

$$z_1\,\boldsymbol{\xi}_m(z_1,z_2)_{i,j} = \frac{\sqrt{m+1-i}\sqrt{m+1-j}}{m+1}\boldsymbol{\xi}_{m+1}(z_1,z_2)_{i,j} + \frac{\sqrt{i}\sqrt{j}}{m+1}\boldsymbol{\xi}_{m-1}(z_1,z_2)_{i-1,j-1} \tag{4.16a}$$

$$z_2\,\boldsymbol{\xi}_m(z_1,z_2)_{i,j} = \frac{\sqrt{m+1-i}\sqrt{j+1}}{m+1}\boldsymbol{\xi}_{m+1}(z_1,z_2)_{i,j+1} - \frac{\sqrt{i}\sqrt{m-j}}{m+1}\boldsymbol{\xi}_{m-1}(z_1,z_2)_{i-1,j} \tag{4.16b}$$

$$\overline{z}_1\,\boldsymbol{\xi}_m(z_1,z_2)_{i,j} = \frac{\sqrt{i+1}\sqrt{j+1}}{m+1}\boldsymbol{\xi}_{m+1}(z_1,z_2)_{i+1,j+1} + \frac{\sqrt{m-i}\sqrt{m-j}}{m+1}\boldsymbol{\xi}_{m-1}(z_1,z_2)_{i,j} \tag{4.16c}$$

$$-\overline{z}_2\,\boldsymbol{\xi}_m(z_1,z_2)_{i,j} = \frac{\sqrt{i+1}\sqrt{m+1-j}}{m+1}\boldsymbol{\xi}_{m+1}(z_1,z_2)_{i+1,j} - \frac{\sqrt{m-i}\sqrt{j}}{m+1}\boldsymbol{\xi}_{m-1}(z_1,z_2)_{i,j-1} \tag{4.16d}$$

where every expression out of domain is interpreted as zero.

The previous recurrence relations can be written in matrix form using suitable matrices filled up with zeros and suitable weights in their entries.

Definition 4.2 (cf. [13])**.** For $m \ge 0$ we define the matrices $a_-(m), a_+(m) \in \mathbb{R}^{(m+1)\times(m+2)}$ by

$$a_-(m) = \frac{1}{\sqrt{m+1}} \begin{pmatrix} \sqrt{m+1} & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{m} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & \sqrt{1} & 0 \end{pmatrix}, \tag{4.17a}$$

$$a_+(m) = \frac{1}{\sqrt{m+1}} \begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \sqrt{m+1} \end{pmatrix}. \tag{4.17b}$$

For $m \geq 1$ we define the matrices $b_-(m), b_+(m) \in \mathbb{R}^{(m+1)\times m}$ by

$$b_-(m) = \frac{1}{\sqrt{m+1}} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \sqrt{1} & 0 & \cdots & 0 \\ 0 & \sqrt{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{m} \end{pmatrix} \tag{4.18a}$$

$$b_+(m) = \frac{1}{\sqrt{m+1}} \begin{pmatrix} \sqrt{m} & 0 & \cdots & 0 \\ 0 & \sqrt{m-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sqrt{1} \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4.18b}$$

and for convenience, we set $b_-(0) = b_+(0) = 0$.

Theorem 4.3 (cf. [13])**.** *For every $m \in \mathbb{N}_0$ the following recurrence relations hold*

$$z_1\, \boldsymbol{\xi}_m(z_1,z_2) = a_-(m)\boldsymbol{\xi}_{m+1}(z_1,z_2)a_-(m)^\top + b_-(m)\boldsymbol{\xi}_{m-1}(z_1,z_2)b_-(m)^\top, \tag{4.19a}$$

$$z_2\, \boldsymbol{\xi}_m(z_1,z_2) = a_-(m)\boldsymbol{\xi}_{m+1}(z_1,z_2)a_+(m)^\top - b_-(m)\boldsymbol{\xi}_{m-1}(z_1,z_2)b_+(m)^\top, \tag{4.19b}$$

$$-\overline{z}_2\, \boldsymbol{\xi}_m(z_1,z_2) = a_+(m)\boldsymbol{\xi}_{m+1}(z_1,z_2)a_-(m)^\top - b_+(m)\boldsymbol{\xi}_{m-1}(z_1,z_2)b_-(m)^\top, \tag{4.19c}$$

$$\overline{z}_1\, \boldsymbol{\xi}_m(z_1,z_2) = a_+(m)\boldsymbol{\xi}_{m+1}(z_1,z_2)a_+(m)^\top + b_+(m)\boldsymbol{\xi}_{m-1}(z_1,z_2)b_+(m)^\top. \tag{4.19d}$$

It is possible to define shift operators acting on the matrix coefficients of a given representation. They are related to left- respectively right-invariant differential operators on $\mathbb{S}^3$.

Definition 4.4 (cf. [13])**.** For differentiable functions $\mathbb{S}^3 \ni (z_1,z_2) \mapsto f(z_1,z_2) \in \mathbb{C}$ we define the following differential operators

$$\partial_+ = -z_2\partial_{z_1} + \overline{z}_1\partial_{\overline{z}_2}, \qquad \partial_- = \overline{z}_2\partial_{\overline{z}_1} - z_1\partial_{z_2}, \tag{4.20a}$$

$$\partial_+^\dagger = \overline{z}_1\partial_{z_2} - \overline{z}_2\partial_{z_1}, \qquad \partial_-^\dagger = -z_1\partial_{\overline{z}_2} + z_2\partial_{\overline{z}_1}. \tag{4.20b}$$

It is easy to see that the operators $\partial_\pm$ and $\partial_\pm^\dagger$ are linear combination of rotational derivatives. The operators $\partial_\pm$ are left invariant and the operators $\partial_\pm^\dagger$ are right invariant. This means that

$$R_{(w_1,w_2)}\partial_\pm^\dagger f(z_1,z_2) = \partial_\pm^\dagger R_{(w_1,w_2)} f(z_1,z_2), \tag{4.21a}$$

$$L_{(w_1,w_2)}\partial_\pm f(z_1,z_2) = \partial_\pm L_{(w_1,w_2)} f(z_1,z_2) \tag{4.21b}$$

holds true for the right translation $R_{(w_1,w_2)}f(z_1,z_2) = f((z_1,z_2)\bullet(w_1,w_2))$ and the left translation $L_{(w_1,w_2)}f(z_1,z_2) = f((w_1,w_2)^{-1}\bullet(z_1,z_2)) = f((\overline{w}_1,-w_2)\bullet(z_1,z_2))$ on the group $\mathbb{S}^3$.

Theorem 4.5 (cf. [13]). *For every $m \in \mathbb{N}_0$ and $0 \le i,j \le m$ the following relations hold*

$$\begin{aligned}
\partial_+^\dagger(\boldsymbol{\xi}_m)_{i,j} &= \sqrt{m-i}\,\sqrt{i+1}\,(\boldsymbol{\xi}_m)_{i+1,j}, && (4.22a)\\
\partial_-^\dagger(\boldsymbol{\xi}_m)_{i,j} &= \sqrt{m+1-i}\,\sqrt{i}\,(\boldsymbol{\xi}_m)_{i-1,j}, && (4.22b)\\
\partial_+(\boldsymbol{\xi}_m)_{i,j} &= -\sqrt{m-j}\,\sqrt{j+1}\,(\boldsymbol{\xi}_m)_{i,j+1}, && (4.22c)\\
\partial_-(\boldsymbol{\xi}_m)_{i,j} &= -\sqrt{m+1-j}\,\sqrt{j}\,(\boldsymbol{\xi}_m)_{i,j-1}, && (4.22d)
\end{aligned}$$

where every matrix coefficient outside of the matrix is understood as zero.

The previous relations can be written in matrix form using two special matrices $\sigma_+(m/2)$ and $\sigma_-(m/2)$ defined as follows. The use of $m/2$ instead of m as argument is related to the parametrisation of representations by weights instead of by homogeneity used later on.

Definition 4.6 (cf. [13]). For $m \ge 0$ we define the matrices $\sigma_+(m/2), \sigma_-(m/2) \in \mathbb{R}^{(m+1)\times(m+1)}$ by

$$\sigma_+(m/2) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0\\ -\sqrt{m} & 0 & \cdots & 0 & 0\\ 0 & -\sqrt{2(m-1)} & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & 0 & 0\\ 0 & \cdots & 0 & -\sqrt{m} & 0 \end{pmatrix}, \tag{4.23a}$$

$$\sigma_-(m/2) = \begin{pmatrix} 0 & -\sqrt{m} & 0 & \cdots & 0\\ 0 & 0 & -\sqrt{2(m-1)} & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 0\\ 0 & 0 & \cdots & 0 & -\sqrt{m}\\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4.23b}$$

They satisfy $\sigma_+(m/2) = \sigma_-(m/2)^\top$.

Corollary 4.7 (cf. [13]). *For every $m \in \mathbb{N}_0$ the following relations hold*

$$\begin{aligned}
\partial_+\boldsymbol{\xi}_m(z_1,z_2) &= \boldsymbol{\xi}_m(z_1,z_2)\,\sigma_+(m/2), && (4.24a)\\
\partial_-\boldsymbol{\xi}_m(z_1,z_2) &= \boldsymbol{\xi}_m(z_1,z_2)\,\sigma_-(m/2), && (4.24b)\\
\partial_+^\dagger\boldsymbol{\xi}_m(z_1,z_2) &= \sigma_+^\dagger(m/2)\,\boldsymbol{\xi}_m(z_1,z_2), && (4.24c)\\
\partial_-^\dagger\boldsymbol{\xi}_m(z_1,z_2) &= \sigma_-^\dagger(m/2)\,\boldsymbol{\xi}_m(z_1,z_2), && (4.24d)
\end{aligned}$$

where $\sigma_+^\dagger(m/2) = -\sigma_+(m/2)^\top = -\sigma_-(m/2)$ *and* $\sigma_-^\dagger(m/2) = -\sigma_-(m/2)^\top = -\sigma_+(m/2)$.

Remark 4.8. The matrices $\sigma_\pm(m/2)$ can be obtained as

$$\sigma_+(m/2) = \partial_+ \boldsymbol{\xi}_m|_{(1,0)} \qquad \text{and} \qquad \sigma_-(m/2) = \partial_- \boldsymbol{\xi}_m|_{(1,0)}, \tag{4.25}$$

where we apply the operators entrywise and use the point evaluation at $(1,0)$. Thus, they are the matrix-valued symbols of the left-invariant differental operators $\partial_\pm \in \mathrm{Diff}^1(\mathbb{S}^3)$ in the sense of Section 2.

4.2 H-representations: spherical harmonics in $\mathbb{R}^3$

We follow the approach of Section 3. By (3.18) and (3.5) the weight vector $\omega_l(\underline{x})$ for $l \in \mathbb{N}_0$ is given by

$$\omega_l(\underline{x}) = \langle \underline{x}, T_1 \rangle^l = \left(\frac{x_1 - \mathbf{i}x_2}{2}\right)^l = \left(\frac{\overline{z}_1}{2}\right)^l. \tag{4.26}$$

We renormalise the weight vector considering $2T_1$ instead of T_1 obtaining the polynomial $\overline{z}_1^l$. To perform the H-action on $\omega_l(\underline{x})$ we consider (4.5b) and we get

$$\mathrm{H}(q)\,\overline{z}_1^l = \langle \overline{q}\underline{x}q, 2T_1 \rangle^l = (q_1^2\,\overline{z}_1 + 2q_1\overline{q}_2 x_3 - \overline{q}_2^2\,z_1)^l. \tag{4.27}$$

Using multi-index notation $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $\alpha! = \alpha_1!\alpha_2!\alpha_3!$, and the multinomial theorem to further expand (4.27) we obtain

$$\begin{aligned} \mathrm{H}(q)\,\overline{z}_1^l &= \sum_{|\alpha|=l} \frac{l!}{\alpha!} (q_1^2\,\overline{z}_1)^{\alpha_1}\,(2q_1\,\overline{q}_2\,x_3)^{\alpha_2}\,(-\overline{q}_2^2\,z_1)^{\alpha_3} \\ &= \sum_{j=0}^{2l} q_1^{2l-j}\,(-\overline{q}_2)^j \sum_{\substack{|\alpha|=l \\ \alpha_2+2\alpha_3=j}} \frac{l!}{\alpha!}\,\overline{z}_1^{\alpha_1}(-2x_3)^{\alpha_2}(-z_1)^{\alpha_3}, \end{aligned} \tag{4.28}$$

where in the last line we put $\alpha_2 + 2\alpha_3 = j$, which yields $2\alpha_1 + \alpha_2 = 2l - j$ from $|\alpha| = l$. The $2l+1$ polynomials given by

$$P_j^l(z_1, \overline{z}_1, x_3) = \sum_{\substack{|\alpha|=l \\ \alpha_2+2\alpha_3=j}} \frac{l!}{\alpha!}\,\overline{z}_1^{\alpha_1}(-2x_3)^{\alpha_2}(-z_1)^{\alpha_3}, \quad j = 0, \ldots, 2l, \tag{4.29}$$

suffice to build the representation space from the weight vector. As the dimension of the representation to weight l is $2l+1$, the polynomials must also belong to the representation space and form a basis. The polynomials are orthogonal with respect to the Fischer inner product (3.7)

$$\langle P_j^l, P_k^l \rangle = [\overline{P_j^l}(2\partial_{\overline{z}_1}, 2\partial_{z_1}, \partial_{x_3}) P_k^l(z_1, \overline{z}_1, x_3)]_0|_{z_1, \overline{z}_1, x_3 = 0} = 0, \qquad j \neq k, \tag{4.30}$$

due to the non-matching orders of the monomials and in order to calculate their norm, we use

$$\langle P_j^l, P_j^l \rangle = [\overline{P_j^l}(2\partial_{\overline{z}_1}, 2\partial_{z_1}, \partial_{x_3}) P_j^l(z_1, \overline{z}_1, x_3)]_0|_{z_1,\overline{z}_1,x_3=0} = \sum_{\substack{|\alpha|=l \\ \alpha_2+2\alpha_3=j}} \frac{(l!)^2}{\alpha!} 2^{l+\alpha_2} = c_{l,j}. \tag{4.31}$$

The inner product is calculated using the Fischer duality $z_1 \mapsto 2\partial_{\overline{z}_1}$, $\overline{z}_1 \mapsto 2\partial_{z_1}$, and $x_3 \mapsto \partial_{x_3}$.

The following lemma provides a relation between trinomial and binomial coefficients.

Lemma 4.9. *For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ it holds*

$$\sum_{\substack{|\alpha|=l \\ \alpha_2+2\alpha_3=j}} \frac{l!}{\alpha!} 2^{\alpha_2} = \binom{2l}{j}. \tag{4.32}$$

Proof. Multiplication by $x^j = x^{\alpha_2+2\alpha_3}$ and summing over j yields

$$\begin{aligned} \sum_{j=0}^{2l} \sum_{\substack{|\alpha|=l \\ \alpha_2+2\alpha_3=j}} \frac{l!}{\alpha!} 2^{\alpha_2} x^{\alpha_2+2\alpha_3} &= \sum_{|\alpha|=l} \frac{l!}{\alpha_1!\alpha_2!\alpha_3!} 1^{\alpha_1} (2x)^{\alpha_2} x^{2\alpha_3} \\ &= (1+2x+x^2)^l = (1+x)^{2l} = \sum_{j=0}^{2l} \binom{2l}{j} x^j \end{aligned} \tag{4.33}$$

based on the multinomial theorem. Comparing powers of x yields the desired result. □

Using Lemma 4.9 we can write the normalising constants $c_{l,j}$ as

$$c_{l,j} = \binom{2l}{j} 2^l \, l!, \qquad j = 0, \dots, 2l \tag{4.34}$$

and obtain $2l+1$ orthonormal spherical harmonics (4.29) on $\mathbb{R}^3$ spanning the representation space $\mathcal{H}_l(\mathbb{R}^3)$ for the given weight $l \in \mathbb{N}$. It remains to express $\mathrm{H}(q)$ in this basis in order to obtain the matrix coefficients and to obtain the relation to the representations constructed in Section 4.1.

For this, we consider the H-action on the orthogonal polynomials P_j^l. For each $j = 0, \dots, 2l$, we use first the actions (4.5a)-(4.5c) and then the multinomial theorem to obtain

$$
\begin{aligned}
\mathrm{H}(q)P_j^l(z_1,\overline{z}_1,x_3) = & \sum_{\substack{|\alpha|=l\\ \alpha_2+2\alpha_3=j}} \binom{l}{\alpha}(q_1^2\overline{z}_1+2q_1\overline{q}_2x_3-\overline{q}_2^2z_1)^{\alpha_1}2^{\alpha_2}\\
& \times (q_1q_2\overline{z}_1-(|q_1|^2-|q_2|^2)x_3+\overline{q}_1\overline{q}_2z_1)^{\alpha_2}(q_2^2\overline{z}_1-2\overline{q}_1q_2x_3-\overline{q}_1^2z_1)^{\alpha_3}\\
= & \sum_{\substack{|\alpha|=l\\ \alpha_2+2\alpha_3=j}} \sum_{|\beta|=\alpha_1}\sum_{|\gamma|=\alpha_2}\sum_{|\delta|=\alpha_3}\sum_{n=0}^{\gamma_2}\binom{l}{\alpha}\binom{\alpha_1}{\beta}\binom{\alpha_2}{\gamma}\binom{\alpha_3}{\delta}\binom{\gamma_2}{n}2^{\alpha_2-\gamma_2}\\
& \times q_1^{2\beta_1+\beta_2+\gamma_1+n}\overline{q}_1^{\gamma_3+\delta_2+2\delta_3+n}\, q_2^{\gamma_1+\gamma_2-n+2\delta_1+\delta_2}(-\overline{q}_2)^{\beta_2+2\beta_3+\gamma_3+\gamma_2-n}\\
& \quad \times \overline{z}_1^{\beta_1+\gamma_1+\delta_1}(-2x_3)^{\beta_2+\gamma_2+\delta_2}\,(-z_1)^{\beta_3+\gamma_3+\delta_3}\,.
\end{aligned} \tag{4.35}
$$

Putting $\beta_\lambda+\gamma_\lambda+\delta_\lambda=\epsilon_\lambda$, $\lambda=1,2,3$ and since $|\alpha|=l$, $|\beta|=\alpha_1$, $|\gamma|=\alpha_2$, $|\delta|=\alpha_3$ we obtain that $|\epsilon|=\epsilon_1+\epsilon_2+\epsilon_3=l$. Moreover, putting $\epsilon_2+2\epsilon_3=i$ and $\beta_2+2\beta_3+\gamma_3+\gamma_2-n=k$ and since $\alpha_2+2\alpha_3=|\gamma|+2|\delta|=j$ we obtain the following identities

$$
\begin{aligned}
2\beta_1+\beta_2+\gamma_1+n &= 2l-j-k\,,\\
\gamma_3+\delta_2+2\delta_3+n &= i-k\,,\\
\gamma_1+\gamma_2+2\delta_1+\delta_2-n &= j-i+k.
\end{aligned} \tag{4.36}
$$

Since $2l-j-k\geq 0$, $i-k\geq 0$, and $j-i+k\geq 0$ we get that $\max(0,i-j)\leq k\leq \min(i,2l-j)$. Therefore, (4.35) can be written as

$$
\begin{aligned}
\frac{1}{\sqrt{c_{l,j}}}\,\mathrm{H}(q)P_j^l(z_1,\overline{z}_1,x_3) = & \sum_{i=0}^{2l}\sum_{k=\max(0,i-j)}^{\min(i,2l-j)} t_{i,j,2l}^k\; q_1^{2l-j-k}\,\overline{q}_1^{i-k}q_2^{j-i+k}(-\overline{q}_2)^k\\
& \times \frac{1}{\sqrt{c_{l,i}}}\sum_{\substack{|\epsilon|=l\\ \epsilon_2+2\epsilon_3=i}}\binom{l}{\epsilon}\,\overline{z}_1^{\epsilon_1}(-2x_3)^{\epsilon_2}(-z_1)^{\epsilon_3},
\end{aligned} \tag{4.37}
$$

for some coefficients $t_{i,j,2l}^k$. Comparing (4.37) with (4.10), we must have the following identifications between the matrix coefficients (4.10) and (4.37): $m\mapsto 2l$, $w_1\mapsto q_1$, and $w_2\mapsto q_2$. Thus, the matrix coefficients related to the (normalised) basis of harmonic polynomials P_j^l associated with the weight $l\in\mathbb{N}_0$ are given by

$$
(\xi_l^{\mathrm{Spin}(3)}(q_1,q_2))_{i,j}=\sum_{k=\max(0,i-j)}^{\min(i,2l-j)} C_{i,j,2l}^k\,q_1^{2l-j-k}\,\overline{q}_1^{i-k}q_2^{j-i+k}(-\overline{q}_2)^k,\quad i,j=0,\dots,2l \tag{4.38}
$$

where $C_{i,j,2l}^k$ is given by (4.11) with $m=2l$. Therefore, it holds $t_{i,j,2l}^k=C_{i,j,2l}^k$. Consequently, the relationship between these H-representations and the representations constructed in Section 4.1 is given by

$$\xi_l^{\mathrm{Spin}(3)}(q_1, q_2) = \boldsymbol{\xi}_{2l}(q_1, q_2). \tag{4.39}$$

Next we give two examples. For $l = 1$, the module $\mathcal{H}_1(\mathbb{R}^3)$ has dimension 3 and an orthonormal basis is given by $\{\frac{1}{\sqrt{2}}\overline{z}_1,\ -x_3,\ -\frac{1}{\sqrt{2}}z_1\}$. The matrix coefficients associated to this basis are

$$\xi_1^{\mathrm{Spin}(3)}(q_1, q_2) = \begin{pmatrix} q_1^2 & \sqrt{2}\,q_1q_2 & q_2^2 \\ -\sqrt{2}\,q_1\overline{q}_2 & |q_1|^2 - |q_2|^2 & \sqrt{2}\,\overline{q}_1q_2 \\ \overline{q}_2^2 & -\sqrt{2}\,\overline{q}_1\overline{q}_2 & \overline{q}_1^2 \end{pmatrix}. \tag{4.40}$$

For $l = 2$, the module $\mathcal{H}_2(\mathbb{R}^3)$ has dimension 5 and an orthonormal basis is given by

$$\{\frac{1}{2\sqrt{2}}\overline{z}_1^2,\ -\frac{1}{\sqrt{2}}\overline{z}_1x_3,\ \frac{1}{2\sqrt{3}}(2x_3^2 - z_1\overline{z}_1),\ \frac{1}{\sqrt{2}}x_3z_1,\ \frac{1}{2\sqrt{2}}z_1^2\}. \tag{4.41}$$

The matrix coefficients $\xi_2^{\mathrm{Spin}(3)}(q_1, q_2)$ associated to this basis are

$$\begin{pmatrix} q_1^4 & 2\,q_1^3q_2 & \sqrt{6}\,q_1^2q_2^2 & 2\,q_1q_2^3 & q_2^4 \\ -2\,q_1^3\overline{q}_2 & q_1^2(|q_1|^2 - 3|q_2|^2) & \sqrt{6}\,q_1q_2(|q_1|^2 - |q_2|^2) & q_2^2(3|q_1|^2 - |q_2|^2) & 2\,\overline{q}_1q_2^3 \\ \sqrt{6}\,q_1^2\overline{q}_2^2 & -\sqrt{6}\,q_1\overline{q}_2(|q_1|^2 - |q_2|^2) & (|q_1|^2 - |q_2|^2)^2 - 2|q_1|^2|q_2|^2 & \sqrt{6}\,\overline{q}_1q_2(|q_1|^2 - |q_2|^2) & \sqrt{6}\,\overline{q}_1^2q_2^2 \\ -2\,q_1\overline{q}_2^3 & \overline{q}_2^2(3|q_1|^2 - |q_2|^2) & -\sqrt{6}\,\overline{q}_1\overline{q}_2(|q_1|^2 - |q_2|^2) & \overline{q}_1^2(|q_1|^2 - 3|q_2|^2) & 2\,\overline{q}_1^3q_2 \\ \overline{q}_2^4 & -2\,\overline{q}_1\overline{q}_2^3 & \sqrt{6}\,\overline{q}_1^2\overline{q}_2^2 & -2\,\overline{q}_1^3\overline{q}_2 & \overline{q}_1^4 \end{pmatrix}. \tag{4.42}$$

4.3 L-representations: spinor-valued monogenics in $\mathbb{R}^3$

In the 3-dimensional case there is only one basic spinor representation $\mathcal{S} \cong \mathcal{S}_4^+ = \mathbb{C}_4^+\mathcal{I}_+ = \mathrm{span}_{\mathbb{C}}\{1, \mathrm{e}_{13}\}\mathcal{I}_+$, where the idempotent $\mathcal{I}_+$ is given in terms of the Witt basis elements (as introduced in Section 3)

$$\begin{aligned} T_1 &= \frac{1}{2}(\mathrm{e}_1 - \mathbf{i}\mathrm{e}_2), \qquad & T_2 &= \frac{1}{2}(\mathrm{e}_3 - \mathbf{i}\mathrm{e}_4), \\ T_1^\dagger &= -\frac{1}{2}(\mathrm{e}_1 + \mathbf{i}\mathrm{e}_2), \qquad & T_2^\dagger &= -\frac{1}{2}(\mathrm{e}_3 + \mathbf{i}\mathrm{e}_4), \end{aligned} \tag{4.43}$$

by $\mathcal{I}_+ = \mathcal{I}_1\mathcal{I}_2$, with $\mathcal{I}_1 = T_1T_1^\dagger$ and $\mathcal{I}_2 = T_2T_2^\dagger$. For $z \in \mathcal{S}_4^+$ we put $z = (z^+, z^-) \in \mathbb{C}^2$ such that $z = z^+\mathcal{I}_+ - z^-\mathrm{e}_{13}\mathcal{I}_+$. Now, considering the left multiplication by $q = a_1 + a_2\mathrm{e}_{12} + a_3\mathrm{e}_{13} + a_4\mathrm{e}_{23} \in \mathrm{Spin}(3)$ on the spinor representation $\mathcal{S}_4^+$ we obtain

$$\begin{aligned} &q(z^+ - z^-\mathrm{e}_{13})\mathcal{I}_+ \\ &\quad = (a_0 + a_1\mathrm{e}_{12} + a_2\mathrm{e}_{13} + a_3\mathrm{e}_{23})(z^+ - z^-\mathrm{e}_{13})\mathcal{I}_+ \\ &\quad = \Big((a_0 + \mathbf{i}a_1)z^+ + (a_2 + \mathbf{i}a_3)z^- - (-(a_2 - a_3\mathbf{i})z^+ + (a_0 - \mathbf{i}a_1)z^-)\mathrm{e}_{13}\Big)\mathcal{I}_+, \end{aligned} \tag{4.44}$$

where we have used the multiplication rules of the Clifford algebra and the identities $e_{12}\mathcal{I}_+ = \mathbf{i}\mathcal{I}_+$ and $e_{23}\mathcal{I}_+ = -\mathbf{i}e_{13}\mathcal{I}_+$ following from the basic rules $e_2\mathcal{I}_+ = -\mathbf{i}e_1\mathcal{I}_+$ and $e_4\mathcal{I}_+ = -\mathbf{i}e_3\mathcal{I}_+$. The linear transformation (4.44) can again be written as an SU(2)-action given by

$$\begin{pmatrix} q_1 & q_2 \\ -\overline{q}_2 & \overline{q}_1 \end{pmatrix} \begin{pmatrix} z^+ \\ z^- \end{pmatrix}, \tag{4.45}$$

where $q_1 = a_0 + \mathbf{i}a_1$ and $q_2 = a_2 + \mathbf{i}a_3$ and $z = z^+\mathcal{I}_+ - z^- e_{13}\mathcal{I}_+$.

Next, we perform the L-action on the weight vector $\omega_l(\underline{x}) = \overline{z}_1^{\,l-\frac{1}{2}}\,\mathcal{I}_+$ for $l \in \mathbb{N}_0 + \frac{1}{2}$ to obtain the representation space for the half-integer weight. Based on (4.5b), (4.45), and (4.28), we get

$$\begin{aligned}
\mathrm{L}(q)\,\omega_l(\underline{x}) &= q\,\langle \overline{q}\,\underline{x}\,q, 2T_1\rangle^{l-\frac{1}{2}}\,\mathcal{I}_+ \\
&= [q_1(q_1^2\overline{z}_1 + 2q_1\overline{q}_2 x_3 - \overline{q}_2^2\,z_1)^{l-\frac{1}{2}} - (-\overline{q}_2(q_1^2\overline{z}_1 + 2q_1\overline{q}_2 x_3 - \overline{q}_2^2 z_1)^{l-\frac{1}{2}})e_{13}]\,\mathcal{I}_+ \\
&= \Big[\sum_{j=0}^{2l-1} q_1^{2l-j}\,(-\overline{q}_2)^j P_j^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3) - \Big(\sum_{j=0}^{2l-1} q_1^{2l-1-j}\,(-\overline{q}_2)^{j+1} P_j^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3)\Big)e_{13}\Big]\mathcal{I}_+ \\
&= \Big[\sum_{j=0}^{2l-1} q_1^{2l-j}\,(-\overline{q}_2)^j P_j^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3) - \Big(\sum_{j=1}^{2l} q_1^{2l-j}\,(-\overline{q}_2)^j P_{j-1}^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3)\Big)e_{13}\Big]\mathcal{I}_+ \\
&= \sum_{j=0}^{2l} q_1^{2l-j}\,(-\overline{q}_2)^j \Big[P_j^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3) - P_{j-1}^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3)e_{13}\Big]\,\mathcal{I}_+.
\end{aligned} \tag{4.46}$$

where we set for convenience $P_{-1}^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3) = P_{2l}^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3) = 0$. Therefore, we have obtained the $2l+1$ $\mathcal{S}_4^+$-valued monogenic polynomials

$$\widetilde{P}_j^l(z_1,\overline{z}_1,x_3) = [P_j^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3) - P_{j-1}^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3)e_{13}]\mathcal{I}_+, \quad j = 0,\dots,2l\,. \tag{4.47}$$

As this is the dimension of the representation space, we know that they are indeed linearly independent and thus must form a basis. Note that for $j = 0$ we obtain $\widetilde{P}_0^l(z_1,\overline{z}_1,x_3) = \overline{z}_1^{\,l-\frac{1}{2}}\,\mathcal{I}_+$, i.e. the weight vector, while for $j = 2l$ we get $\widetilde{P}_{2l}^l(z_1,\overline{z}_1,x_3) = -(-z_1)^{l-\frac{1}{2}}e_{13}\mathcal{I}_+$.

To obtain an orthonormal basis for $\mathcal{M}_l(\mathbb{R}^3)$, we still need to compute the Fischer inner products (3.7) of the polynomials. By (4.30)

$$\langle \widetilde{P}_j^l, \widetilde{P}_k^l\rangle = \langle P_j^{l-\frac{1}{2}}, P_k^{l-\frac{1}{2}}\rangle + \langle P_{j-1}^{l-\frac{1}{2}}, P_{k-1}^{l-\frac{1}{2}}\rangle = 0, \qquad j \neq k, \tag{4.48}$$

follows and the normalizing constants

$$d_{l,j} = \langle \widetilde{P}^l_j, \widetilde{P}^l_j \rangle = \langle P_j^{l-\frac{1}{2}}, P_j^{l-\frac{1}{2}} \rangle + \langle P_{j-1}^{l-\frac{1}{2}}, P_{j-1}^{l-\frac{1}{2}} \rangle = c_{l-\frac{1}{2},j} + c_{l-\frac{1}{2},j-1} = \binom{2l}{j} 2^{l-\frac{1}{2}} \, (l-\frac{1}{2})! \tag{4.49}$$

are obtained for $j = 0, \ldots, 2l$. We will endow the representation space $\mathcal{M}_l(\mathbb{R}^3)$ with the basis obtained by normalising $\widetilde{P}^l_j$.

To obtain the matrix coefficients associated to (4.47), we calculate the L-action on these orthogonal polynomials $\widetilde{P}^l_j$. Using (4.45) we obtain for each $j = 0, \ldots, 2l$

$$\begin{aligned} \mathrm{L}(q)\widetilde{P}^l_j(\underline{x}) =& q\, \widetilde{P}^l_j(\overline{q}\underline{x}q) \\ =& \big[q_1 P_j^{l-\frac{1}{2}}(\overline{q}\underline{x}q) + q_2 P_{j-1}^{l-\frac{1}{2}}(\overline{q}\underline{x}q) + \big(\overline{q}_2 P_j^{l-\frac{1}{2}}(\overline{q}\underline{x}q) - \overline{q}_1 P_{j-1}^{l-\frac{1}{2}}(\overline{q}\underline{x}q)\big)\mathrm{e}_{13}\big]\mathcal{I}_+. \end{aligned} \tag{4.50}$$

Using (4.37) we see that the maximal exponents of q_1 and q_2 are q_1^{2l-j-k} and q_2^{j-i+k}. Therefore, we can write (4.50) using the renormalised basis (4.47) in the form

$$\begin{aligned} \frac{1}{\sqrt{d_{l,j}}}\,\mathrm{L}(q)\widetilde{P}^l_j(\underline{x}) = \sum_{i=0}^{2l} \sum_{k=\max(0,i-j)}^{\min(i,2l-j)} & t^k_{i,j,2l}\; q_1^{2l-j-k}\, \overline{q}_1^{\,i-k} q_2^{j-i+k}(-\overline{q}_2)^k \\ & \times \frac{1}{\sqrt{d_{l,i}}}\Big(P_i^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3) - P_{i-1}^{l-\frac{1}{2}}(z_1,\overline{z}_1,x_3)\mathrm{e}_{13}\Big)\mathcal{I}_+, \end{aligned} \tag{4.51}$$

for some still to be determined coefficients $t^k_{i,j,2l}$. Comparing (4.51) with (4.10), we must have the identifications $m \mapsto 2l$, $w_1 \mapsto q_1$, and $w_2 \mapsto q_2$ between the matrix coefficients (4.10) and (4.51). Therefore, the matrix coefficients related to the basis of monogenic polynomials $\widetilde{P}^l_j$ associated with the weight $l \in \mathbb{N}_0 + \frac{1}{2}$ are given by

$$(\xi_l^{\mathrm{Spin}(3)}(q_1,q_2))_{i,j} = \sum_{k=\max(0,i-j)}^{\min(i,2l-j)} C^k_{i,j,2l}\, q_1^{2l-j-k}\, \overline{q}_1^{\,i-k} q_2^{j-i+k}(-\overline{q}_2)^k, \tag{4.52}$$

for $i,j = 0,\ldots,2l$ and with $C^k_{i,j,2l}$ given by (4.11) for $m = 2l$. Thus, $t^k_{i,j,2l} = C^k_{i,j,2l}$ and consequently, the relationship between the spinor-valued representations and the representations constructed in Section 4.1 is

$$\xi_l^{\mathrm{Spin}(3)}(q_1,q_2) = \boldsymbol{\xi}_{2l}(q_1,q_2). \tag{4.53}$$

Next we give two examples. For $l = \frac{1}{2}$, the representation space $\mathcal{M}_{\frac{1}{2}}(\mathbb{R}^3)$ has dimension 2 and an orthonormal basis is given by $\{\mathcal{I}_+, -\mathrm{e}_{13}\mathcal{I}_+\}$. The matrix coefficients associated to this basis are

$$\xi_{\frac{1}{2}}^{\mathrm{Spin}(3)}(q_1,q_2) = \begin{pmatrix} q_1 & q_2 \\ -\overline{q}_2 & \overline{q}_1 \end{pmatrix}. \tag{4.54}$$

For $l = \frac{3}{2}$, the module $\mathcal{M}_{\frac{3}{2}}(\mathbb{R}^3)$ has dimension 4 and an orthonormal basis is given by

$$\{\frac{1}{\sqrt{2}}\overline{z}_1\mathcal{I}_+,\ \frac{1}{\sqrt{6}}(-2x_3 - \overline{z}_1\mathrm{e}_{13})\mathcal{I}_+,\ \frac{1}{\sqrt{6}}(-z_1 + 2x_3\mathrm{e}_{13})\mathcal{I}_+,\ \frac{1}{\sqrt{2}}z_1\mathrm{e}_{13}\mathcal{I}_+\}. \tag{4.55}$$

The associated matrix coefficients are

$$\xi^{\mathrm{Spin}(3)}_{\frac{3}{2}}(q_1,q_2) = \begin{pmatrix} q_1^3 & \sqrt{3}\,q_1^2q_2 & \sqrt{3}\,q_1q_2^2 & q_2^3 \\ -\sqrt{3}\,q_1^2\overline{q}_2 & q_1(|q_1|^2 - 2|q_2|^2) & q_2(2|q_1|^2 - |q_2|^2) & \sqrt{3}\,\overline{q}_1q_2^2 \\ \sqrt{3}\,q_1\overline{q}_2^2 & -\overline{q}_2(2|q_1|^2 - |q_2|^2) & \overline{q}_1(|q_1|^2 - 2|q_2|^2) & \sqrt{3}\,\overline{q}_1^2q_2 \\ -\overline{q}_2^3 & \sqrt{3}\,\overline{q}_1\overline{q}_2^2 & -\sqrt{3}\,\overline{q}_1^2\overline{q}_2 & \overline{q}_1^3 \end{pmatrix}. \tag{4.56}$$

5 Spin(4) representations

5.1 Prerequisites

We start by recalling the notation from Section 3. We use an orthonormal basis $\{\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_4\}$ of $\mathbb{R}^4$ and denote by $\mathbb{R}_{0,4}$ the real 2^4-dimensional Clifford algebra over $\mathbb{R}^4$ generated by the relations $\mathrm{e}_i^2 = -1$, $i = 1,\dots,4$ and $\mathrm{e}_i\mathrm{e}_j = -\mathrm{e}_j\mathrm{e}_i$, $i \neq j$. The group Spin(4) consists of even products of unit vectors

$$\mathrm{Spin}(4) = \{\prod_{j=1}^{2k} s_j : s_j \in \mathbb{R}^4,\ |s_j| = 1,\ k \in \mathbb{N}\} \subset \mathbb{R}^+_{0,4}. \tag{5.1}$$

Later on we will make use of the isomorphism $\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3)$ following from the identifications $\mathbb{R}^+_{0,4} \cong \mathbb{R}_{0,3} \cong \mathbb{H} \oplus \mathbb{H}$, which we recall next. Due to [5, Section 0.5.4] we can write $a \in \mathbb{R}^+_{0,4}$ in the form $a = \omega_+ a_+ + \omega_- a_-$ with $a_\pm \in \mathbb{R}^+_{0,3} = \mathrm{span}\{1,\mathrm{e}_{12},\mathrm{e}_{13},\mathrm{e}_{23}\} \cong \mathbb{H}$ and $\omega_\pm = \frac{1}{2}(1 \pm \mathrm{e}_{1234})$. These elements $\omega_\pm$ are mutually annihilating idempotents satisfying

$$\omega_\pm^2 = \omega_\pm, \qquad \omega_+\omega_- = \omega_-\omega_+ = 0, \qquad \omega_+ + \omega_- = 1, \qquad \overline{\omega}_\pm = \omega_\pm. \tag{5.2}$$

As $\omega_\pm$ commute with a_+ and a_- we can also write $a = a_+\omega_+ + a_-\omega_-$. Applying this decomposition to elements $\mathbf{s} \in \mathrm{Spin}(4)$ we obtain

$$\mathbf{s} = s\omega_+ + q\omega_- \qquad \text{with} \quad q, s \in \mathrm{Spin}(3) \tag{5.3}$$

together with

$$\mathbf{s}\mathbf{s}' = ss'\omega_+ + qq'\omega_- \qquad \text{for} \quad \mathbf{s}' = s'\omega_+ + q'\omega_- \quad \text{with} \quad q', s' \in \mathrm{Spin}(3) \tag{5.4}$$

making the isomorphism explicit. For an arbitrary element $\underline{x} \in \mathbb{R}^4$ putting $\underline{x} = \sum_{i=1}^4 x_i\mathrm{e}_i$ and observing that for each $i = 1,\dots,4$ it holds $\omega_+\mathrm{e}_i = \mathrm{e}_i\omega_-$ and

$\omega_- e_i = e_i \omega_+$ we obtain

$$\begin{aligned} \bar{\mathbf{s}}\,\underline{x}\,\mathbf{s} &= (\bar{s}\omega_+ + \bar{q}\omega_-)\,\underline{x}\,(s\omega_+ + q\omega_-) \\ &= \bar{s}\underline{x}s\omega_-\omega_+ + \bar{s}\underline{x}q\omega_-^2 + \bar{q}\underline{x}s\omega_+^2 + \bar{q}\underline{x}q\omega_+\omega_- \\ &= \bar{s}\underline{x}q\,\omega_- + \bar{q}\underline{x}s\,\omega_+, \end{aligned} \tag{5.5}$$

which describes a rotation in $\mathbb{R}^4$ induced by $(q,s) \in \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. Due to the isomorphism $\mathrm{Spin}(3) \cong \mathbb{S}^3$ and the identification of $\mathbb{R}^4$ with $\mathbb{C}^2$ we can write the action $\bar{\mathbf{s}}\,\underline{x}\,\mathbf{s}$ inside $\mathbb{C}^2$. Putting $q = (q_1, q_2), s = (s_1, s_2) \in \mathbb{S}^3$, and $\underline{x} = (z_1, z_2) \in \mathbb{C}^2$ yields after a lengthy calculation

$$\bar{\mathbf{s}}\,\underline{x}\,\mathbf{s} = \big(\bar{q}_1(\bar{s}_1 z_1 + s_2 z_2) + q_2(\bar{s}_1\bar{z}_2 - s_2\bar{z}_1)\big) + \big(\bar{q}_1(s_1 z_2 - \bar{s}_2 z_1) - q_2(s_1\bar{z}_1 + \bar{s}_2\bar{z}_2)\big)\mathbf{j} \tag{5.6}$$

identifying $\mathbb{C}^2$ with $\mathbb{H}$. Next, we describe the Spin(4) action on the spinor spaces $\mathcal{S}_4^{\pm}$. A realisation of $\mathcal{S}_4^+$ was already described in Section 4.3. Considering $\mathbf{s} = s\omega_+ + q\omega_-$ with $q, s \in \mathrm{Spin}(3)$ we obtain for $\mathcal{S}_4^+$ the half-spin action

$$(s\omega_+ + q\omega_-)(z^+ - z^- e_{13})\mathcal{I}_+ = q(z^+ - z^- e_{13})\mathcal{I}_+ \tag{5.7}$$

as $\omega_+\mathcal{I}_+ = 0$, $\omega_-\mathcal{I}_+ = \mathcal{I}_+$, and ω_+, ω_- commute with e_{13}. Thus on $\mathcal{S}_4^+$ the action corresponds to the left multiplication by $q \in \mathrm{Spin}(3) \cong \mathbb{S}^3$ and hence by (4.45) also to the matrix multiplication

$$\begin{pmatrix} q_1 & q_2 \\ -\bar{q}_2 & \bar{q}_1 \end{pmatrix} \begin{pmatrix} z^+ \\ z^- \end{pmatrix}. \tag{5.8}$$

For $\mathcal{S}_4^-$, we get the half-spin action

$$(s\omega_+ + q\omega_-)(z^+ - z^- e_{13})\mathcal{I}_- = s(z^+ - z^- e_{13})\mathcal{I}_-. \tag{5.9}$$

Thus, this action corresponds to the left multiplication by $s \in \mathrm{Spin}(3) \cong \mathbb{S}^3$ and as in (4.45) we see that it corresponds to the matrix-multiplication

$$\begin{pmatrix} s_1 & s_2 \\ -\bar{s}_2 & \bar{s}_1 \end{pmatrix} \begin{pmatrix} z^+ \\ z^- \end{pmatrix}. \tag{5.10}$$

5.2 H-representations: spherical harmonics in $\mathbb{R}^4$

Following Section 3, we construct explicitly the representations associated to the weights (l_1, l_2), with $l_1, l_2 \in \mathbb{N}_0$, and $0 \le l_2 \le l_1$. Considering $\underline{x} = \sum_{i=1}^4 x_i e_i$, $\underline{y} = \sum_{i=1}^4 y_i e_i$ by (3.18) the (normalised) weight vector is given by

$$\omega_{(l_1,l_2)}(\underline{x}, \underline{y}) = \langle \underline{x}, 2T_1 \rangle^{l_1 - l_2} \langle \underline{x} \wedge \underline{y}, 4T_1 \wedge T_2 \rangle^{l_2} = \bar{z}_1^{\,l_1 - l_2} \, (\bar{z}_1 \overline{w}_2 - \overline{w}_1 \bar{z}_2)^{l_2}, \tag{5.11}$$

where $z_1 = x_1 + \mathbf{i}x_2$, $z_2 = x_3 + \mathbf{i}x_4$, $w_1 = y_1 + \mathbf{i}y_2$ and $w_2 = y_3 + \mathbf{i}y_4$. Representing an element $\mathbf{s} = s\omega_+ + q\omega_- \in \mathrm{Spin}(4)$ by two unit quaternions $q = (q_1, q_2)$, $s = (s_1, s_2) \in \mathbb{S}^3$ and using complex coordinates for $\underline{x} = (z_1, z_2)$, $\underline{y} = (w_1, w_2) \in \mathbb{C}^2$ equation (5.6) yields

$$\langle \overline{\mathbf{s}}\,\underline{x}\,\mathbf{s}, 2T_1\rangle = q_1 s_1 \overline{z}_1 + q_1 \overline{s}_2 \overline{z}_2 + \overline{q}_2 s_1 z_2 - \overline{q}_2 \overline{s}_2 z_1 = q_1 \underbrace{(s_1 \overline{z}_1 + \overline{s}_2 \overline{z}_2)}_{Q_1} - \overline{q}_2 \underbrace{(\overline{s}_2 z_1 - s_1 z_2)}_{Q_2} \tag{5.12}$$

and

$$\begin{aligned}\langle \overline{\mathbf{s}}\,\underline{x} \wedge \underline{y}\,\mathbf{s}, 4\,T_1 \wedge T_2\rangle &= \langle (\overline{\mathbf{s}}\,\underline{x}\,\mathbf{s}) \wedge (\overline{\mathbf{s}}\,\underline{y}\,\mathbf{s}), 4\,T_1 \wedge T_2\rangle \\ &= q_1^2 \underbrace{(\overline{z}_1 \overline{w}_2 - \overline{z}_2 \overline{w}_1)}_{Q_3} - q_1 \overline{q}_2 \underbrace{(\overline{z}_1 w_1 - z_1 \overline{w}_1 + \overline{z}_2 w_2 - z_2 \overline{w}_2)}_{Q_4} + \overline{q}_2^2 \underbrace{(z_1 w_2 - z_2 w_1)}_{Q_5}.\end{aligned} \tag{5.13}$$

Thus, we obtain for the H-action on the weight vector $\omega_{(l_1,l_2)}(\underline{x}, \underline{y})$ and by using shorthand notations Q_j for the terms marked above

$$\begin{aligned}\mathrm{H}(\mathbf{s})\omega_{(l_1,l_2)}(\underline{x}, \underline{y}) &= \langle \overline{\mathbf{s}}\,\underline{x}\,\mathbf{s}, 2T_1\rangle^{l_1-l_2} \langle \overline{\mathbf{s}}\,\underline{x} \wedge \underline{y}\,\mathbf{s}, 4\,T_1 \wedge T_2\rangle^{l_2} \\ &= \sum_{i=0}^{l_1-l_2} \binom{l_1-l_2}{i} q_1^{l_1-l_2-i} (-\overline{q}_2)^i Q_1^{l_1-l_2-i} Q_2^i \sum_{j=0}^{2l_2} q_1^{2l_2-j} (-\overline{q}_2)^j \sum_{\substack{|\alpha|=l_2 \\ \alpha_2+2\alpha_3=j}} \frac{l_2!}{\alpha!} Q_3^{\alpha_1} Q_4^{\alpha_2} Q_5^{\alpha_3} \\ &= \sum_{i=0}^{l_1-l_2} \sum_{j=0}^{2l_2} \binom{l_1-l_2}{i} q_1^{l_1+l_2-i-j} (-\overline{q}_2)^{i+j} Q_1^{l_1-l_2-i} Q_2^i \sum_{\substack{|\alpha|=l_2 \\ \alpha_2+2\alpha_3=j}} \frac{l_2!}{\alpha!} Q_3^{\alpha_1} Q_4^{\alpha_2} Q_5^{\alpha_3}.\end{aligned} \tag{5.14}$$

Collecting the powers of q this can be written as

$$\mathrm{H}(\mathbf{s})\omega_{(l_1,l_2)}(\underline{x}, \underline{y}) = \sum_{k=0}^{l_1+l_2} q_1^{l_1+l_2-k} (-\overline{q}_2)^k \sum_{i+j=k} \binom{l_1-l_2}{i} Q_1^{l_1-l_2-i} Q_2^i \sum_{\substack{|\alpha|=l_2 \\ \alpha_2+2\alpha_3=j}} \frac{l_2!}{\alpha!} Q_3^{\alpha_1} Q_4^{\alpha_2} Q_5^{\alpha_3}. \tag{5.15}$$

The terms Q_3, Q_4 and Q_5 are independent of the spinor $\mathbf{s} = s\omega_+ + q\omega_-$. For Q_1 and Q_2 we extract the dependence on s and obtain (with $l = l_1 - l_2$)

$$\begin{aligned}Q_1^{l-i} Q_2^i &= (s_1 \overline{z}_1 + \overline{s}_2 \overline{z}_2)^{l-i} (\overline{s}_2 z_1 - s_1 z_2)^i \\ &= \sum_{m=0}^{l} s_1^{l-m} (-\overline{s}_2)^m \sum_{n=\max(0,m+i-l)}^{\min(i,m)} (-1)^i \binom{l-i}{m-n} \binom{i}{n} \overline{z}_1^{l-i-m+n} z_1^n z_2^{i-n} (-\overline{z}_2)^{m-n}.\end{aligned} \tag{5.16}$$

Thus, defining the following polynomials

$$S^l_{m,i}(\underline{z}) = (-1)^i \binom{l}{i} \sum_{n=\max(0,m+i-l)}^{\min(i,m)} \binom{l-i}{m-n}\binom{i}{n} \overline{z}_1^{\,l-i-m+n} z_1^n z_2^{i-n} (-\overline{z}_2)^{m-n}, \tag{5.17}$$

$$Q^{l_2}_j(\underline{z}\wedge\underline{w}) = \sum_{\substack{|\alpha|=l_2\\ \alpha_2+2\alpha_3=j}} \frac{l_2!}{\alpha!} Q_3^{\alpha_1} Q_4^{\alpha_2} Q_5^{\alpha_3}, \tag{5.18}$$

where $\underline{z} = z_1T_1 - \overline{z}_1T_1^\dagger + z_2T_2 - \overline{z}_2T_2^\dagger$ and $\underline{w} = w_1T_1 - \overline{w}_1T_1^\dagger + w_2T_2 - \overline{w}_2T_2^\dagger$ are the Hermitian variables constructed from $\underline{x}$ and $\underline{y}$, we can write (5.15) as

$$\mathrm{H}(\mathbf{s})\omega_{(l_1,l_2)}(\underline{x},\underline{y}) = \sum_{k=0}^{l_1+l_2} q_1^{l_1+l_2-k}(-\overline{q}_2)^k \sum_{m=0}^{l_1-l_2} s_1^{l_1-l_2-m}(-\overline{s}_2)^m \sum_{i+j=k} S^{l_1-l_2}_{m,i}(\underline{z})\, Q^{l_2}_j(\underline{z}\wedge\underline{w}). \tag{5.19}$$

From (5.19) the tensor product structure of the Spin(4)-representations of weight (l_1, l_2), $l_i \in \mathbb{N}_0$, $l_1 \geq l_2$ can be seen.

Lemma 5.1. *Let $l_1, l_2 \in \mathbb{N}_0$ with $l_1 \geq l_2$. Then the polynomials*

$$\sum_{i+j=k} S^{l_1-l_2}_{m,i}(\underline{z})\, Q^{l_2}_j(\underline{z}\wedge\underline{w}), \tag{5.20}$$

for $k = 0, \ldots, l_1 + l_2$ and $m = 0, \ldots, l_1 - l_2$, form an orthogonal basis of $\mathcal{H}_{(l_1,l_2)}(\mathbb{R}^4)$. In particular the dimension of the (l_1, l_2)-representation is $(l_1 + l_2 + 1)(l_1 - l_2 + 1)$.

Proof. By construction, these polynomials generate the representation space $\mathcal{H}_{(l_1,l_2)}(\mathbb{R}^4)$. To obtain orthogonality with respect to the Fischer inner product we consider their degrees of homogeneity in certain sets of variables. First we distinguish holomorphic variables z_1, z_2, w_1, w_2 from their antiholomorphic counterparts $\overline{z}_1, \overline{z}_2, \overline{w}_1, \overline{w}_2$. Then due to (5.18) the polynomial $S^{l_1-l_2}_{m,i}(\underline{z})$ is of holomorphic degree i and of antiholomorphic degree $l_1 - l_2 - i$, while $Q^{l_2}_j(\underline{z}\wedge\underline{w})$ is of holomorphic degree j and antiholomorphic degree $2l_2 - j$. Thus, the polynomials (5.20) are holomorphic of degree k and antiholomorphic of degree $l_1 + l_2 - k$.

Next we consider a mixed form of degree, considering the variables $z_1, \overline{z}_2, w_1, \overline{w}_2$ the mixed degree of the polynomial $S^{l_1-l_2}_{m,i}(\underline{z})$ is of degree m and $Q^{l_2}_j(\underline{z}\wedge\underline{w})$ of mixed degree l_2 and thus the above polynomials (5.20) are of mixed degree $m + l_2$.

It follows that for given l_1 and l_2 and different k and m the polynomials (5.20) have different homogeneities and are thus Fischer orthogonal. □

Next, we construct the representations associated to the weights (l_1, l_2) with $l_1, l_2 \in \mathbb{Z}$ and $l_1 \geq -l_2 > 0$. By (3.21) the corresponding (normalised) weight vector is given by

$$\omega_{(l_1,l_2)}(\underline{x},\underline{y}) = \langle \underline{x}, 2T_1\rangle^{l_1+l_2} \langle \underline{x}\wedge\underline{y}, -4T_1\wedge T_2^\dagger\rangle^{-l_2} = \overline{z}_1^{\,l_1+l_2}\,(z_2\overline{w}_1 - \overline{z}_1 w_2)^{-l_2}. \quad (5.21)$$

Using again $q = (q_1, q_2), s = (s_1, s_2) \in \mathbb{S}^3$ to describe elements of the spin group and $\underline{x} = (z_1, z_2)$, $\underline{y} = (w_1, w_2) \in \mathbb{C}^2$ as complex coordinates, equation (5.6) yields again

$$\langle \overline{\mathbf{s}}\,\underline{x}\,\mathbf{s}, 2T_1\rangle = s_1\,\underbrace{(q_1\overline{z}_1 + \overline{q}_2 z_2)}_{\widetilde{Q}_1} - \overline{s}_2\,\underbrace{(\overline{q}_2 z_1 - q_1\overline{z}_2)}_{\widetilde{Q}_2} \quad (5.22)$$

and now

$$\begin{aligned}\langle \overline{\mathbf{s}}\,\underline{x}\wedge\underline{y}\,\mathbf{s}, -4T_1\wedge T_2^\dagger\rangle &= \langle(\overline{\mathbf{s}}\,\underline{x}\,\mathbf{s})\wedge(\overline{\mathbf{s}}\,\underline{y}\,\mathbf{s}), -4T_1\wedge T_2^\dagger\rangle \\ &= s_1^2\,\underbrace{(\overline{z}_1 w_2 - z_2\overline{w}_1)}_{\widetilde{Q}_3} - s_1\overline{s}_2\,\underbrace{(\overline{z}_1 w_1 - z_1\overline{w}_1 + z_2\overline{w}_2 - \overline{z}_2 w_2)}_{\widetilde{Q}_4} + \overline{s}_2^2\,\underbrace{(z_1\overline{w}_2 - \overline{z}_2 w_1)}_{\widetilde{Q}_5}.\end{aligned} \quad (5.23)$$

Using these short-hand notations we compute the H-action on the weight vector $\omega_{(l_1,l_2)}(\underline{x},\underline{y})$ following the lines of (5.14) as

$$\begin{aligned}\mathrm{H}(\mathbf{s})\omega_{(l_1,l_2)}(\underline{x},\underline{y}) &= \langle \overline{\mathbf{s}}\,\underline{x}\,\mathbf{s}, 2T_1\rangle^{l_1+l_2}\,\langle \overline{\mathbf{s}}\,\underline{x}\wedge\underline{y}\,\mathbf{s}, -4T_1\wedge T_2^\dagger\rangle^{-l_2} \\ &= \sum_{m=0}^{l_1-l_2} s_1^{l_1-l_2-m}\,(-\overline{s}_2)^m \sum_{i+j=m}\binom{l_1+l_2}{i}\widetilde{Q}_1^{l_1+l_2-i}\,\widetilde{Q}_2^i \sum_{\substack{|\alpha|=|l_2|\\ \alpha_2+2\alpha_3=j}} \frac{|l_2|!}{\alpha!}\widetilde{Q}_3^{\alpha_1}\widetilde{Q}_4^{\alpha_2}\widetilde{Q}_5^{\alpha_3} \\ &= \sum_{k=0}^{l_1+l_2} q_1^{l_1+l_2-k}\,(-\overline{q}_2)^k \sum_{m=0}^{l_1-l_2} s_1^{l_1-l_2-m}(-\overline{s}_2)^m \sum_{i+j=m} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\,\widetilde{Q}_j^{l_2}(\underline{z}\wedge\underline{w})\end{aligned} \quad (5.24)$$

with

$$\widetilde{S}_{k,i}^l(\underline{z}) = (-1)^i\binom{l}{i}\sum_{n=\max(0,k+i-l)}^{\min(i,k)}\binom{l-i}{k-n}\binom{i}{n}\overline{z}_1^{\,l-i-k+n} z_1^n \overline{z}_2^{\,i-n}\,(-z_2)^{k-n}, \quad (5.25)$$

and obtained from $S_{k,i}^l(\underline{z})$ by complex conjugating z_2 and

$$\widetilde{Q}_j^{l_2}(\underline{z}\wedge\underline{w}) = \sum_{\substack{|\alpha|=|l_2|\\ \alpha_2+2\alpha_3=j}} \frac{|l_2|!}{\alpha!}\,\widetilde{Q}_3^{\alpha_1}\widetilde{Q}_4^{\alpha_2}\widetilde{Q}_5^{\alpha_3} \quad (5.26)$$

obtained from $Q_j^{|l_2|}(\underline{z}\wedge\underline{w})$ by complex conjugating z_2 and w_2.

Lemma 5.2. *Let $l_1, l_2 \in \mathbb{Z}$ with $l_1 \geq -l_2 > 0$. Then the polynomials*

$$\sum_{i+j=m} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\, \widetilde{Q}_j^{l_2}(\underline{z} \wedge \underline{w}), \tag{5.27}$$

for $k = 0, \ldots, l_1 + l_2$ and $m = 0, \ldots, l_1 - l_2$, form an orthogonal basis of $\mathcal{H}_{(l_1,l_2)}(\mathbb{R}^4)$.

Proof. The proof follows again by considering holomorphic, antiholomorphic and mixed degrees of the polynomials. □

5.3 L-representations: spinor-valued monogenics in $\mathbb{R}^4$

In this section we construct the representations associated to the weights (l_1, l_2) with $l_1, l_2 \in \mathbb{Z} + \frac{1}{2}$, and $l_1 \geq |l_2|$. First we consider the case $l_2 \geq 0$ and the corresponding L-representation on the space of $\mathcal{S}_4^+$-valued simplicial monogenic polynomials. From (5.7), (5.8), and (5.19) we obtain

$$\begin{aligned}
&\mathrm{L}(\mathbf{s})\omega_{(l_1,l_2)}(\underline{x}, \underline{y}) = q\, \langle \overline{\mathbf{s}}\, \underline{x}\, \mathbf{s}, 2T_1 \rangle^{l_1-l_2} \langle \overline{\mathbf{s}}\, \underline{x} \wedge \underline{y}\, \mathbf{s}, 4\, T_1 \wedge T_2 \rangle^{l_2-\frac{1}{2}}\, \mathcal{I}_+ \\
&= \Big[q_1 \sum_{k=0}^{l_1+l_2-1} q_1^{l_1+l_2-k-1} (-\overline{q}_2)^k \sum_{m=0}^{l_1-l_2} s_1^{l_1-l_2-m} (-\overline{s}_2)^m \sum_{i+j=k} S_{m,i}^{l_1-l_2}(\underline{z})\, Q_j^{l_2-\frac{1}{2}}(\underline{z} \wedge \underline{w}) \\
&+ \overline{q}_2 \sum_{k=0}^{l_1+l_2-1} q_1^{l_1+l_2-k-1} (-\overline{q}_2)^k \sum_{m=0}^{l_1-l_2} s_1^{l_1-l_2-m} (-\overline{s}_2)^m \sum_{i+j=k} S_{m,i}^{l_1-l_2}(\underline{z})\, Q_j^{l_2-\frac{1}{2}}(\underline{z} \wedge \underline{w}) \mathrm{e}_{13} \Big] \mathcal{I}_+ \\
&= \sum_{k=0}^{l_1+l_2} q_1^{l_1+l_2-k} (-\overline{q}_2)^k \sum_{m=0}^{l_1-l_2} s_1^{l_1-l_2-m} (-\overline{s}_2)^m \\
&\quad \times \Big[\sum_{i+j=k} S_{m,i}^{l_1-l_2}(\underline{z})\, Q_j^{l_2-\frac{1}{2}}(\underline{z} \wedge \underline{w}) - \sum_{i+j=k-1} S_{m,i}^{l_1-l_2}(\underline{z})\, Q_j^{l_2-\frac{1}{2}}(\underline{z} \wedge \underline{w}) \mathrm{e}_{13} \Big] \mathcal{I}_+ .
\end{aligned} \tag{5.28}$$

where we used $\sum_{i+j=-1} S_{m,i}^{l_1-l_2}(\underline{z})\, Q_j^{l_2-\frac{1}{2}}(\underline{z} \wedge \underline{w}) = \sum_{i+j=l_1+l_2} S_{m,i}^{l_1-l_2}(\underline{z})\, Q_j^{l_2-\frac{1}{2}}(\underline{z} \wedge \underline{w}) = 0$. Due to the orthogonality between our basis functions of $\mathcal{H}_{(l_1,l_2)}(\mathbb{R}^4)$ we conclude:

Corollary 5.3. *Let $l_1, l_2 \in \mathbb{N}_0 + \frac{1}{2}$ with $l_1 \geq l_2$. Then the monogenic polynomials*

$$\Big[\sum_{i+j=k} S_{m,i}^{l_1-l_2}(\underline{z})\, Q_j^{l_2-\frac{1}{2}}(\underline{z} \wedge \underline{w}) - \sum_{i+j=k-1} S_{m,i}^{l_1-l_2}(\underline{z})\, Q_j^{l_2-\frac{1}{2}}(\underline{z} \wedge \underline{w}) \mathrm{e}_{13} \Big] \mathcal{I}_+ \tag{5.29}$$

for $k = 0, \ldots, l_1 + l_2$ and $m = 0, \ldots, l_1 - l_2$ form an orthogonal basis of $\mathcal{M}_{(l_1,l_2)}(\mathbb{R}^4)$.

The calculations for the representations of weight $\big(l_1, l_2\big)$ with $l_1, l_2 \in \mathbb{Z} + \frac{1}{2}$ and $l_1 \geq -l_2 > 0$ on the space of $\mathcal{S}_4^-$-valued polynomials are similar. Indeed,

$$
\begin{aligned}
&\mathrm{L}(\mathbf{s})\omega_{(l_1,l_2)}(\underline{x},\underline{y}) = s\, \langle \overline{\mathbf{s}}\, \underline{x}\, \mathbf{s}, 2T_1\rangle^{l_1+l_2} \, \langle \overline{\mathbf{s}}\, \underline{x} \wedge \underline{y}\, \mathbf{s}, -4\, T_1 \wedge T_2^\dagger\rangle^{-l_2-\frac{1}{2}}\, \mathcal{I}_- \\
&= \Big[s_1 \sum_{k=0}^{l_1+l_2} q_1^{l_1+l_2-k}\, (-\overline{q}_2)^k \sum_{m=0}^{l_1-l_2-1} s_1^{l_1-l_2-m-1} (-\overline{s}_2)^m \sum_{i+j=m} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\, \widetilde{Q}_j^{l_2+\frac{1}{2}}(\underline{z}\wedge\underline{w}) \\
&+ \overline{s}_2 \sum_{k=0}^{l_1+l_2} q_1^{l_1+l_2-k} (-\overline{q}_2)^k \sum_{m=0}^{l_1-l_2-1} s_1^{l_1-l_2-m-1} (-\overline{s}_2)^m \sum_{i+j=m} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\, \widetilde{Q}_j^{l_2+\frac{1}{2}}(\underline{z}\wedge\underline{w})\mathrm{e}_{13} \Big] \mathcal{I}_- \\
&= \sum_{k=0}^{l_1+l_2} q_1^{l_1+l_2-k}\, (-\overline{q}_2)^k \sum_{m=0}^{l_1-l_2} s_1^{l_1-l_2-m} (-\overline{s}_2)^m \\
&\quad \times \Big[\sum_{i+j=m} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\, \widetilde{Q}_j^{l_2+\frac{1}{2}}(\underline{z}\wedge\underline{w}) - \sum_{i+j=m-1} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\, \widetilde{Q}_j^{l_2+\frac{1}{2}}(\underline{z}\wedge\underline{w})\mathrm{e}_{13} \Big] \mathcal{I}_-
\end{aligned} \tag{5.30}
$$

where we used $\sum_{i+j=-1} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\, \widetilde{Q}_j^{l_2+\frac{1}{2}}(\underline{z}\wedge\underline{w}) = \sum_{i+j=l_1-l_2} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\, \widetilde{Q}_j^{l_2+\frac{1}{2}}(\underline{z}\wedge\underline{w}) = 0$. Again we conclude:

Corollary 5.4. *Let $l_1, l_2 \in \mathbb{Z} + \frac{1}{2}$ with $l_1 \geq -l_2 > 0$. Then the monogenic polynomials*

$$
\Big[\sum_{i+j=m} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\, \widetilde{Q}_j^{l_2+\frac{1}{2}}(\underline{z}\wedge\underline{w}) - \sum_{i+j=m-1} \widetilde{S}_{k,i}^{l_1+l_2}(\underline{z})\, \widetilde{Q}_j^{l_2+\frac{1}{2}}(\underline{z}\wedge\underline{w})\mathrm{e}_{13} \Big] \mathcal{I}_- \tag{5.31}
$$

for $k = 0, \ldots, l_1 + l_2$ and $m = 0, \ldots, l_1 - l_2$ form an orthogonal basis of $\mathcal{M}_{(l_1,l_2)}(\mathbb{R}^4)$.

5.4 Matrix coefficients

To describe all Spin(4) representations $\xi_{(l_1,l_2)}$, we consider the lattice

$$
\Gamma_{\mathrm{Spin}(4)} = \{(l_1, l_2) \in \mathbb{Z}^2 \cup (\mathbb{Z} + \frac{1}{2})^2 \mid l_1 \geq |l_2|\} \tag{5.32}
$$

consisting of pairs of integers and pairs of half-integers. In Sections 5.2 and 5.3 we constructed orthogonal bases for the representation spaces of the Spin(4) representations $\xi_{(l_1,l_2)}$ for all $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$. This choice of bases allows to make direct use of the tensor product structure of these representations. By construction, we have

$$
\xi_{(l_1,l_2)}(\mathbf{s}) = \xi_{(l_1,l_2)}(q, s) = \xi^{\mathrm{Spin}(3)}_{\frac{l_1+l_2}{2}}(q) \otimes \xi^{\mathrm{Spin}(3)}_{\frac{l_1-l_2}{2}}(s) = \boldsymbol{\xi}_{l_1+l_2}(q) \otimes \boldsymbol{\xi}_{l_1-l_2}(s), \tag{5.33}
$$

where $\mathbf{s} = s\omega_+ + q\omega_-$ and $\otimes$ denotes the Kronecker product of the matrices. Each $\xi_{(l_1,l_2)}(\mathbf{s})$ is a unitary matrix of dimension

$$
d_{(l_1,l_2)} = d_{\frac{l_1+l_2}{2}}\, d_{\frac{l_1-l_2}{2}} = (l_1 + l_2 + 1)(l_1 - l_2 + 1). \tag{5.34}
$$

Figure 1 shows the infinite triangle of Spin(4) representations acting on harmonic modules $\mathcal{H}$ and monogenic modules $\mathcal{M}$. At the borders of the infinite triangle, we can observe the Spin(3) representations $\xi^{\mathrm{Spin}(3)}_{\frac{l_1+l_2}{2}} \otimes 1$ in red and the Spin(3) representations $1 \otimes \xi^{\mathrm{Spin}(3)}_{\frac{l_1-l_2}{2}}$ in blue.

$\mathcal{H}_{(0,0)}$	$\mathcal{M}_{(\frac{1}{2},l_2)}$	$\mathcal{H}_{(1,l_2)}$	$\mathcal{M}_{(\frac{3}{2},l_2)}$	$\mathcal{H}_{(2,l_2)}$	$\mathcal{M}_{(\frac{5}{2},l_2)}$	$\mathcal{H}_{(3,l_2)}$	$\cdots$
						$(3,3)$	$\cdots$
					$\left(\frac{5}{2},\frac{5}{2}\right)$		
				$(2,2)$		$(3,2)$	$\cdots$
			$\left(\frac{3}{2},\frac{3}{2}\right)$		$\left(\frac{5}{2},\frac{3}{2}\right)$		
		$(1,1)$		$(2,1)$		$(3,1)$	$\cdots$
	$\left(\frac{1}{2},\frac{1}{2}\right)$		$\left(\frac{3}{2},\frac{1}{2}\right)$		$\left(\frac{5}{2},\frac{1}{2}\right)$		
$(0,0)$		$(1,0)$		$(2,0)$		$(3,0)$	$\cdots$
	$\left(\frac{1}{2},-\frac{1}{2}\right)$		$\left(\frac{3}{2},-\frac{1}{2}\right)$		$\left(\frac{5}{2},-\frac{1}{2}\right)$		
		$(1,-1)$		$(2,-1)$		$(3,-1)$	$\cdots$
			$\left(\frac{3}{2},-\frac{3}{2}\right)$		$\left(\frac{5}{2},-\frac{3}{2}\right)$		
				$(2,-2)$		$(3,-2)$	$\cdots$
					$\left(\frac{5}{2},-\frac{5}{2}\right)$		
						$(3,-3)$	$\cdots$

Fig. 1 Harmonic and monogenic modules of Spin(4) representations $\xi_{(l_1,l_2)}$.

5.5 Recurrence relations

In the following two sections we discuss the general structure of matrix coefficients of the representations we have constructed. First, we prove recurrence relations for the matrix coefficients and then we define differential operators corresponding (up to factors) to shifts in the matrix coefficients of a given representation.

Definition 5.5. For all $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$ we define the following matrices:

$$\begin{aligned}
A_\pm(l_1,l_2) &= a_\pm(l_1+l_2) \otimes \mathrm{I}_{l_1-l_2+1},\\
B_\pm(l_1,l_2) &= b_\pm(l_1+l_2) \otimes \mathrm{I}_{l_1-l_2+1},\\
C_\pm(l_1,l_2) &= \mathrm{I}_{l_1+l_2+1} \otimes a_\pm(l_1-l_2),\\
D_\pm(l_1,l_2) &= \mathrm{I}_{l_1+l_2+1} \otimes b_\pm(l_1-l_2),
\end{aligned}$$

where $a_\pm(l_1 \pm l_2)$ and $b_\pm(l_1 \pm l_2)$ are the matrices given in Definition 4.2.

Using the matrices just defined we can transfer the recurrence relations given in Theorem 4.3 to the Spin(4) case. To simplify the expressions we use $A_\pm$ instead of

$A_\pm(l_1, l_2)$ and the same for the other matrices. The notation introduced will also be used later on when dealing with difference operators and the group Fourier transform.
Theorem 5.6. *For all* $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$ *the three term recurrence relations*

$$
\begin{aligned}
q_1\, \xi_{(l_1,l_2)}(q,s) \;&=\; A_-\, \xi_{(l_1^+,l_2^+)}(q,s)\, A_-^\top \;+\; B_-\, \xi_{(l_1^-,l_2^-)}(q,s)\, B_-^\top, && (5.35a)\\
q_2\, \xi_{(l_1,l_2)}(q,s) \;&=\; A_-\, \xi_{(l_1^+,l_2^+)}(q,s)\, A_+^\top \;-\; B_-\, \xi_{(l_1^-,l_2^-)}(q,s)\, B_+^\top, && (5.35b)\\
-\overline{q}_2\, \xi_{(l_1,l_2)}(q,s) \;&=\; A_+\, \xi_{(l_1^+,l_2^+)}(q,s)\, A_-^\top \;-\; B_+\, \xi_{(l_1^-,l_2^-)}(q,s)\, B_-^\top, && (5.35c)\\
\overline{q}_1\, \xi_{(l_1,l_2)}(q,s) \;&=\; A_+\, \xi_{(l_1^+,l_2^+)}(q,s)\, A_+^\top \;+\; B_+\, \xi_{(l_1^-,l_2^-)}(q,s)\, B_+^\top, && (5.35d)\\
s_1\, \xi_{(l_1,l_2)}(q,s) \;&=\; C_-\, \xi_{(l_1^+,l_2^-)}(q,s)\, C_-^\top \;+\; D_-\, \xi_{(l_1^-,l_2^+)}(q,s)\, D_-^\top, && (5.35e)\\
s_2\, \xi_{(l_1,l_2)}(q,s) \;&=\; C_-\, \xi_{(l_1^+,l_2^-)}(q,s)\, C_+^\top \;-\; D_-\, \xi_{(l_1^-,l_2^+)}(q,s)\, D_+^\top, && (5.35f)\\
-\overline{s}_2\, \xi_{(l_1,l_2)}(q,s) \;&=\; C_+\, \xi_{(l_1^+,l_2^-)}(q,s)\, C_-^\top \;-\; D_+\, \xi_{(l_1^-,l_2^+)}(q,s)\, D_-^\top, && (5.35g)\\
\overline{s}_1\, \xi_{(l_1,l_2)}(q,s) \;&=\; C_+\, \xi_{(l_1^+,l_2^-)}(q,s)\, C_+^\top \;+\; D_+\, \xi_{(l_1^-,l_2^+)}(q,s)\, D_+^\top && (5.35h)
\end{aligned}
$$

hold true, where we used the notation

$$
l_j^+ = l_j + \frac{1}{2} \qquad \text{and} \qquad l_j^- = l_j - \frac{1}{2}, \qquad j = 1, 2. \tag{5.36}
$$

for the neighbouring weights in the weight lattice.

Proof. Theorem 4.3 together with the mixed product property $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ of the Kronecker product yields

$$
\begin{aligned}
q_1\, \xi_{(l_1,l_2)}(q,s) &= q_1 \boldsymbol{\xi}_{l_1+l_2}(q) \otimes \boldsymbol{\xi}_{l_1-l_2}(s)\\
&= \Big(a_-(l_1+l_2)\, \boldsymbol{\xi}_{l_1+l_2+1}(q)\, a_-(l_1+l_2)^\top\Big) \otimes \boldsymbol{\xi}_{l_1-l_2}(s)\\
&\qquad + \Big(b_-(l_1+l_2)\, \boldsymbol{\xi}_{l_1+l_2-1}(q)\, b_-(l_1+l_2)^\top\Big) \otimes \boldsymbol{\xi}_{l_1-l_2}(s)\\
&= \big(a_-(l_1+l_2) \otimes \mathrm{I}_{l_1-l_2+1}\big)\, \big(\boldsymbol{\xi}_{l_1+l_2+1}(q) \otimes \boldsymbol{\xi}_{l_1-l_2}(s)\big)\, \big(a_-(l_1+l_2)^\top \otimes \mathrm{I}_{l_1-l_2+1}\big)\\
&\qquad + \big(b_-(l_1+l_2) \otimes \mathrm{I}_{l_1-l_2+1}\big)\, \big(\boldsymbol{\xi}_{l_1+l_2-1}(q) \otimes \boldsymbol{\xi}_{l_1-l_2}(s)\big)\, \big(b_-(l_1+l_2)^\top \otimes \mathrm{I}_{l_1-l_2+1}\big)\\
&= A_-\, \xi_{(l_1^+,l_2^+)}(q,s)\, A_-^\top \;+\; B_-\, \xi_{(l_1^-,l_2^-)}(q,s)\, B_-^\top
\end{aligned}
\tag{5.37}
$$

and thus the first identity follows. The remaining identities are proved analogously. Note that for the last four formulas the sum of the weights stays the same while the difference is altered by ± 1. ☐

Example 5.7. The matrix recurrence relations yield three term relations for coefficients. We provide one example obtained by plugging in the definition of the matrices $a_\pm$ and

$b_\pm$ from (4.2). It follows that

$$q_1\,\xi_{(l_1,l_2)}(q,s)_{i,j} = \frac{\sqrt{r-\lfloor i/p\rfloor}\,\sqrt{r-\lfloor j/p\rfloor}}{r}\xi_{(l_1^+,l_2^+)}(q,s)_{i,j} + \frac{\sqrt{\lfloor i/p\rfloor}\,\sqrt{\lfloor j/p\rfloor}}{r}\xi_{(l_1^-,l_2^-)}(q,s)_{i-p,j-p} \tag{5.38}$$

with $r = l_1+l_2+1$ and $p = l_1-l_2+1$.

The next theorem provides second order recurrence relations following from Theorem 5.6.

Corollary 5.8. *For all $(l_1,l_2) \in \Gamma_{\mathrm{Spin}(4)}$ the following matrix recurrence relations hold*

$$\begin{aligned} q_1s_1\,\xi_{(l_1,l_2)}(q,s) = {} & \big(a_-(l_1+l_2)\otimes a_-(l_1-l_2)\big)\xi_{(l_1+1,l_2)}(q,s)\big(a_-(l_1+l_2)\otimes a_-(l_1-l_2)\big)^\top \\ & + \big(a_-(l_1+l_2)\otimes b_-(l_1-l_2)\big)\xi_{(l_1,l_2+1)}(q,s)\big(a_-(l_1+l_2)\otimes b_-(l_1-l_2)\big)^\top \\ & + \big(b_-(l_1+l_2)\otimes a_-(l_1-l_2)\big)\xi_{(l_1,l_2-1)}(q,s)\big(b_-(l_1+l_2)\otimes a_-(l_1-l_2)\big)^\top \\ & + \big(b_-(l_1+l_2)\otimes b_-(l_1-l_2)\big)\xi_{(l_1-1,l_2)}(q,s)\big(b_-(l_1+l_2)\otimes b_-(l_1-l_2)\big)^\top, \end{aligned} \tag{5.39a}$$

$$\begin{aligned} q_2s_2\,\xi_{(l_1,l_2)}(q,s) = {} & \big(a_-(l_1+l_2)\otimes a_-(l_1-l_2)\big)\xi_{(l_1+1,l_2)}(q,s)\big(a_+(l_1+l_2)\otimes a_+(l_1-l_2)\big)^\top \\ & - \big(a_-(l_1+l_2)\otimes b_-(l_1-l_2)\big)\xi_{(l_1,l_2+1)}(q,s)\big(a_+(l_1+l_2)\otimes b_-(l_1-l_2)\big)^\top \\ & - \big(b_-(l_1+l_2)\otimes a_-(l_1-l_2)\big)\xi_{(l_1,l_2-1)}(q,s)\big(b_+(l_1+l_2)\otimes a_+(l_1-l_2)\big)^\top \\ & + \big(b_-(l_1+l_2)\otimes b_-(l_1-l_2)\big)\xi_{(l_1-1,l_2)}(q,s)\big(b_+(l_1+l_2)\otimes b_-(l_1-l_2)\big)^\top, \end{aligned} \tag{5.39b}$$

$$\begin{aligned} \overline{q}_2\overline{s}_2\,\xi_{(l_1,l_2)}(q,s) = {} & \big(a_+(l_1+l_2)\otimes a_+(l_1-l_2)\big)\xi_{(l_1+1,l_2)}(q,s)\big(a_-(l_1+l_2)\otimes a_-(l_1-l_2)\big)^\top \\ & - \big(a_+(l_1+l_2)\otimes b_+(l_1-l_2)\big)\xi_{(l_1,l_2+1)}(q,s)\big(a_-(l_1+l_2)\otimes b_-(l_1-l_2)\big)^\top \\ & - \big(b_+(l_1+l_2)\otimes a_+(l_1-l_2)\big)\xi_{(l_1,l_2-1)}(q,s)\big(b_-(l_1+l_2)\otimes a_-(l_1-l_2)\big)^\top \\ & + \big(b_+(l_1+l_2)\otimes b_+(l_1-l_2)\big)\xi_{(l_1-1,l_2)}(q,s)\big(b_-(l_1+l_2)\otimes b_-(l_1-l_2)\big)^\top, \end{aligned} \tag{5.39c}$$

$$\begin{aligned} \overline{q}_1\overline{s}_1\,\xi_{(l_1,l_2)}(q,s) = {} & \big(a_+(l_1+l_2)\otimes a_+(l_1-l_2)\big)\xi_{(l_1+1,l_2)}(q,s)\big(a_+(l_1+l_2)\otimes a_+(l_1-l_2)\big)^\top \\ & + \big(a_+(l_1+l_2)\otimes b_+(l_1-l_2)\big)\xi_{(l_1,l_2+1)}(q,s)\big(a_+(l_1+l_2)\otimes b_+(l_1-l_2)\big)^\top \\ & + \big(b_+(l_1+l_2)\otimes a_+(l_1-l_2)\big)\xi_{(l_1,l_2-1)}(q,s)\big(b_+(l_1+l_2)\otimes a_+(l_1-l_2)\big)^\top \\ & + \big(b_+(l_1+l_2)\otimes b_+(l_1-l_2)\big)\xi_{(l_1-1,l_2)}(q,s)\big(b_+(l_1+l_2)\otimes b_+(l_1-l_2)\big)^\top, \end{aligned} \tag{5.39d}$$

where every expression out of domain is interpreted as zero.

5.6 Differential relations for matrix coefficients

The matrix coefficients of the Spin(4) representations can be generated using particular first order differential operators. This is a consequence of Theorem 4.5, for which we provide the details now. We use again a pair $q = (q_1, q_2) \in \mathbb{S}^3$ and $s = (s_1, s_2) \in \mathbb{S}^3$ of unit quaternions as coordinates for $\mathbf{s} = s\omega_+ + q\omega_- \in \mathrm{Spin}(4)$, cf. Section 5.

Definition 5.9. On Spin(4) we define the following first order differential operators

$$
\begin{aligned}
\partial_{q+} &= q_2\partial_{q_1} - \overline{q}_1\partial_{\overline{q}_2}, & \partial_{s+} &= s_2\partial_{s_1} - \overline{s}_1\partial_{\overline{s}_2},\\
\partial_{q-} &= -\overline{q}_2\partial_{\overline{q}_1} + q_1\partial_{q_2}, & \partial_{s-} &= -\overline{s}_2\partial_{\overline{s}_1} + s_1\partial_{s_2},\\
\partial^\dagger_{q+} &= -\overline{q}_1\partial_{q_2} + \overline{q}_2\partial_{q_1}, & \partial^\dagger_{s+} &= -\overline{s}_1\partial_{s_2} + \overline{s}_2\partial_{s_1},\\
\partial^\dagger_{q-} &= q_1\partial_{\overline{q}_2} - q_2\partial_{\overline{q}_1}, & \partial^\dagger_{s-} &= s_1\partial_{\overline{s}_2} - s_2\partial_{\overline{s}_1}.
\end{aligned}
$$

These operators are complex linear combinations of rotational derivatives and, moreover, are left or right invariant operators.

Let $\mathbf{s}, \mathbf{s}_0 \in \mathrm{Spin}(4)$ be given by $\mathbf{s} = s\omega_+ + q\omega_-$ and $\mathbf{s}_0 = s_0\omega_+ + q_0\omega_-$ with $q, s, q_0, s_0 \in \mathbb{S}^3$. Then by the properties of the idempotents $\omega_\pm$ it is easy to see that the left translation on Spin(4) is given by $\mathbf{s}_0^{-1}\mathbf{s} = \overline{s}_0 s\,\omega_+ + \overline{q}_0 q\,\omega_-$, while the right translation is given by $\mathbf{s}\mathbf{s}_0 = ss_0\,\omega_+ + qq_0\,\omega_-$. Consequently, left and right translations for functions $f(q, s)$ are given by

$$L_{(q_0,s_0)}f(q, s) = f(\overline{q}_0 q, \overline{s}_0 s), \tag{5.40a}$$
$$R_{(q_0,s_0)}f(q, s) = f(qq_0, ss_0) \tag{5.40b}$$

in these coordinates. The following statement follows by direct calculation similar to (4.21).

Theorem 5.10. *The operators $\partial_{q\pm}, \partial_{s\pm}$ are left invariant and the operators $\partial^\dagger_{q\pm}, \partial^\dagger_{s\pm}$ are right invariant on* Spin(4)*, that is,*

$$L_{(q_0,s_0)} \circ \partial_{q\pm} = \partial_{q\pm} \circ L_{(q_0,s_0)}, \qquad L_{(q_0,s_0)} \circ \partial_{s\pm} = \partial_{s\pm} \circ L_{(q_0,s_0)} \tag{5.41a}$$

together with

$$R_{(q_0,s_0)} \circ \partial^\dagger_{q\pm} = \partial^\dagger_{q\pm} \circ R_{(q_0,s_0)}, \qquad R_{(q_0,s_0)} \circ \partial^\dagger_{s\pm} = \partial^\dagger_{s\pm} \circ R_{(q_0,s_0)}. \tag{5.41b}$$

We now determine derivatives of our representations. Using the matrix relations (4.24a)-(4.24d) we can obtain the following differential relations.

Theorem 5.11. *For all $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$ the following identities*

$$\partial^\dagger_{q\pm}\,\xi_{(l_1,l_2)}(q, s) = \big(\sigma^\dagger_\pm(\tfrac{l_1 + l_2}{2}) \otimes \mathrm{I}_{l_1-l_2+1}\big)\xi_{(l_1,l_2)}(q, s), \tag{5.42a}$$
$$\partial_{q\pm}\,\xi_{(l_1,l_2)}(q, s) = \xi_{(l_1,l_2)}(q, s)\big(\sigma_\pm(\tfrac{l_1 + l_2}{2}) \otimes \mathrm{I}_{l_1-l_2+1}\big), \tag{5.42b}$$
$$\partial^\dagger_{s\pm}\,\xi_{(l_1,l_2)}(q, s) = \big(\mathrm{I}_{l_1+l_2+1} \otimes \sigma^\dagger_\pm(\tfrac{l_1 - l_2}{2})\big)\xi_{(l_1,l_2)}(q, s), \tag{5.42c}$$

$$\partial_{s\pm}\, \xi_{(l_1,l_2)}(q,s) = \xi_{(l_1,l_2)}(q,s)\big(\mathrm{I}_{l_1+l_2+1} \otimes \sigma_\pm(\frac{l_1-l_2}{2})\big) \tag{5.42d}$$

hold true with

$$\sigma_\pm^\dagger(\frac{l_1+l_2}{2}) = -\sigma_\mp(\frac{l_1+l_2}{2}) \qquad \textit{and} \qquad \sigma_\pm^\dagger(\frac{l_1-l_2}{2}) = -\sigma_\mp(\frac{l_1-l_2}{2}). \tag{5.43}$$

and for $\sigma_\pm$ *given by* (4.23).

Proof. We prove only the identity for $\partial_{q+}^\dagger$, the remaining ones are obtained analogously. By using (5.33), (4.24c), and the mixed-product property for the Kronecker product, we obtain

$$\begin{aligned}
\partial_{q+}^\dagger\, \xi_{(l_1,l_2)}(q,s) &= \Big(\partial_{q+}\, \boldsymbol{\xi}_{l_1+l_2}(q)\Big) \otimes \boldsymbol{\xi}_{l_1-l_2}(s) \\
&= \Big(\sigma_+^\dagger(\frac{l_1+l_2}{2})\, \boldsymbol{\xi}_{l_1+l_2}(q)\Big) \otimes \boldsymbol{\xi}_{l_1-l_2}(s) \\
&= \Big(\sigma_+^\dagger(\frac{l_1+l_2}{2}) \otimes \mathrm{I}_{l_1-l_2+1}\Big)\Big(\boldsymbol{\xi}_{l_1+l_2}(q) \otimes \boldsymbol{\xi}_{l_1-l_2}(s)\Big) \\
&= \Big(\sigma_+^\dagger(\frac{l_1+l_2}{2}) \otimes \mathrm{I}_{l_1-l_2+1}\Big)\, \xi_{(l_1,l_2)}(q,s).
\end{aligned} \tag{5.44}$$

□

Example 5.12. On the level of matrix coefficients this allows to switch around between different indices. We only give some formulas

$$\partial_{q\pm}^\dagger\, \xi_{(l_1,l_2)}(q,s)_{i,j} \;=\; \sqrt{r-\delta^\pm - \lfloor i/p \rfloor}\, \sqrt{\lfloor i/p \rfloor + \delta^\pm}\, \xi_{(l_1,l_2)}(q,s)_{i\pm p,j} \tag{5.45}$$

$$\partial_{q\pm}\, \xi_{(l_1,l_2)}(q,s)_{i,j} \;=\; -\sqrt{r-\delta^\pm - \lfloor j/p \rfloor}\, \sqrt{\lfloor j/p \rfloor + \delta^\pm}\, \xi_{(l_1,l_2)}(q,s)_{i,j\pm p} \tag{5.46}$$

where $r = l_1 + l_2 + 1$, $p = l_1 - l_2 + 1$, $\delta^+ = 1$, $\delta^- = 0$, and every expression out of domain is interpreted as zero.

Corollary 5.13. *For all* $(l_1,l_2) \in \Gamma_{\mathrm{Spin}(4)}$ *the following relations hold*

$$\partial_{q+}^\dagger\, \partial_{s\pm}^\dagger\, \xi_{(l_1,l_2)}(q,s) = \big(\sigma_+^\dagger(\frac{l_1+l_2}{2}) \otimes \sigma_\pm^\dagger(\frac{l_1-l_2}{2})\big)\xi_{(l_1,l_2)}(q,s), \tag{5.47a}$$

$$\partial_{q-}^\dagger\, \partial_{s\pm}^\dagger\, \xi_{(l_1,l_2)}(q,s) = \big(\sigma_-^\dagger(\frac{l_1+l_2}{2}) \otimes \sigma_\pm^\dagger(\frac{l_1-l_2}{2})\big)\xi_{(l_1,l_2)}(q,s), \tag{5.47b}$$

$$\partial_{q+}\, \partial_{s\pm}\, \xi_{(l_1,l_2)}(q,s) = \xi_{(l_1,l_2)}(q,s)\big(\sigma_+(\frac{l_1+l_2}{2}) \otimes \sigma_\pm(\frac{l_1-l_2}{2})\big), \tag{5.47c}$$

$$\partial_{q-}\, \partial_{s\pm}\, \xi_{(l_1,l_2)}(q,s) = \xi_{(l_1,l_2)}(q,s)\big(\sigma_-(\frac{l_1+l_2}{2}) \otimes \sigma_\pm(\frac{l_1-l_2}{2})\big) \tag{5.47d}$$

for $\sigma_\pm$ *and* $\sigma_\pm^\dagger$ *given by* (4.23).

6 Calculus

6.1 Left invariant differential operators

The differential operators $\partial_{q\pm}$, $\partial_{s\pm}$ appearing in the recurrence relations for matrix coefficients allow to construct all left-invariant differential operators. To see this, we first provide a basis for the space of left-invariant vector fields on Spin(4), i.e. for the Lie algebra $\mathfrak{spin}(4)$.

Note, that the differential operators $\partial_{q\pm}$ and $\partial_{s\pm}$ are complex derivatives and that on a formal level $\partial_{q+}^* = -\partial_{q-}$. To obtain elements of the Lie algebra, we form combinations of them.

Definition 6.1. We define the left invariant differential operators

$$\mathrm{D}_{1q} = -\frac{\mathbf{i}}{2}(\partial_{q-} + \partial_{q+}), \qquad \mathrm{D}_{1s} = -\frac{\mathbf{i}}{2}(\partial_{s-} + \partial_{s+}) \tag{6.1a}$$

$$\mathrm{D}_{2q} = \frac{1}{2}(\partial_{q-} - \partial_{q+}), \qquad \mathrm{D}_{2s} = \frac{1}{2}(\partial_{s-} - \partial_{s+}) \tag{6.1b}$$

$$\mathrm{D}_{3q} = [\mathrm{D}_{1q}, \mathrm{D}_{2q}] = -\frac{\mathbf{i}}{2}[\partial_{q+}, \partial_{q-}], \qquad \mathrm{D}_{3s} = [\mathrm{D}_{1s}, \mathrm{D}_{2s}] = -\frac{\mathbf{i}}{2}[\partial_{s+}, \partial_{s-}]. \tag{6.1c}$$

Remark 6.2. The operators ∂_{q+} and ∂_{s+} are sometimes called creation operators, while the operators ∂_{q-} and ∂_{s-} are called annihilator operators (cf. [23, Remark 12.2.3.] where the operators are denoted by ∂_+ and ∂_-, and [28, p. 140] where the operators are denoted by $\hat{H}_+$ and $\hat{H}_-$). It is also customary to define

$$\partial_{q0} = \frac{1}{2}[\partial_{q+}, \partial_{q-}] \tag{6.2}$$

and denote this as the neutral operator. The operator D_{3q} can thus be written as $\mathrm{D}_{3q} = -\mathbf{i}\partial_{q0}$.

Proposition 6.3. *The commutator relations hold*

$$[\mathrm{D}_{1q}, \mathrm{D}_{2q}] = \mathrm{D}_{3q}, \qquad [\mathrm{D}_{2q}, \mathrm{D}_{3q}] = \mathrm{D}_{1q}, \qquad [\mathrm{D}_{3q}, \mathrm{D}_{1q}] = \mathrm{D}_{2q}, \tag{6.3a}$$

$$[\mathrm{D}_{1s}, \mathrm{D}_{2s}] = \mathrm{D}_{3s}, \qquad [\mathrm{D}_{2s}, \mathrm{D}_{3s}] = \mathrm{D}_{1s}, \qquad [\mathrm{D}_{3s}, \mathrm{D}_{1s}] = \mathrm{D}_{2s}, \tag{6.3b}$$

together with $[\mathrm{D}_{\nu q}, \mathrm{D}_{\mu s}]$ *for all* $\mu, \nu \in \{1, 2, 3\}$. *Furthermore, the set* $\{D_{\mu q}, \mathrm{D}_{\nu s} \mid \mu, \nu = 1, 2, 3\}$ *forms an (orthonormal) basis of the Lie algebra* $\mathfrak{spin}(4)$.

Proof. The last identity follows from the direct product decomposition of Spin(4). For the remaining identities we are concentrating on derivatives with respect to the component q, the remaining ones are obtained similarly. The first commutator relation holds by definition. For the second commutator relation we observe that by straightforward calculation

$$[\partial_{q+}, \partial_{q-}] = -q_1\partial_{q_1} + q_2\partial_{q_2} + \overline{q}_1\partial_{\overline{q}_1} - \overline{q}_2\partial_{\overline{q}_2} \tag{6.4}$$

such that

$$[\partial_{q+}, [\partial_{q+}, \partial_{q-}]] = -2\partial_{q+} \quad \text{and} \quad [\partial_{q-}, [\partial_{q+}, \partial_{q-}]] = 2\partial_{q-}. \tag{6.5}$$

Therefore, we obtain

$$[\mathrm{D}_{2q},\mathrm{D}_{3q}] = -\frac{\mathbf{i}}{4}\big([\partial_{q-},[\partial_{q+},\partial_{q-}]]-[\partial_{q+},[\partial_{q+},\partial_{q-}]]\big) = -\frac{\mathbf{i}}{4}(2\partial_{q-}+2\partial_{q+}) = \mathrm{D}_{1q} \quad (6.6)$$

together with

$$[\mathrm{D}_{3q},\mathrm{D}_{1q}] = -[\mathrm{D}_{1q},\mathrm{D}_{3q}] = \frac{1}{4}([\partial_{q-},[\partial_{q+},\partial_{q-}]]+[\partial_{q+},[\partial_{q+},\partial_{q-}]]) = \frac{1}{4}(2\partial_{q-}-2\partial_{q+}) = \mathrm{D}_{2q}. \quad (6.7)$$

□

The Laplace–Beltrami operator $\mathcal{L}$ on Spin(4) is given by

$$\begin{aligned}\mathcal{L} &= \mathrm{D}_{1q}^2+\mathrm{D}_{2q}^2+\mathrm{D}_{3q}^2+\mathrm{D}_{1s}^2+\mathrm{D}_{2s}^2+\mathrm{D}_{3s}^2\\ &= -\frac{1}{2}(\partial_{q+}\partial_{q-}+\partial_{q-}\partial_{q+}+\partial_{s+}\partial_{s-}+\partial_{s-}\partial_{s+})-\partial_{q0}^2-\partial_{s0}^2.\end{aligned} \quad (6.8)$$

If we denote by $\mathcal{H}^{(l_1,l_2)}$ the complex linear span of the matrix coefficients of $\xi_{(l_1,l_2)}$,

$$\mathcal{H}^{(l_1,l_2)} = \mathrm{span}\{\mathbf{s}\mapsto \xi_{(l_1,l_2)}(\mathbf{s})_{i,j} : 0\le i,j\le d_{(l_1,l_2)}-1\}, \quad (6.9)$$

the following theorem holds true.

Theorem 6.4. *The space $\mathcal{H}^{(l_1,l_2)}$ is an eigenspace of $\mathcal{L}$ with eigenvalue*

$$\lambda_{l_1,l_2} = -\Big(\frac{(l_1+l_2)(l_1+l_2+2)}{4}+\frac{(l_1-l_2)(l_1-l_2+2)}{4}\Big) = -\frac{l_1^2+2l_1+l_2^2}{2}. \quad (6.10)$$

Proof. This follows directly from (6.8) in combination with the formulas of Theorem 4.5 for the matrix coefficients of the Spin(3)-representation. □

For later use we define

$$\langle\xi_{l_1,l_2}\rangle = \sqrt{1-\lambda_{l_1,l_2}} = \sqrt{1+l_1+\frac{l_1^2+l_2^2}{2}}. \quad (6.11)$$

6.2 The group Fourier transform on Spin(4)

First, we introduce notation and recall some basic facts. Then the characterisations of function spaces on Spin(4) follows from abstract arguments, as presented in [21] and [23]. See also [16] for the relation between certain function spaces on direct products of groups.

The group Fourier transform on Spin(4) is given in terms of all equivalence classes of irreducible representations

$$\xi_{(l_1,l_2)}(s\omega_+ + q\omega_-) = \xi^{\mathrm{Spin}(3)}_{\frac{l_1+l_2}{2}}(q)\otimes\xi^{\mathrm{Spin}(3)}_{\frac{l_1-l_2}{2}}(s). \quad (6.12)$$

For an integrable function $f \in \mathrm{L}^1(\mathrm{Spin}(4))$ we define

$$\widehat{f}(l_1, l_2) = \int_{\mathrm{Spin}(4)} f(\mathbf{s})\, \xi_{(l_1,l_2)}(\mathbf{s})^*\, \mathrm{d}\mathbf{s}, \tag{6.13}$$

where we integrate with respect to the normalised Haar measure on the group Spin(4). Note that by uniqueness of the Haar measure and by the direct product structure $\mathrm{Spin}(4) \simeq \mathrm{Spin}(3) \times \mathrm{Spin}(3)$ the Haar measure on Spin(4) is also the tensor product of the normalised Haar measures on both factors.

The Fourier transform maps $\mathrm{L}^2(\mathrm{Spin}(4))$ unitarily onto a sequence space. For this we define $\ell^2 := \ell^2(\widehat{\mathrm{Spin}(4)})$ to be the space of all sequences

$$\sigma : \Gamma_{\mathrm{Spin}(4)} \ni (l_1, l_2) \mapsto \sigma(l_1, l_2) \in \mathbb{C}^{d_{(l_1,l_2)} \times d_{(l_1,l_2)}} \tag{6.14}$$

such that

$$\|\sigma\|_{\ell^2}^2 = \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}} d_{(l_1,l_2)} \|\sigma(l_1, l_2)\|_{\mathrm{HS}}^2 < \infty. \tag{6.15}$$

This space is clearly a Hilbert space and we endow it with its natural inner product. Now, Peter–Weyl Theorem 2.1 implies

Theorem 6.5. *The Fourier transform is unitary from* $\mathrm{L}^2(\mathrm{Spin}(4))$ *to* $\ell^2(\widehat{\mathrm{Spin}(4)})$ *with inverse*

$$f(\mathbf{s}) = \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}} d_{(l_1,l_2)} \operatorname{tr}(\widehat{f}(l_1, l_2)\xi_{(l_1,l_2)}(\mathbf{s})) \tag{6.16}$$

and Plancherel identity

$$\|f\|_{L^2(\mathrm{Spin}(4))}^2 = \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}} d_{(l_1,l_2)}\, \|\widehat{f}(l_1, l_2)\|_{\mathrm{HS}}^2. \tag{6.17}$$

Remark 6.6. In the particular case $f(s\omega_+ + q\omega_-) = f_1(q)f_2(s)$ with $f_i \in \mathrm{L}^2(\mathrm{Spin}(3))$, the Kronecker product representation of the Fourier coefficients

$$\begin{aligned}
\widehat{f}(l_1, l_2) &= \int_{\mathrm{Spin}(4)} f(\mathbf{s})\, \xi_{(l_1,l_2)}(\mathbf{s})^*\, \mathrm{d}\mathbf{s} \\
&= \int_{\mathrm{Spin}(3)} f_1(q)\, \xi^{\mathrm{Spin}(3)}_{\frac{l_1+l_2}{2}}(q)^*\, \mathrm{d}q \otimes \int_{\mathrm{Spin}(3)} f_2(s)\, \xi^{\mathrm{Spin}(3)}_{\frac{l_1-l_2}{2}}(s)^*\, \mathrm{d}s \\
&= \widehat{f}_1(\frac{l_1+l_2}{2}) \otimes \widehat{f}_2(\frac{l_1-l_2}{2}).
\end{aligned} \tag{6.18}$$

in terms of the Spin(3) Fourier transforms imply

$$\begin{aligned}
f(s\omega_+ + q\omega_-) = &\sum_{m_1\in\frac{1}{2}\mathbb{N}_0} (2m_1+1) \operatorname{tr}\big(\widehat{f}_1(m_1)\, \xi^{\mathrm{Spin}(3)}_{m_1}(q)\big) \\
&\times \sum_{m_2\in\frac{1}{2}\mathbb{N}_0} (2m_2+1) \operatorname{tr}\big(\widehat{f}_2(m_2)\, \xi^{\mathrm{Spin}(3)}_{m_2}(s)\big).
\end{aligned} \tag{6.19}$$

Here we made use of $\operatorname{tr}(A\otimes B)=\operatorname{tr}(A)\operatorname{tr}(B)$ together with (5.34). This also allows to split the Plancherel formula into a double sum

$$\|f\|^2_{L^2(\mathrm{Spin}(4))}=\sum_{m_1\in\frac12\mathbb{N}_0}(2m_1+1)\|\widehat{f}_1(m_1)\|^2_{\mathrm{HS}}\sum_{m_2\in\frac12\mathbb{N}_0}(2m_2+1)\|\widehat{f}_2(m_2)\|^2_{\mathrm{HS}} \tag{6.20}$$

based on the product formula $\|A\otimes B\|_{\mathrm{HS}}=\|A\|_{\mathrm{HS}}\|B\|_{\mathrm{HS}}$ for Hilbert-Schmidt norms.

The group Fourier transform extends naturally to distributions. The space $\mathcal{D}'(\mathrm{Spin}(4))$ of distributions is the topological dual space of smooth functions $C^\infty(\mathrm{Spin}(4))$. As usual for a function $f\in L^1(\mathrm{Spin}(4))$ and $\varphi\in C^\infty(\mathrm{Spin}(4))$ we define $T_f\in\mathcal{D}'(\mathrm{Spin}(4))$ by

$$\langle T_f,\varphi\rangle=\int_{\mathrm{Spin}(4)}f(\mathbf{s})\varphi(\mathbf{s})\,\mathrm{d}\mathbf{s} \tag{6.21}$$

and use the same notation for the dual pairing between distributions and functions. For a distribution $T\in\mathcal{D}'(\mathrm{Spin}(4))$ its Fourier transform $\widehat{T}$ is defined by $\widehat{T}(l_1,l_2)=\langle T,\xi^*_{(l_1,l_2)}\rangle$.

6.3 Function spaces

Sobolev spaces are characterised in terms of the Laplacian. Thus, for $r\in\mathbb{R}$ the space $\mathrm{H}^r(\mathrm{Spin}(4))$ has the familiar characterisation

$$f\in\mathrm{H}^r(\mathrm{Spin}(4))\iff(-\mathcal{L})^rf\in\mathrm{L}^2(\mathrm{Spin}(4))\iff\langle\xi_{(l_1,l_2)}\rangle^r\,\widehat{f}(l_1,l_2)\in\ell^2(\widehat{\mathrm{Spin}(4)}) \tag{6.22}$$

in terms of the group Fourier transform. This allows to characterise spaces of smooth functions and of distributions. In the following, we denote by $\mathfrak{s}:=\mathfrak{s}(\widehat{\mathrm{Spin}(4)})$ the space of rapidly decaying matrix sequences

$$\rho:\Gamma_{\mathrm{Spin}(4)}\to\bigcup_d\mathbb{C}^{d\times d} \tag{6.23}$$

such that the dimension of $\rho(l_1,l_2)$ equals $d_{(l_1,l_2)}=(l_1+l_2+1)(l_1-l_2+1)$ and

$$\sup_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}}\|\rho(l_1,l_2)\|_{\mathrm{HS}}\langle\xi_{(l_1,l_2)}\rangle^N<\infty \tag{6.24}$$

for any number N. The particular choice of matrix norm does not matter due to the polynomial growth of $\langle\xi_{(l_1,l_2)}\rangle$ in the dimension $d_{(l_1,l_2)}$. We also denote by $\mathfrak{s}':=\mathfrak{s}'(\widehat{\mathrm{Spin}(4)})$ the space of all such matrix sequences with

$$\sup_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}}\|\rho(l_1,l_2)\|_{\mathrm{HS}}\langle\xi_{(l_1,l_2)}\rangle^{-N}<\infty \tag{6.25}$$

for one number N. Both are equipped with their natural locally convex topology arising from (6.24) and (6.25), respectively. Using this topology the spaces $\mathfrak{s}$ and $\mathfrak{s}'$ are dual to each other.

Theorem 6.7. *The Fourier transform provides isomorphisms*

1. $\mathcal{F} : C^\infty(\mathrm{Spin}(4)) \to \mathfrak{s}$
2. $\mathcal{F} : \mathcal{D}'(\mathrm{Spin}(4)) \to \mathfrak{s}'$

and thus characterises smooth functions in terms of rapidly decaying sequences of Fourier coefficients and distributions in terms of moderately growing sequences of Fourier coefficients.

6.4 Differential and pseudo-differential operators on Spin(4)

In a next step we provide details on the differential and pseudo-differential calculus on the group Spin(4).

6.4.1 Symbolic calculus of invariant operators

Now, we discuss the symbolic calculus for operators on Spin(4). We recall from Section 2 that an operator $A : C^\infty \to \mathcal{D}'$ is said to be left-invariant if it commutes with left translations, i.e. if $A \circ L_{\mathbf{s}_0} = L_{\mathbf{s}_0} \circ A$ for any $\mathbf{s}_0 = q_0\omega_+ + s_0\omega_-$ with L given by (5.40a). Any such operator A can be expressed in terms of a left-symbol σ_A in the form of a Fourier multiplier

$$Af(\mathbf{s}) = \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}} d_{(l_1,l_2)} \operatorname{tr}(\sigma_A(l_1,l_2)\, \widehat{f}(l_1,l_2)\, \xi_{(l_1,l_2)}(\mathbf{s})). \tag{6.26}$$

Similarly, right invariant operators $B : C^\infty \to \mathcal{D}'$ with $B \circ R_{\mathbf{s}_0} = R_{\mathbf{s}_0} \circ B$ correspond to right multiplication of the Fourier coefficients by a right-symbol $\sigma_B^\dagger$

$$Bf(\mathbf{s}) = \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}} d_{(l_1,l_2)} \operatorname{tr}(\widehat{f}(l_1,l_2)\, \sigma_B^\dagger(l_1,l_2)\, \xi_{(l_1,l_2)}(\mathbf{s})). \tag{6.27}$$

It is important to distinguish between left and right-symbols here, right-invariant operators also posses (variable coefficient) left-symbols.

In Table 1 and Table 2 we present symbols for some left-invariant respectively right-invariant differential operators on Spin(4). The formulas for symbols are a consequence of Theorem 5.11 and Corollary 5.13 and given in terms of $\sigma_\pm$ from (4.23) and with $\sigma_0 = [\sigma_+, \sigma_-]$, i.e. with $\sigma_0(m/2)_{ij} = \frac{1}{2}(m-2j)\delta_{ij}$ for every $m \in \mathbb{N}_0$ and $0 \le i,j \le m$.

Mapping properties of left-invariant operators are characterised in terms of difference operators acting on their symbols. We recall the definition first before providing properties of the difference operators of our choice. A difference operator $\Delta : \mathfrak{s}' \to \mathfrak{s}'$ acting on moderate matrix sequences is defined in terms of a function $\varphi \in C^\infty(\mathrm{Spin}(4))$ via $\Delta\widehat{f} = \widehat{\varphi f}$ using the group Fourier transform $\widehat{f}$ of distributions $f \in \mathcal{D}'(\mathrm{Spin}(4))$. If φ vanishes to first order at the identity, we call Δ a first order difference operator.

There are different ways to construct first-order difference operators. At first glance the concept of difference operators introduced in [8] seems to be a natural choice for difference operators defined over tensor products of compact Lie groups, but it has a major drawback. In general, arbitrary tensor products of representations are not

left-invariant		right-invariant	
operator	left symbol	operator	right symbol
$\partial_{q\pm}$	$\sigma_\pm(\frac{l_1+l_2}{2})\otimes \mathrm{I}_{l_1-l_2+1}$	$\partial^\dagger_{q\pm}$	$-\sigma_\mp(\frac{l_1+l_2}{2})\otimes \mathrm{I}_{l_1-l_2+1}$
∂_{q0}	$\sigma_0(\frac{l_1+l_2}{2})\otimes \mathrm{I}_{l_1-l_2+1}$	$\partial^\dagger_{q0}$	$-\sigma_0(\frac{l_1+l_2}{2})\otimes \mathrm{I}_{l_1-l_2+1}$
$\partial_{s\pm}$	$\mathrm{I}_{l_1+l_2+1}\otimes\sigma_\pm(\frac{l_1-l_2}{2})$	$\partial^\dagger_{s\pm}$	$-\mathrm{I}_{l_1+l_2+1}\otimes\sigma_\mp(\frac{l_1-l_2}{2})$
∂_{s0}	$\mathrm{I}_{l_1+l_2+1}\otimes\sigma_0(\frac{l_1-l_2}{2})$	$\partial^\dagger_{s0}$	$-\mathrm{I}_{l_1+l_2+1}\otimes\sigma_0(\frac{l_1-l_2}{2})$

Table 1 Symbols for some left and right invariant first order differential operators.

left-invariant		right-invariant	
operator	left symbol	operator	right symbol
$\partial_{q+}\,\partial_{s\pm}$	$\sigma_+(\frac{l_1+l_2}{2})\otimes\sigma_\pm(\frac{l_1-l_2}{2})$	$\partial^\dagger_{q+}\,\partial^\dagger_{s\pm}$	$\sigma_-(\frac{l_1+l_2}{2})\otimes\sigma_\mp(\frac{l_1-l_2}{2})$
$\partial_{q-}\,\partial_{s\pm}$	$\sigma_-(\frac{l_1+l_2}{2})\otimes\sigma_\pm(\frac{l_1-l_2}{2})$	$\partial^\dagger_{q-}\,\partial^\dagger_{s\pm}$	$\sigma_+(\frac{l_1+l_2}{2})\otimes\sigma_\mp(\frac{l_1-l_2}{2})$

Table 2 Symbols for some left and right invariant second order differential operators.

irreducible and require another decomposition making the construction of an admissible collection rather difficult. In the present case this approach leads to the same difference operators which we introduce here in a more direct way.

We use particular difference operators related to the matrix entries of the representations $\xi_{(\frac12,\pm\frac12)}$. As pointed out in [23] this construction leads to difference operators satisfying a finite Leibniz rule. To fix notation, we collect them in Table 3.

difference operator	associated function	difference operator	associated function
Δ_q^{--}	$\varphi(q,s)=q_1-1$	Δ_s^{--}	$\varphi(q,s)=s_1-1$
Δ_q^{-+}	$\varphi(q,s)=q_2$	Δ_s^{-+}	$\varphi(q,s)=s_2$
Δ_q^{+-}	$\varphi(q,s)=-\overline{q}_2$	Δ_s^{+-}	$\varphi(q,s)=-\overline{s}_2$
Δ_q^{++}	$\varphi(q,s)=\overline{q}_1-1$	Δ_s^{++}	$\varphi(q,s)=\overline{s}_1-1$

Table 3 Difference operators of order 1 for $\mathbf{s}=s\omega_+ + q\omega_- \in \mathrm{Spin}(4)$.

We recall the following result from [25]. We will use multi-index notation for difference operators and write $\Delta^\alpha=(\Delta_q^{--})^{\alpha_1}\cdots(\Delta_s^{++})^{\alpha_8}$ for $\alpha\in\mathbb{N}_0^8$ and formulate the multiplier theorem [25, Theorem 3.5] for the particular case of Spin(4).

Theorem 6.8 (Multiplier theorem, [25])**.** *Let $A: C^\infty(\mathrm{Spin}(4))\to\mathcal{D}'(\mathrm{Spin}(4))$ be a left-invariant operator on* Spin(4) *with left-symbol σ_A satisfying*

$$\|\sigma_A(l_1,l_2)\|_{\mathrm{op}}+\sum_{|\alpha|\le 3}\langle\xi_{(l_1,l_2)}\rangle^{|\alpha|}\|\Delta^\alpha\sigma_A(l_1,l_2)\|_{\mathrm{op}}+\langle\xi_{l_1,l_2}\rangle^4\|\triangle^2\sigma_A(l_1,l_2)\|_{\mathrm{op}}\le C \quad (6.28)$$

with the particular second order difference $\triangle=\Delta_q^{++}+\Delta_q^{--}+\Delta_s^{++}+\Delta_s^{--}\in\mathrm{diff}^2(\widehat{\mathrm{Spin}(4)})$. Then A is bounded on $\mathrm{L}^p(\mathrm{Spin}(4))$ for $1<p<\infty$ and of weak type $(1,1)$.

Explicit formulas for the difference operators from Table 3 follow from the recurrence relations given in Theorem 5.6.

Theorem 6.9. *The difference operators of order 1 given in Table 3 are explicitly given by*

$$\begin{aligned}\Delta_q^{\pm\pm}\,\sigma(l_1,l_2) =& \frac{l_1+l_2}{l_1+l_2+1}\,A_\pm(l_1^-,l_2^-)^\top\sigma(l_1^-,l_2^-)\,A_\pm(l_1^-,l_2^-) \\ &+ \frac{l_1+l_2+2}{l_1+l_2+1}\,B_\pm(l_1^+,l_2^+)^\top\sigma(l_1^+,l_2^+)\,B_\pm(l_1^+,l_2^+) - \sigma(l_1,l_2),\end{aligned} \tag{6.29a}$$

$$\begin{aligned}\Delta_q^{\pm\mp}\,\sigma(l_1,l_2) =& \frac{l_1+l_2}{l_1+l_2+1}\,A_\pm(l_1^-,l_2^-)^\top\sigma(l_1^-,l_2^-)\,A_\mp(l_1^-,l_2^-) \\ &- \frac{l_1+l_2+2}{l_1+l_2+1}\,B_\pm(l_1^+,l_2^+)^\top\sigma(l_1^+,l_2^+)\,B_\mp(l_1^+,l_2^+),\end{aligned} \tag{6.29b}$$

$$\begin{aligned}\Delta_s^{\pm\pm}\,\sigma(l_1,l_2) =& \frac{l_1-l_2}{l_1-l_2+1}\,C_\pm(l_1^+,l_2^-)^\top\sigma(l_1^+,l_2^-)\,C_\pm(l_1^+,l_2^-) \\ &+ \frac{l_1-l_2+2}{l_1-l_2+1}\,D_\pm(l_1^-,l_2^+)^\top\sigma(l_1^-,l_2^+)\,D_\pm(l_1^-,l_2^+) - \sigma(l_1,l_2),\end{aligned} \tag{6.29c}$$

$$\begin{aligned}\Delta_s^{\pm\mp}\,\sigma(l_1,l_2) =& \frac{l_1-l_2}{l_1-l_2+1}\,C_\pm(l_1^+,l_2^-)^\top\sigma(l_1^+,l_2^-)\,C_\mp(l_1^+,l_2^-) \\ &- \frac{l_1-l_2+2}{l+1}\,D_\pm(l_1^-,l_2^+)^\top\sigma(l_1^-,l_2^+)\,D_\mp(l_1^-,l_2^+)\,.\end{aligned} \tag{6.29d}$$

Proof. We consider the first difference operator Δ_q^{--}. By using the first matrix recurrence relation from Theorem 5.6 and the cyclic property of the trace, we obtain

$$\begin{aligned} q_1 f(\mathbf{s}) \;=&\; \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}} d_{(l_1,l_2)}\;\operatorname{tr}\left(\widehat{f}(l_1,l_2)\,q_1\,\xi_{(l_1,l_2)}(\mathbf{s})\right) \\ =& \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}} d_{(l_1,l_2)}\;\operatorname{tr}\left(\widehat{f}(l_1,l_2)\,A_-(l_1,l_2)\,\xi_{(l_1^+,l_2^+)}(\mathbf{s})\,A_-(l_1,l_2)^\top\right) \\ &+ \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}} d_{(l_1,l_2)}\;\operatorname{tr}\left(\widehat{f}(l_1,l_2)\,B_-(l_1,l_2)\,\xi_{(l_1^-,l_2^-)}(\mathbf{s})\,B_-(l_1,l_2)^\top\right) \\ =& \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}+(\frac12,\frac12)} d_{(l_1^-,l_2^-)}\;\operatorname{tr}\left(\widehat{f}(l_1^-,l_2^-)\,A_-(l_1^-,l_2^-)\,\xi_{(l_1,l_2)}(\mathbf{s})\,A_-(l_1^-,l_2^-)^\top\right) \\ &+ \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}-(\frac12,\frac12)} d_{(l_1^+,l_2^+)}\;\operatorname{tr}\left(\widehat{f}(l_1^+,l_2^+)\,B_-(l_1^+,l_2^+)\,\xi_{(l_1,l_2)}(\mathbf{s})\,B_-(l_1^+,l_2^+)^\top\right) \\ =& \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}+(\frac12,\frac12)} d_{(l_1^-,l_2^-)}\;\operatorname{tr}\left(A_-(l_1^-,l_2^-)^\top\,\widehat{f}(l_1^-,l_2^-)\,A_-(l_1^-,l_2^-)\,\xi_{(l_1,l_2)}(\mathbf{s})\right) \\ &+ \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}-(\frac12,\frac12)} d_{(l_1^+,l_2^+)}\;\operatorname{tr}\left(B_-(l_1^+,l_2^+)^\top\,\widehat{f}(l_1^+,l_2^+)\,B_-(l_1^+,l_2^+)\,\xi_{(l_1,l_2)}(\mathbf{s})\right).\end{aligned} \tag{6.30}$$

As for indices just outside $\Gamma_{\mathrm{Spin}(4)}$ the matrices $A_\pm$ and $B_\pm$ vanish, the last sums can be rewritten as sums over $\Gamma_{\mathrm{Spin}(4)}$. Making use of the identities

$$\frac{d_{(l_1^+,l_2^+)}}{d_{(l_1,l_2)}} = \frac{l_1+l_2+2}{l_1+l_2+1}, \qquad \frac{d_{(l_1^-,l_2^-)}}{d_{(l_1,l_2)}} = \frac{l_1+l_2}{l_1+l_2+1} \tag{6.31}$$

and subtracting by $\sigma(l_1,l_2)$ we obtain the expression for Δ_q^{--}. The remaining formulas are obtained similarly from Theorem 5.6. □

Corollary 6.10. *When $\sigma(l_1,l_2) = \rho(\frac{l_1+l_2}{2}) \otimes \tau(\frac{l_1-l_2}{2})$ is of tensor product form, the difference operators of order 1 given in Table 3 are explicitly given by*

$$\Delta_q^{\pm\pm}\rho(\frac{l_1+l_2}{2}) \otimes \tau(\frac{l_1-l_2}{2}) = (\Delta^{\pm\pm}\rho)(\frac{l_1+l_2}{2}) \otimes \tau(\frac{l_1-l_2}{2}), \tag{6.32a}$$

$$\Delta_q^{\pm\mp}\rho(\frac{l_1+l_2}{2}) \otimes \tau(\frac{l_1-l_2}{2}) = (\Delta^{\pm\mp}\rho)(\frac{l_1+l_2}{2}) \otimes \tau(\frac{l_1-l_2}{2}), \tag{6.32b}$$

$$\Delta_s^{\pm\pm}\rho(\frac{l_1+l_2}{2}) \otimes \tau(\frac{l_1-l_2}{2}) = \rho(\frac{l_1+l_2}{2}) \otimes (\Delta^{\pm\pm}\tau)(\frac{l_1-l_2}{2}), \tag{6.32c}$$

$$\Delta_s^{\pm\mp}\rho(\frac{l_1+l_2}{2}) \otimes \tau(\frac{l_1-l_2}{2}) = \rho(\frac{l_1+l_2}{2}) \otimes (\Delta^{\pm\mp}\tau)(\frac{l_1-l_2}{2}), \tag{6.32d}$$

where

$$\Delta^{\pm\pm}\rho(l) = \frac{2l}{2l+1}\, a_\pm(2\,l^-)^\top \rho(l^-)\, a_\pm(2\,l^-) + \frac{2l+2}{2l+1}\, b_\pm(2\,l^+)^\top \rho(l^+)\, b_\pm(2\,l^+) - \rho(l), \tag{6.33a}$$

$$\Delta^{\pm\mp}\rho(l) = \frac{2l}{2l+1}\, a_\pm(2\,l^-)^\top \rho(l^-)\, a_\mp(2\,l^-) - \frac{2l+2}{2l+1}\, b_\pm(2\,l^+)^\top \rho(l^+)\, b_\mp(2\,l^+) \tag{6.33b}$$

are the difference operators on Spin(3) *given in terms of $a_\pm$ and $b_\pm$ from Definition 4.2.*

Corollary 6.11. *A matrix sequence is of the form $\sigma(l_1,l_2) = \rho(\frac{l_1+l_2}{2}) \otimes \mathrm{I}_{l_1-l_2+1}$ if and only if*

$$\Delta_s^{\pm\pm}\sigma = \Delta_s^{\pm\mp}\sigma = 0. \tag{6.34}$$

Similarly, it is of the form $\sigma(l_1,l_2) = \mathrm{I}_{l_1+l_2+1} \otimes \tau(\frac{l_1+l_2}{2})$ if and only if

$$\Delta_q^{\pm\pm}\sigma = \Delta_q^{\pm\mp}\sigma = 0. \tag{6.35}$$

In Table 4 we show the difference operators applied to the symbols of the elementary first order differential operators $\partial_{q\nu}, \partial_{s\mu},\ \nu,\mu \in \{0,+,-\}$ and the Laplacian $\mathcal{L}$ on the group. The table can be computed using Corollary 6.10 in combination with the first columns of [23, Table 1], the symbol of the Laplacian is corrected here.

By construction, difference operators are mutually commuting operators. Differences acting on symbols of differential operators are best calculated using the Leibniz

	$\sigma_{\partial_{q0}}$	$\sigma_{\partial_{q+}}$	$\sigma_{\partial_{q-}}$	$\sigma_{\mathcal{L}}$
Δ_q^{--}	$\frac{1}{2}\mathrm{I}_d$	0	0	$-\sigma_{\partial_{q0}} - \frac{3}{4}\mathrm{I}_d$
Δ_q^{-+}	0	I_d	0	$-\sigma_{\partial_{q-}}$
Δ_q^{+-}	0	0	I_d	$-\sigma_{\partial_{q+}}$
Δ_q^{++}	$-\frac{1}{2}\mathrm{I}_d$	0	0	$\sigma_{\partial_{q0}} - \frac{3}{4}\mathrm{I}_d$

	$\sigma_{\partial_{s0}}$	$\sigma_{\partial_{s+}}$	$\sigma_{\partial_{s-}}$	$\sigma_{\mathcal{L}}$
Δ_s^{--}	$\frac{1}{2}\mathrm{I}_d$	0	0	$-\sigma_{\partial_{s0}} - \frac{3}{4}\mathrm{I}_d$
Δ_s^{-+}	0	I_d	0	$-\sigma_{\partial_{s-}}$
Δ_s^{+-}	0	0	I_d	$-\sigma_{\partial_{s+}}$
Δ_s^{++}	$-\frac{1}{2}\mathrm{I}_d$	0	0	$\sigma_{\partial_{s0}} - \frac{3}{4}\mathrm{I}_d$

Table 4 Difference operators acting on some symbols.

rule for difference operators combined with Table 4. From [23] it follows that

$$\Delta_q^{ij}(\sigma\tau) = (\Delta_q^{ij}\sigma)\tau + \sigma(\Delta_q^{ij}\tau) - \sum_{k\in\{+,-\}} (\Delta_q^{ik}\sigma)(\Delta_q^{kj}\tau), \qquad i,j\in\{+,-\} \tag{6.36}$$

and similarly for Δ_s^{ij}. We show how to apply this to compute the difference operators acting on the symbol of the partial Laplacians

$$\mathcal{L}_s = -\frac{1}{2}(\partial_{s+}\partial_{s-}+\partial_{s-}\partial_{s+})-\partial_{s0}^2 \qquad \text{and} \qquad \mathcal{L}_q = -\frac{1}{2}(\partial_{q+}\partial_{q-}+\partial_{q-}\partial_{q+})-\partial_{q0}^2 \tag{6.37}$$

and thus the Laplacian $\mathcal{L}$. As $\sigma_{\mathcal{L}_q} = (-\frac{1}{2}(\sigma_+\sigma_- + \sigma_-\sigma_+) - \sigma_0^2)\otimes \mathrm{I}$ it follows that

$$\begin{aligned}
-2\Delta_q^{++}\sigma_{\mathcal{L}_q} &= (\Delta^{++}(\sigma_+\sigma_- + \sigma_-\sigma_+ + 2\sigma_0^2))\otimes \mathrm{I}_{l_1-l_2+1}\\
&= \Big((\Delta^{++}\sigma_+)\sigma_- + \sigma_+(\Delta^{++}\sigma_-) + \sum_{j\in\{+,-\}} (\Delta^{+j}\sigma_+)(\Delta^{j+}\sigma_-)+\\
&\qquad + (\Delta^{++}\sigma_-)\sigma_+ + \sigma_-(\Delta^{++}\sigma_+) + \sum_{j\in\{+,-\}} (\Delta^{+j}\sigma_-)(\Delta^{j+}\sigma_+)+\\
&\qquad + 2\big((\Delta^{++}\sigma_0)\sigma_0 + \sigma_0(\Delta^{++}\sigma_0 + \sum_{j\in\{+,-\}} (\Delta^{+j}\sigma_0)(\Delta^{j+}\sigma_0)\big)\Big)\otimes \mathrm{I}_{l_1-l_2+1}\\
&= \Big((\Delta^{+-}\sigma_-)(\Delta^{-+}\sigma_+) + 2\big((\Delta^{++}\sigma_0)\sigma_0 + \sigma_0(\Delta^{++}\sigma_0) + (\Delta^{++}\sigma_0)(\Delta^{++}\sigma_0)\big)\Big)\otimes \mathrm{I}\\
&= (\mathrm{I}_{l_1+l_2+1} - 2\sigma_0 + \frac{1}{2}\mathrm{I}_{l_1+l_2+1})\otimes \mathrm{I}_{l_1-l_2+1}
\end{aligned} \tag{6.38}$$

where we made use of $\Delta^{++}\sigma_+ = 0$ and $\Delta^{+-}\sigma_0 = 0$ to simplify the expression. The calculation for the remaining differences $\Delta_q^{\pm\pm}\sigma_{\mathcal{L}_q}$ and $\Delta_s^{\pm\pm}\sigma_{\mathcal{L}_s}$ is similar and due to $\Delta_s^{\pm\pm}\sigma_{\mathcal{L}_q} = \Delta_q^{\pm\pm}\sigma_{\mathcal{L}_s} = 0$ the formulas for differences applied to $\sigma_{\mathcal{L}}$ follow.

6.4.2 Symbolic calculus of pseudodifferential operators

A continuous linear operator A mapping $C^\infty(\mathrm{Spin}(4))$ to $\mathcal{D}'(\mathrm{Spin}(4))$ can be characterised by its matrix-valued full left-symbol

$$\sigma_A : \mathrm{Spin}(4)\times\Gamma_{\mathrm{Spin}(4)} \ni (\mathbf{s}, l_1, l_2) \mapsto \sigma_A(\mathbf{s}, l_1, l_2) \in \mathbb{C}^{d_{(l_1,l_2)}\times d_{(l_1,l_2)}} \tag{6.39}$$

defined by

$$\sigma_A(\mathbf{s}, l_1, l_2) := \xi_{(l_1,l_2)}(\mathbf{s})^*(A\,\xi_{(l_1,l_2)})(\mathbf{s}). \tag{6.40}$$

By definition

$$Af(\mathbf{s}) = \sum_{(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}} d_{(l_1,l_2)}\ \mathrm{tr}\left(\sigma_A(\mathbf{s}, l_1, l_2)\,\widehat{f}(l_1, l_2)\,\xi_{(l_1,l_2)}(\mathbf{s})\right) \tag{6.41}$$

holds true as $\mathcal{D}'$-convergent series. For A and σ_A related by (6.41) we write $A = \mathrm{Op}(\sigma_A)$.

In [21], [23] the Hörmander class $\Psi^k(G)$ of pseudo-differential operators of order k on a compact Lie group G was characterised in terms of these full symbols, also (ρ,δ)-classes $\Psi^k_{\rho,\delta}$ have been introduced there. We recall this and the resulting characterisations of ellipticity and hypoellipticity of operators for the particular case of Spin(4).

Adapted to the difference operators Δ^α, $\alpha \in \mathbb{N}_0^8$, we find left-invariant differential operators $\partial^{(\alpha)}$ of order $|\alpha|$ such that Taylor's formula (2.11) and Theorem 2.5 hold true. Although these differential operators play a crucial role for the calculus, we will use a different set of differential operators for our purposes. We use the multi-index notation

$$\partial^\beta = (\partial_{q+})^{\beta_1}(\partial_{q0})^{\beta_2}(\partial_{q-})^{\beta_3}(\partial_{s+})^{\beta_4}(\partial_{s0})^{\beta_5}(\partial_{s-})^{\beta_6}, \qquad \beta \in \mathbb{N}_0^6, \tag{6.42}$$

and point out that $\partial^\alpha \neq \partial^{(\alpha)}$ are different operators. However, $\partial^{(\alpha)}$ is a complex linear combination of ∂^β with $|\beta| \le |\alpha|$.

Theorem 6.12 ([23, Theorems 2.2 and 2.6])**.** *Let A be a linear continuous operator from $C^\infty(\mathrm{Spin}(4))$ to $\mathcal{D}'(\mathrm{Spin}(4))$ with matrix-valued full symbol $\sigma_A(\mathbf{s}, l_1, l_2) \in \mathbb{C}^{d_{(l_1,l_2)}\times d_{(l_1,l_2)}}$. Then $A \in \Psi^k(\mathrm{Spin}(4))$ if and only if*

$$\|\Delta^\alpha\partial^\beta\,\sigma_A(\mathbf{s}, l_1, l_2)\|_{\mathrm{op}} \le C_{\alpha,\beta}\langle\xi_{(l_1,l_2)}\rangle^{k-|\alpha|} \tag{6.43}$$

for all multi-indices α and β uniformly in $\mathbf{s} \in \mathrm{Spin}(4)$ and $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$.

Moreover, the rapid off-diagonal decay property of the symbol

$$|\Delta^\alpha\partial^\beta\sigma_A(\mathbf{s}, l_1, l_2)_{i,j}| \le C_{A,\alpha,\beta,N}(1 + |i-j|)^{-N}\,\langle\xi_{(l_1,l_2)}\rangle^{k-|\alpha|} \tag{6.44}$$

holds true uniformly in $\mathbf{s} \in \mathrm{Spin}(4)$, $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$ and $0 \le i, j < d_{(l_1,l_2)}$.

In order to obtain symbolic estimates for the (multiplicative) inverse of a symbol $\sigma = \sigma(\mathbf{s}, l_1, l_2)$ we will use the following result taking from [23]. It implies ellipticity and also hypoellipticity of certain operators within the calculus.

Lemma 6.13 ([23, Lem. 4.5])**.** *Let $k \ge k_0$ and $0 \le \delta < \rho \le 1$. Let the matrix symbol $\sigma(\mathbf{s}, l_1, l_2)$ satisfy the (ρ,δ)-estimates of order k*

$$\|\Delta^\alpha\partial^\beta\sigma(\mathbf{s}, l_1, l_2)\|_{\mathrm{op}} \le C_{\alpha,\beta}\,\langle\xi_{(l_1,l_2)}\rangle^{k-\rho|\alpha|+\delta|\beta|} \tag{6.45}$$

for all multi-indices α and β and uniformly in $\mathbf{s} \in \mathrm{Spin}(4)$ and $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$. Assume further that $\sigma(\mathbf{s}, l_1, l_2)$ is invertible for all $\mathbf{s} \in \mathrm{Spin}(4)$ and $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$

and satisfies

$$\|\sigma(\mathbf{s},l_1,l_2)^{-1}\|_{\mathrm{op}} \le C\langle\xi_{(l_1,l_2)}\rangle^{-k_0} \tag{6.46}$$

for all $\mathbf{s}\in\mathrm{Spin}(4)$ *and* $(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}$ *and if* $k_0<k$ *in addition that*

$$\|\sigma(\mathbf{s},l_1,l_2)^{-1}\big(\Delta^\alpha\partial^\beta\sigma(\mathbf{s},l_1,l_2)\big)\|_{\mathrm{op}} \le C\,\langle\xi_{(l_1,l_2)}\rangle^{-\rho|\alpha|+\delta|\beta|} \tag{6.47}$$

for all $\mathbf{s}\in\mathrm{Spin}(4)$ *and* $(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}$*.*

Then the symbol $\sigma^{-1}(\mathbf{s},l_1,l_2)=\sigma(\mathbf{s},l_1,l_2)^{-1}$ *satisfies the* (ρ,δ)*-estimates of order* $-k_0$

$$\|\Delta^\alpha\partial^\beta\sigma^{-1}(\mathbf{s},l_1,l_2)\|_{\mathrm{op}} \le C'_{\alpha,\beta}\,\langle\xi_{(l_1,l_2)}\rangle^{-k_0-\rho|\alpha|+\delta|\beta|} \tag{6.48}$$

for all multi-indices α *and* β *and uniformly in* $\mathbf{s}\in\mathrm{Spin}(4)$ *and* $(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}$*.*

As consequence, one obtains combined with Theorem 2.5 and the standard construction of parametrices within the calculus characterisations of ellipticity and local hypoellipticity. We recall these theorems before giving examples on Spin(4) later on. First we give a characterisation of the elliptic operators in $\Psi^k(\mathrm{Spin}(4))$ in terms of their global symbols.

Theorem 6.14. *[23, Thm. 4.1] An operator* $A\in\Psi^k(\mathrm{Spin}(4))$ *is elliptic if and only if its matrix valued symbol* $\sigma_A(\mathbf{s},l_1,l_2)$ *is invertible for all but finitely many* $(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}$ *and for all such* (l_1,l_2) *satisfies*

$$\|\sigma_A^{-1}(\mathbf{s},l_1,l_2)\|_{\mathrm{op}} \le C\,\langle\xi_{(l_1,l_2)}\rangle^{-k} \tag{6.49}$$

uniformly in $\mathbf{s}\in\mathrm{Spin}(4)$*.*

We say that a symbol σ_A belongs to the symbol class $S^k_{\rho,\delta}(\mathrm{Spin}(4))$ if estimate (6.45) holds true for all multi-indices α, β and uniformly in the remaining variables. We recall the existence of parametrices for operators with such symbols under suitable conditions on the multiplicative inverse of the symbol. It is again taken from [23] and provides an analogue of the well-known hypoellipticity result of Hörmander [15], requiring conditions on lower order terms of the symbol.

Theorem 6.15 ([23, Thm. 5.1.])**.** *Let* $k\ge k_0$ *and* $1\ge\rho>\delta\ge 0$*. Let* $A\in\mathrm{Op}(S^k_{\rho,\delta}(\mathrm{Spin}(4)))$ *be a pseudo-differential operator with the matrix-valued symbol* $\sigma_A=\sigma_A(\mathbf{s},l_1,l_2)\in S^k_{\rho,\delta}(\mathrm{Spin}(4))$ *which is invertible for all but finitely many* $(l_1,l_2)\in\Gamma_{\mathrm{Spin}(4)}$ *and satisfies for all such* (l_1,l_2)

$$\|\sigma_A(\mathbf{s},l_1,l_2)^{-1}\|_{\mathrm{op}} \le C\,\langle\xi_{(l_1,l_2)}\rangle^{-k_0} \tag{6.50}$$

uniformly in $\mathbf{s}\in\mathrm{Spin}(4)$*. Assume further that*

$$\|\,\sigma_A^{-1}(\mathbf{s},l_1,l_2)\big(\Delta^\alpha\partial^\beta\,\sigma_A(\mathbf{s},l_1,l_2)\big)\,\|_{\mathrm{op}} \le C\,\langle\xi_{(l_1,l_2)}\rangle^{-\rho|\alpha|+\delta|\beta|} \tag{6.51}$$

for all multi-indices α *and* β*, all* $\mathbf{s}\in\mathrm{Spin}(4)$*, and all but finitely many* (l_1,l_2)*. Then there exists an operator* $B\in\mathrm{Op}(S^{-k_0}_{\rho,\delta}(\mathrm{Spin}(4))$ *such that* $AB-I$ *and* $BA-I$ *map* $\mathcal{D}'(\mathrm{Spin}(4))$ *to* $C^\infty(\mathrm{Spin}(4))$*.*

Consequently, A *is locally hypoelliptic and*

$$\operatorname{sing\,supp} Au = \operatorname{sing\,supp} u \quad \textit{for all } u\in\mathcal{D}'(\mathrm{Spin}(4)). \tag{6.52}$$

Note, that the parametrix B provided by this theorem satisfies the subelliptic estimates $\|Bf\|_{H^r} \lesssim \|f\|_{H^{r+k_0}}$ with k_0 independent of $r \in \mathbb{R}$.

7 Examples

To conclude this paper we provide examples of operators on Spin(4) having interesting ellipticity and hypoellipticity properties.

Example 7.1 (See [23, Example 5.2]). Consider the operator $\mathrm{D}_{3q}+c$, with $c \in \mathbb{C}$. Using the identity $\mathrm{D}_{3q} = -\frac{\mathbf{i}}{2}\partial_{q0}$ and the symbols given in Table 1, we have

$$\begin{aligned} \sigma_{D_{3q}+c}(l_1,l_2) &= c - \frac{\mathbf{i}}{2}\sigma_0(\frac{l_1+l_2}{2}) \otimes \mathrm{I}_{l_1-l_2+1} \\ &= \Big([c - \frac{\mathbf{i}}{2}(l_1+l_2-2j)]\delta_{ij}\Big)_{(l_1+l_2+1)\times(l_1+l_2+1)} \otimes \mathrm{I}_{l_1-l_2+1} \end{aligned} \tag{7.1}$$

and this symbol is invertible for all (l_1,l_2)-representations if and only if $\mathbf{i}c \notin \frac{1}{2}\mathbb{Z}$. The inverse satisfies $(\mathrm{D}_{3q}+c)^{-1} \in \mathrm{Op}(S^0_{0,0}(\mathrm{Spin}(4)))$ since

$$\begin{aligned} \Delta_q^{--}\sigma_{(\mathrm{D}_{3q}+c)^{-1}} &= \sigma_{(\mathrm{D}_{3q}+c+\frac{\mathbf{i}}{2})^{-1}}, \\ \Delta_q^{++}\sigma_{(\mathrm{D}_{3q}+c)^{-1}} &= \sigma_{(\mathrm{D}_{3q}+c-\frac{\mathbf{i}}{2})^{-1}}, \\ \Delta_q^{+-}\sigma_{(\mathrm{D}_{3q}+c)^{-1}} &= \Delta_q^{-+}\sigma_{(\mathrm{D}_{3q}+c)^{-1}} = 0 \end{aligned} \tag{7.2}$$

together with

$$\Delta_s^{--}\sigma_{(\mathrm{D}_{3q}+c)^{-1}} = \Delta_s^{++}\sigma_{(\mathrm{D}_{3q}+c)^{-1}} = \Delta_s^{+-}\sigma_{(\mathrm{D}_{3q}+c)^{-1}} = \Delta_s^{-+}\sigma_{(\mathrm{D}_{3q}+c)^{-1}} = 0 \tag{7.3}$$

by using Corollary 6.10. Therefore, the operators $\mathrm{D}_{3q}+c$ satisfy subelliptic estimates with loss of one derivative and are thus globally hypoelliptic.

Example 7.2. Any left-invariant vector field on Spin(4) can be conjugated by an inner automorphism to a constant multiple of

$$\mathrm{X} = \mathrm{D}_{3q} + \varkappa \mathrm{D}_{3s} \tag{7.4}$$

with parameter $\varkappa \in \mathbb{R}$. We investigate this in dependence of $\varkappa$. Considering the symbol of the operator we obtain

$$2\mathbf{i}\,\sigma_{\mathrm{X}}(l_1,l_2) = \sigma_0(\frac{l_1+l_2}{2}) \otimes \mathrm{I}_{l_1-l_2+1} + \varkappa \mathrm{I}_{l_1+l_2+1} \otimes \sigma_0(\frac{l_1-l_2}{2}), \tag{7.5}$$

where $\sigma_0(\frac{m}{2}) = \frac{1}{2}(m-2j)\delta_{ij} = \mathrm{diag}(-\frac{m}{2}, -\frac{m}{2}+1, \ldots, \frac{m}{2}-1, \frac{m}{2})$. This matrix is diagonal with entries given as sums

$$(\frac{l_1+l_2}{2} - j) + \varkappa(\frac{l_1-l_2}{2} - k) \tag{7.6}$$

for $0 \le j \le l_1 + l_2$ and $0 \le k \le l_1 - l_2$. Both terms vanish for some admissible k and l if both $l_1 + l_2$ and $l_1 - l_2$ are even. Thus, there are always infinitely many $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$ for which $\sigma_{\mathrm{X}}(l_1, l_2)$ is not invertible and none of the left-invariant vector fields on Spin(4) can be globally hypoelliptic.

This is in contrast to tori $\mathbb{T}^n$, where hypoelliptic left-invariant vector fields are known to exist, see [12] and [6]. In [17] a new notion of hypoellipticity was introduced. We say that X is hypoelliptic modulo ker X, if for $f \in \mathcal{D}'$ with $\mathrm{X}f \in C^\infty$ we find an element $g \in \ker \mathrm{X}$ such that $f - g \in C^\infty$.

The vector field X is globally hypoelliptic modulo ker X if (and only if) $\varkappa$ is irrational and non-Liouville. Indeed, such a number $\varkappa$ can only be approximated to finite order by rationals (cf. [14, §11.7]) and hence

$$\left| \left(\frac{l_1 + l_2}{2} - j \right) + \varkappa \left(\frac{l_1 - l_2}{2} - k \right) \right| \ge c \left(\left| \frac{l_1 + l_2}{2} - j \right| + \left| \frac{l_1 - l_2}{2} - k \right| \right)^{-\mu} \tag{7.7}$$

with constants c and μ for all $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)}$ and $0 \le j \le l_1 + l_2$ and $0 \le k \le l_1 - l_2$ such that $(\frac{l_1+l_2}{2} - j) + \varkappa(\frac{l_1-l_2}{2} - k) \ne 0$. The term in parantheses on the right hand side is clearly bounded by $\langle \xi_{(l_1,l_2)} \rangle$ and thus we obtain a lower bound on the symbol by a multiple of $\langle \xi_{(l_1,l_2)} \rangle^{-\mu}$. This implies the subelliptic estimate

$$\|f - g\|_{\mathrm{H}^r(\mathrm{Spin}(4))} \le C \|\mathrm{X}f\|_{\mathrm{H}^{r+\mu-1}(\mathrm{Spin}(4))}, \tag{7.8}$$

where the Fourier coefficients of g equal those of f on the zeros of σ_{X} and in turn the global hypoellipticity modulo ker X follows.

Example 7.3. Next, we consider the (partial) Laplacians

$$\mathcal{L}_q = \partial_{q+}\partial_{q-} + \partial_{q-}\partial_{q+} + \partial_{q0}^2 \tag{7.9}$$

and

$$\mathcal{L}_s = \partial_{s+}\partial_{s-} + \partial_{s-}\partial_{s+} + \partial_{s0}^2 \tag{7.10}$$

together with the Laplacian $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_q$. Their symbols are given by

$$\sigma_{\mathcal{L}_q}(l_1, l_2) = -\frac{(l_1 + l_2)(l_1 + l_2 + 2)}{4}\mathrm{I}, \qquad \sigma_{\mathcal{L}_s}(l_1, l_2) = -\frac{(l_1 - l_2)(l_1 - l_2 + 2)}{4}\mathrm{I} \tag{7.11}$$

and

$$\sigma_{\mathcal{L}}(l_1, l_2) = -\frac{l_1^2 + 2l_1 + l_2^2}{2}\mathrm{I} \tag{7.12}$$

(see also Theorem 6.4). While the latter one is elliptic, the symbols of the two partial Laplacians $\mathcal{L}_q$ and $\mathcal{L}_s$ vanish for $l_1 = -l_2$ and for $l_1 = l_2$, respectively.

Clearly any combination $\mathcal{L}_q + \varkappa\mathcal{L}_s$ with $\varkappa > 0$ is elliptic. For $\varkappa < 0$ the behaviour of this operator depends on number theoretic properties of $\varkappa$ as the next example shows.

Example 7.4. The ultra-hyperbolic operator

$$\mathcal{L}_q - \varkappa\mathcal{L}_s \tag{7.13}$$

with $\varkappa > 0$ is hypoelliptic for irrational non-Liouville numbers $\varkappa$. To see this, we consider its symbol

$$\sigma_{\mathcal{L}_q-\varkappa\mathcal{L}_s}(l_1,l_2) = -\frac{(l_1+l_2)(l_1+l_2+2) - \varkappa(l_1-l_2)(l_1-l_2+2)}{4}\mathrm{I}. \tag{7.14}$$

This is clearly invertible for irrational $\varkappa$ and $(l_1,l_2) \neq (0,0)$. Furthermore, if $\varkappa$ is non-Liouville we find a maximal approximation order and hence there is a number μ such that $\varkappa$ can not be rationally approximated to this order (see e.g. [14, Chapter §11.4 und §11.7]) and hence there is a constant $C > 0$ such that there are at most finitely many couples (l_1,l_2) with $l_1 \neq l_2$ and

$$\left|\varkappa - \frac{(l_1+l_2)(l_1+l_2+2)}{(l_1-l_2)(l_1-l_2+2)}\right| < \frac{1}{C((l_1-l_2)(l_1-l_2+2))^{\mu}}. \tag{7.15}$$

Thus, apart from those (l_1,l_2), it follows that

$$\|\sigma^{-1}_{\mathcal{L}_q-\varkappa\mathcal{L}_s}(l_1,l_2)\|_{\mathrm{op}} \leq 4C((l_1-l_2)(l_1-l_2+2))^{\mu-1} \leq \tilde{C}\langle\xi_{(l_1,l_2)}\rangle^{2\mu-2}, \qquad l_1 \neq l_2 \tag{7.16}$$

together with the estimate

$$\|\sigma^{-1}_{\mathcal{L}_q-\varkappa\mathcal{L}_s}(l,l)\|_{\mathrm{op}} \leq \frac{1}{l(l+1)}, \qquad l_1 = l_2 = l \neq 0. \tag{7.17}$$

Thus, the operator satisfies the subelliptic estimate

$$\|f\|_{\mathrm{H}^r(\mathrm{Spin}(4))} \leq C\|\mathcal{L}_q f - \varkappa\mathcal{L}_s f\|_{\mathrm{H}^{r+2\mu-2}(\mathrm{Spin}(4))} \tag{7.18}$$

between Sobolev spaces with loss of 2μ derivatives. We leave it open whether the parametrix to this operator is pseudodifferential in our calculus.

Example 7.5. We consider the following sub-Laplacian on Spin(4)

$$\begin{aligned}\mathcal{L}_{\mathrm{sub}} &= \mathrm{D}^2_{1q} + \mathrm{D}^2_{2q} + \mathrm{D}^2_{1s} + \mathrm{D}^2_{2s} + \mathrm{D}^2_{3s} \\ &= -\frac{1}{2}(\partial_{q+}\partial_{q-} + \partial_{q-}\partial_{q+} + \partial_{s+}\partial_{s-} + \partial_{s-}\partial_{s+}) - \partial^2_{s0}.\end{aligned} \tag{7.19}$$

with symbol given by the diagonal matrix

$$\sigma_{\mathcal{L}_{\mathrm{sub}}}(l_1,l_2) = \left(\left(\frac{(l_1+l_2-2i)^2 - (2l_1^2+4l_1+2l_2^2)}{4}\right)\delta_{ij}\right)_{(l_1+l_2+1)\times(l_1+l_2+1)} \otimes \mathrm{I}_{l_1-l_2+1}. \tag{7.20}$$

As all the entries are non-zero, we conclude that the operator $\mathcal{L}_{\mathrm{sub}}$ has trivial null space. As symbol of a second order differential operator it belongs to the class $S^2(\mathrm{Spin}(4))$ and thus satisfies the norm estimate

$$\|\Delta^{\alpha}\,\sigma_{\mathcal{L}_{\mathrm{sub}}}(l_1,l_2)\|_{\mathrm{op}} \leq C_\alpha\,\langle\xi_{(l_1,l_2)}\rangle^{2-|\alpha|} \tag{7.21}$$

for all multi-indices α. The operator is not elliptic, but it is subelliptic with loss of one derivative. We are going to show this next by appealing to Theorem 6.14. The pointwise inverse of the symbol satisfies the norm estimate

$$\|\sigma_{\mathcal{L}_{\mathrm{sub}}}^{-1}(l_1,l_2)\|_{\mathrm{op}} = \frac{4}{l_1^2+2l_1(2-l_2)+l_2^2} \leq C\,\langle\xi_{(l_1,l_2)}\rangle^{-1} \tag{7.22}$$

for all admissible (l_1,l_2) with $C=\sqrt{7}$ being the sharp constant. In particular, we obtain the subelliptic estimate

$$\|f\|_{\mathrm{H}^r(\mathrm{Spin}(4))} \leq C\|\mathcal{L}_{\mathrm{sub}}f\|_{\mathrm{H}^{r-1}(\mathrm{Spin}(4))} \tag{7.23}$$

for all Sobolev regularities $r\in\mathbb{R}$.

The condition (6.51) of Theorem 6.14 is satisfied with $\rho=\frac{1}{2}$ and for $\alpha=0$ or $|\alpha|\geq 2$. Hence, it remains to check the case of first order differences. For this we use $\Delta_q^{--}\sigma_{\mathcal{L}_{\mathrm{sub}}}=\Delta_q^{++}\sigma_{\mathcal{L}_{\mathrm{sub}}}=-\frac{1}{2}\mathrm{I}_d$ together with

$$\begin{aligned}\|\sigma_{\mathcal{L}_{\mathrm{sub}}}^{-1}(l_1,l_2)[\Delta_q^{+-}\sigma_{\mathcal{L}_{\mathrm{sub}}}(l_1,l_2)]\|_{\mathrm{op}} &= \|\sigma_{\mathcal{L}_{\mathrm{sub}}}^{-1}(l_1,l_2)\sigma_{\partial_{q+}}(l_1,l_2)]\|_{\mathrm{op}}\\ &= \frac{4\sqrt{l_1+l_2}}{l_1^2+2l_1(2-l_2)+l_2^2} \leq C\,\langle\xi_{(l_1,l_2)}\rangle^{-\frac{1}{2}}\end{aligned} \tag{7.24}$$

and an analogous statement for Δ_q^{-+}. Concerning the differences in s we can make use of the relations $\Delta_s^{--}\sigma_{\mathcal{L}_{\mathrm{sub}}}=-\sigma_{\partial_{s0}}-\frac{3}{4}\mathrm{I}_d$, $\Delta_s^{++}\sigma_{\mathcal{L}_{\mathrm{sub}}}=\sigma_{\partial_{s0}}-\frac{3}{4}\mathrm{I}_d$, $\Delta_s^{+-}\sigma_{\mathcal{L}_{\mathrm{sub}}}=-\sigma_{\partial_{s+}}$, and $\Delta_s^{--}\sigma_{\mathcal{L}_{\mathrm{sub}}}=-\sigma_{\partial_{s-}}$ directly arising from the partial Laplacian.

Hence, $\mathcal{L}_{\mathrm{sub}}$ has a pseudo-differential parametrix $\mathcal{L}_{\mathrm{sub}}^{\natural}\in\mathrm{Op}(S^{-1}_{\frac{1}{2},0}(\mathrm{Spin}(4)))$.

Further examples can be constructed along the lines of [23, Section 5]. As proofs are similar, we only provide the results.

Example 7.6. The following analogue of the heat operator on Spin(4)

$$\mathrm{H}=\mathrm{D}_{3q}-\mathrm{D}_{1q}^2-\mathrm{D}_{2q}^2-\mathrm{D}_{1s}^2-\mathrm{D}_{2s}^2-\mathrm{D}_{3s}^2=\mathrm{D}_{3q}-\mathcal{L}_{\mathrm{sub}} \tag{7.25}$$

has a parametrix $\mathrm{H}^{\natural}\in\mathrm{Op}(S^{-1}_{\frac{1}{2},0}(\mathrm{Spin}(4)))$ and, consequently, it satisfies the sub-elliptic estimate

$$\|f\|_{\mathrm{H}^r(\mathrm{Spin}(4))} \leq C\|\mathrm{H}f\|_{\mathrm{H}^{r-1}(\mathrm{Spin}(4))} \tag{7.26}$$

for all Sobolev regularities $r\in\mathbb{R}$.

Example 7.7. The operators

$$\mathrm{S}_{\pm}=\pm\mathbf{i}\mathrm{D}_{3q}-\mathcal{L}_{\mathrm{sub}}. \tag{7.27}$$

are analogues of the Schrödinger operator on Spin(4). These operators have non-trivial distributions in their null-spaces and can thus be not hypoelliptic.

However, the operators $\mathrm{S}_{\pm}+c$ with $c\in\mathbb{C}\setminus\mathbb{R}$ are globally hypoelliptic.

Our last example shows a differential operator with non-constant coefficients which cannot be written as tensor product of Spin(3) operators.

Example 7.8. Let us consider the operator

$$\mathrm{A} = a(\mathbf{s})(\mathrm{D}_{1q}^2 + \mathrm{D}_{2q}^2) + b(\mathbf{s})(\mathrm{D}_{1s}^2 + \mathrm{D}_{2s}^2) \tag{7.28}$$

with $a, b \in C^\infty(\mathrm{Spin}(4))$ such that $\Re(\theta a(\mathbf{s})) \geq c_a > 0$ and $\Re(\theta b(\mathbf{s})) \geq c_b > 0$ for all $\mathbf{s} \in \mathrm{Spin}(4)$ and for a fixed complex number $\theta \in \mathbb{C} \setminus \{0\}$. The full symbol of the operator A can be written as the Kronecker sum

$$\sigma_{\mathrm{A}}(\mathbf{s}, l_1, l_2) = C \oplus D = C \otimes \mathrm{I}_{l_1 - l_2 + 1} + \mathrm{I}_{l_1 + l_2 + 1} \otimes D \tag{7.29}$$

with

$$\begin{aligned} C &= \left(a(\mathbf{s}) \left(\frac{(l_1 + l_2 - 2j)^2}{4} - \frac{(l_1 + l_2)(l_1 + l_2 + 2)}{4} \right) \delta_{ij} \right)_{(l_1+l_2+1)\times(l_1+l_2+1)}, \\ D &= \left(b(\mathbf{s}) \left(\frac{(l_1 - l_2 - 2j)^2}{4} - \frac{(l_1 - l_2)(l_1 - l_2 + 2)}{4} \right) \delta_{ij} \right)_{(l_1-l_2+1)\times(l_1-l_2+1)}. \end{aligned} \tag{7.30}$$

Since $\Re(\theta a)$ and $\Re(\theta b)$ are both positive and $(l - 2j)^2 - l(l+2) \leq -2l$ and $0 \leq j \leq l$ and every $l \in \mathbb{N}_0$, we conclude that the symbol is invertible for all $\mathbf{s} \in \mathrm{Spin}(4)$ and all $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)} \setminus \{(0,0)\}$ with eigenvalues satisfying

$$\Re(\theta \operatorname{Spec} \sigma_A(\mathbf{s}, l_1, l_2)) \geq \Re(\theta a(\mathbf{s}))(l_1 + l_2) + \Re(\theta b(\mathbf{s}))(l_1 - l_2), \tag{7.31}$$

where $\operatorname{Spec} \sigma_A$ denotes the spectrum of the matrix σ_A. As symbol of a second order differential operator we also know that $\sigma_{\mathrm{A}} \in S^2(\mathrm{Spin}(4))$ and it remains to ask for symbolic properties of $\sigma_{\mathrm{A}}^{-1}(\mathbf{s}, l_1, l_2)$ for $(l_1, l_2) \neq (0,0)$. As the absolutely smallest eigenvalue corresponds to the operator norm of the inverse, we observe

$$\|\sigma_{\mathrm{A}}(\mathbf{s}, l_1, l_2)^{-1}\|_{\mathrm{op}} = \frac{2}{|c_a(l_1 + l_2) + c_b(l_1 - l_2)|} \leq C \, \langle \xi_{(l_1, l_2)} \rangle^{-1} \tag{7.32}$$

for all $(l_1, l_2) \in \Gamma_{\mathrm{Spin}(4)} \setminus \{(0,0)\}$. Hence, condition (6.51) of Theorem 6.15 holds with $\rho = \frac{1}{2}$ and $\delta = 0$. Thus, the operator A has a pseudo-differential parametrix $\mathrm{A}^\natural \in \mathrm{Op}(S^{-1}_{\frac{1}{2},0}(\mathrm{Spin}(4)))$ and therefore satisfies the subelliptic estimate

$$\|f\|_{H^s(\mathrm{Spin}(4))} \leq C \|Af\|_{H^{s-1}(\mathrm{Spin}(4))} \tag{7.33}$$

for all $s \in \mathbb{R}$.

A

We comment on some choices of bases used in this paper and their relation to the irreducible representations constructed here.

A.1 Representations of $\mathbb{S}^3$ and identification of $\mathbb{C}^2$ with $\mathbb{H}$

The representations of the quaternionic unit sphere $\mathbb{S}^3 \subset \mathbb{H}$ quoted from [13] fix an isomorphism between $\mathbb{S}^3$ and SU(2). This choice is reflected in our identification between $\mathbb{H}$ and $\mathbb{C}^2$.

A.2 Representations of Spin(3) and the identification of $\mathbb{R}^+_{0,3}$ with $\mathbb{C}^2$ and $\mathbb{H}$

The representation with weight $(\frac{1}{2})$ fixes an isomorphism of Spin(3) and SU(2). This is made explicit by the choice of isomorphism between $\mathbb{R}^+_{3,0}$ and $\mathbb{C}^2$ and thus also between $\mathbb{R}^+_{3,0}$ and $\mathbb{H}$.

A.3 Representations of Spin(4) and choice of bases for the spin modules $\mathcal{S}^\pm_4$

The representation $(\frac{1}{2}, \frac{1}{2})$ specifies the choice of the basis for $\mathcal{S}^+_4$ such that the left multiplication with spinors on $\mathcal{S}^+_4$ is given in this basis by the SU(2) representation of Spin(3).

The $(1, 0)$ representation of Spin(4) gives an embedding of Spin(4) into unitary 4×4 matrices or better into unitary linear maps on $\mathbb{C}^2\otimes\mathbb{C}^2$ and thus $\mathbb{H}\otimes\mathbb{H}$. Fixing the form of the $(1, 0)$ representation specifies an isomorphism between the even Clifford algebra $\mathbb{R}^+_{0,4}$ and $\mathbb{H} \otimes \mathbb{H}$ and gives thus an identification of basis elements.

Our choice was based on the identification of the representations $(1, 0) = (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, -\frac{1}{2})$.

A.4 Fixing the bases in representation spaces

The bases in the spin modules $\mathcal{H}_l(\mathbb{R}^3)$ and $\mathcal{M}_l(\mathbb{R}^3)$ were done in such a way that the neutral differential operator ∂_0 is represented by a diagonal matrix. Similarly for $\mathcal{H}_{(l_1,l_2)}(\mathbb{R}^4)$ and $\mathcal{M}_{(l_1,l_2)}(\mathbb{R}^4)$ and the neutral operator $\partial_{p,0} \otimes \partial_{s,0}$.

A.5 Choice of difference operators

The choice of the admissible selection of difference operators on Spin(3) and on Spin(4) is based on the matrix coefficients of the root representations. For Spin(3) this is the $(\frac{1}{2})$-representation and for Spin(4) the representations to weights $(\frac{1}{2}, \pm\frac{1}{2})$. This is compatible with the choices in [23] and [25].

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