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Moments of losses during busy-periods of regular and nonpreemptive oscillating $M^X/G/1/n$ systems

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Abstract This work addresses loss characteristics associated to busy-periods of regular and nonpreemptive oscillating $M^X/G/1/n$ systems. By taking advantage of the Markov regenerative structure of the number of customers in the system and resorting to results on moments of compound mixed Poisson distributions, it proposes a fast and easy to implement recursive procedure to compute integer moments of the number of customers lost in busy-periods initiated with multiple customers in the system.

Keywords Batch arrivals · Oscillating system · Busy-periods · Customer losses · Moments · Compound mixed-Poisson

1 Introduction

The goal of this paper is to compute integer moments for the number of customers lost during busy-periods of regular and nonpreemptive oscillating $M^X/G/1/n$ systems. These are queueing systems at which customers arrive in batches according to a compound Poisson process with rate λ. The sequences of batch sizes and batch interarrival times are independent,

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and the systems have finite capacity n, including the customer in service—if any. The batch sizes, $X_1, X_2, \ldots$, are independent and identically distributed to a random variable X having probability mass function $(f_l)_{l\in\mathbb{N}_+}$, with finite mean $\bar{f}$, where $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$. The partial blocking customer acceptance policy is considered (see, e.g., Laxmi and Gupta 2000), so that the batches which upon arrival are unable to find enough space in the buffer to accommodate all the customers of the batch are partially rejected. Specifically, if a batch of size l arrives and finds i customers in the system, then the first $\min(l, n-i)$ customers of the batch enter the system and the remaining customers of the batch (if any) are blocked. Customers accepted in the system are served by a single server, in a first come first served manner. In contrast with the regular queueing systems, in the oscillating queueing systems customer service times are not independent of each other, depending on the evolution of the number of customers in the system, with impact on the service characteristics, as described below.

Due to the importance of customer losses for the analysis of queueing systems, customer loss characteristics in finite capacity Markovian arrival queues have been addressed in several works, including Abramov (1997, 2011), Al Hanbali and Boxma (2010), Ferreira et al. (2013), Pacheco and Ribeiro (2006, 2008), Peköz (1999), Peköz et al. (2003), Righter (1999), Wolff (2002). Their analysis during busy-period cycles, i.e., during effective system utilization periods, is relevant from the operator's point of view, providing crucial information for the management of congested systems. In this respect, Abramov (1997), Wolff (2002), and Peköz et al. (2003) showed that, in $M^X/G/1/n$ queues with unitary traffic intensity ($\rho = 1$), the mean number of customers lost during a busy-period initiated with i customers in the system is i (invariant with n). In addition, case $\rho < 1$ ($\rho > 1$), the mean number of customers lost in such busy-period is smaller (greater) than i, being a decreasing (increasing) function of n, for $n \geq 1$. The mean invariance property does not extends to general arrival processes with unitary traffic intensity. In particular, Peköz et al. (2003) showed that, in a $GI/M/1/n$ system, if the mean number of losses in a busy-period initiated with a single customer is unitary and invariant on the system capacity, then the system must be a $M/M/1/n$ queue with $\rho = 1$. Moments of higher order for the number of customers lost in busy-periods of $M/G/1/n$ and $GI/M/1/n$ systems were derived in Peköz (1999). Righter (1999) showed that customer loss probabilities in busy-periods initiated with a single customer in a $M/G/1/1$ system could be used to recursively compute the mean number of losses in busy-periods of $M/G/1/m$ systems, $m \geq 1$, all with the same parameters except the queue capacity. Pacheco and Ribeiro (2006, 2008) recursively compute the probability of consecutive losses in busy-periods of $GI/M(m)//n$ and regular and oscillating $M^X/G/1/n$ queueing systems. Recently, Ferreira et al. (2013) evaluated the probability mass function of the number of losses in busy-periods of $M^X/G/1/n$ queues.

In regular $M^X/G/1/n$ systems, customer service times are independent random variables, identically distributed to a random variable S_A having (general) distribution function $A(\cdot)$ with mean μ^{-1}, and are independent of the compound Poisson customer arrival process. By contrast, in oscillating systems the distribution of customer service times depends on the evolution of the number of customers in the system. This evolution determines the operating phase, 1 or 2, of the system at each time instant, according to crossings of two barriers: a lower barrier a and an upper barrier b, with $0 \leq a < b \leq n$. In particular, when the system is empty it is (operating) under phase 1, and when the system is in phase 1 it remains in phase 1 until the number of customers in the system reaches or goes above the upper barrier b. After that instant, and once the customer in service ends being served, the service is upgraded (in a nonpreemptive manner) changing to operating phase 2. The system then remains in phase 2 until the subsequent moment at which the number of customers in the system falls to or below the lower barrier a. After that instant, the service is downgraded in a nonpreemptive manner, changing again to operating phase 1. Service times initiated in phase 1 have duration S_{A_1} with distribution function A_1 and mean μ_1^{-1}, and services initiated in phase 2 have duration S_{A_2} with distribution function A_2 and mean μ_2^{-1}, usually smaller than μ_1^{-1}. In this paper, we denote by M^X/G_1–$G_2/1/\mathbf{n}$ system, with $\mathbf{n} = (n, a, b)$, the oscillating systems with the barriers a and b.

Oscillating systems with one or more barriers are found in the queueing literature. We refer, e.g., to Choi and Choi (1996) and Choi et al. (1999) that propose the use of oscillating systems with a single barrier (threshold) to analyse cell-discarding schemes for voice packets in ATM networks, by allowing dropping of low-priority (less significant) bits of information during congestion periods. Sriram et al. (1991) modeled a cell discarding scheme as a M^X/D_1–$D_2/1/(n, b-1, b)$ system with D denoting a deterministic distribution. Oscillating systems with two barriers, in which a hysteresis (retardation) in service type changes is proposed to avoid costly oscillations around a single threshold, are considered, e.g., in Bratiychuk and Chydzinski (2003), Chydzinski (2002, 2004), and Federgruen and Tijms (1980). We stress that other queueing models identified in the literature as queueing systems with state dependent parameters (cf. Dshalalow 1997) can be modeled as oscillating systems; see, for instance, Bahary and Kolesar (1972). Similar in the spirit of improving the cost/performance ratio of a system, reacting to changes in workload through the use of thresholds, oscillating systems differ from threshold queues with hysteresis (see, e.g., Golubchik and Lui 2002; Sriram and Lucantoni 1989; Yadin and Naor 1967, and references therein) in the fact that they maintain a single server, although with changing service time distributions.

In the paper, we propose to investigate moments of the number of customers lost in busy-periods of regular and nonpreemptive oscillating queueing systems. We consider multi busy-periods, i.e., busy-periods initiated with multiple customers in the system. Specifically, by a busy-period initiated by i customers we mean the period of time that starts at an arrival instant that makes the system stay with i customers, and ends at the first subsequent time at which the system becomes empty, with a customer initiating service after the arrival instant. This definition is in line with that of remaining busy-period from state i given in Harris (1971), of residual busy-period provided in Al Hanbali (2011), and of busy-period initiated with i customers considered in Peköz et al. (2003).

To our knowledge, apart the particular cases of queues with single arrivals ($M/G/1/n$ and $GI/M/1/n$) and queues with unitary traffic intensity ($\rho = 1$), no formulas are available in the literature to compute moments of the mean number of customers lost in busy-periods of regular or oscillating $M^X/G/1/n$ systems. The main contribution of this work goes in this direction, providing an efficient recursive procedure to compute integer moments for the number of customers lost during busy-periods initiated by multiple customers in regular $M^X/G/1/n$ and nonpreemptive oscillating M^X/G_1–$G_2/1/\mathbf{n}$ queueing systems.

The remainder of the paper is organized as follows. In Sect. 2 we address the recursive computation of the moments of the number of customers lost in multi busy-periods of regular $M^X/G/1/n$ systems, and in Sect. 3 we extend the results of Sect. 2 to nonpreemptive oscillating M^X/G_1–$G_2/1/\mathbf{n}$ systems. Finally, we illustrate our results by means of some numerical examples in Sect. 4 and draw some conclusions in Sect. 5.

2 Moments of losses in a busy-period of regular systems

In this section, we propose a general procedure for the recursive evaluation of the moments of the number of customers lost in i-busy-periods of regular $M^X/G/1/n$ systems that does not require the computation of the loss probabilities in busy-periods.

We let $L_i^{(n)}$ denote the number of customers lost during a busy-period initiated with i customers in the system. Due to the Markov regenerative property of $M^X/G/1/n$ systems at service completion epochs, $L_i^{(n)}$ satisfies (c.f. Ferreira et al. 2013)

$$L_i^{(n)} \stackrel{\mathrm{d}}{=} L_{i-1}^{(n-1)} \oplus L_1^{(n)}, \qquad 1 \le i \le n \tag{1}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution, $\oplus$ denotes the sum of independent random variables, and $L_0^{(m)} = 0$. As a consequence, the number of losses in an i-busy-period of an $M^X/G/1/n$ system can be expressed as a direct function of the distribution of the number of customers lost in 1-busy-periods of $M^X/G/1/m$ systems with smaller or equal system capacity, namely $n+1-i \le m \le n$, but otherwise with the same parameters as the former system,

$$L_i^{(n)} \stackrel{\mathrm{d}}{=} \bigoplus_{m=n+1-i}^{n} L_1^{(m)}.$$

Denoting by C_A the number of customers that arrive to the system during a customer service time with distribution A, the distribution of the number of customers lost in 1-busy-periods of $M^X/G/1/n$ systems conditional to the number of customers that arrive to the system during the service of the customer that initiates that 1-busy-period verifies

$$[L_1^{(n)} | C_A = l] \stackrel{\mathrm{d}}{=} \begin{cases} 0 & l = 0 \\ L_l^{(n)} & 1 \le l \le n-1\,, \\ l-(n-1)+L_{n-1}^{(n)} & l \ge n \end{cases} \quad \text{for } n \ge 1. \tag{2}$$

From the total probability law, it immediately follows that the kth moment of the number of losses in a 1-busy-period of an $M^X/G/1/n$ system is such that

$$\mathbb{E}\left[\left(L_1^{(n)}\right)^k\right] = \sum_{l=1}^{n-1} r_l(A)\,\mathbb{E}\left[\left(L_l^{(n)}\right)^k\right] + \sum_{l\ge n} r_l(A)\,\mathbb{E}\left[\left(l-(n-1)+L_{n-1}^{(n)}\right)^k\right], \tag{3}$$

with $r_l(A) = P(C_A = l)$ and $k \in \mathbb{N}_+$. This enables us to propose the following recursive procedure to compute integer moments for the number of customers lost in i-busy-periods of $M^X/G/1/n$ systems.

Theorem 1 *The integer moments of the number of customers lost in i-busy-periods of $M^X/G/1/n$ systems are such that, for $k \in \mathbb{N}_+$,*

$$\mathbb{E}\left[\left(L_1^{(1)}\right)^k\right] = \mathbb{E}\left[C_A^k\right], \tag{4}$$

and, for $n \ge 2$,

$$\mathbb{E}\left[\left(L_1^{(n)}\right)^k\right] = \tfrac{1}{r_0(A)}\left(\textstyle\sum_{l\ge 1} r_l(A)\,\Phi_{\min(l,n-1)}^{(n,k)} + \sum_{j=0}^{k-1}\binom{k}{j}\Upsilon_{n-1}^{(k-j)}(A)\,\mathbb{E}\left[\left(L_{n-1}^{(n)}\right)^j\right]\right) \tag{5}$$

$$\mathbb{E}\left[\left(L_i^{(n)}\right)^k\right] = \textstyle\sum_{j=0}^{k}\binom{k}{j}\mathbb{E}\left[\left(L_{i-1}^{(n-1)}\right)^{k-j}\right]\mathbb{E}\left[\left(L_1^{(n)}\right)^j\right], \quad i \ge 2 \tag{6}$$

where

$$\Phi_i^{(p,u)} \stackrel{def}{=} \sum_{l=0}^{u-1}\binom{u}{l}\mathbb{E}\left[\left(L_{i-1}^{(p-1)}\right)^{u-l}\right]\mathbb{E}\left[\left(L_1^{(p)}\right)^l\right] \tag{7}$$

with $L_0^{(p)} = 0$*, and*

$$\Upsilon_p^{(q)}(A) \stackrel{def}{=} \sum_{l \geq p+1} r_l(A)(l-p)^q = \sum_{j=0}^{q} \binom{q}{j} (-1)^{q-j} p^{q-j} \left[\mathbb{E}\left[C_A^j\right] - \sum_{l=0}^{p} l^j r_l(A) \right]. \quad (8)$$

Proof We start by noting that statement (4) trivially follows from the fact that $L_1^{(1)} \stackrel{\mathrm{d}}{=} C_A$, while (6) follows directly from (1) and the fact that the binomial formula along with the linearity of the expected value operator imply that,

$$\mathbb{E}\left[(U \oplus V)^k\right] = \sum_{s=0}^{k} \binom{k}{s} \mathbb{E}\left[U^{k-s}\right] \mathbb{E}\left[V^s\right] \quad (9)$$

for independent random variables U and V. For the expression for the computation of $\Upsilon_p^{(q)}(A)$ in the last part of (8) it suffices to note that, for $p, q \in \mathbb{N}_+$,

$$\begin{aligned} \Upsilon_p^{(q)}(A) &= \sum_{l \geq p+1} r_l(A)(l-p)^q = \sum_{l \geq p+1} r_l(A) \sum_{j=0}^{q} \binom{q}{j} (-p)^{q-j} l^j \\ &= \sum_{j=0}^{q} \binom{q}{j} (-p)^{q-j} \left[\sum_{l \geq 0} l^j r_l(A) - \sum_{l=0}^{p} l^j r_l(A) \right] \\ &= \sum_{j=0}^{q} \binom{q}{j} (-1)^{q-j} p^{q-j} \left[\mathbb{E}\left[C_A^j\right] - \sum_{l=0}^{p} l^j r_l(A) \right]. \end{aligned}$$

Additionally, from (1), (3), (6), and (9), it follows that, for $k \in \mathbb{N}_+$,

$$\begin{aligned} \mathbb{E}\left[\left(L_1^{(n)}\right)^k\right] &= \sum_{l=1}^{n-1} r_l(A) \sum_{j=0}^{k} \binom{k}{j} \mathbb{E}\left[\left(L_{l-1}^{(n-1)}\right)^{k-j}\right] \mathbb{E}\left[\left(L_1^{(n)}\right)^j\right] \\ &\quad + \sum_{l \geq n} r_l(A) \sum_{j=0}^{k} \binom{k}{j} (l-(n-1))^{k-j} \sum_{i=0}^{j} \binom{j}{i} \mathbb{E}\left[\left(L_{n-2}^{(n-1)}\right)^{j-i}\right] \mathbb{E}\left[\left(L_1^{(n)}\right)^i\right]. \end{aligned}$$

Finally, the statement (5) follows, by taking into account (7)–(8), after separating in the previous equation the terms for which $j = k$ and $i = k$ containing $\mathbb{E}[(L_1^{(n)})^k]$, and isolating them in the first member of the equation. □

We note that for $M/G/1/n$ systems the mean number of losses in 1-busy-periods obtained in Eq. (5) agrees with the following result previously derived in Peköz (1999):

$$\mathbb{E}\left[L_1^{(n)}\right] = \frac{1}{r_0(A)} \left[\sum_{m=2}^{n-1} \mathbb{E}\left[L_1^{(m)}\right] \sum_{l \geq n+1-m} r_l(A) + \sum_{l \geq n} r_l(A)(l-(n-1)) \right].$$

To compute the probability that exactly l customers arrive during a customer service time, $r_l(A) = P(C_A = l)$, we note that $C_A = \sum_{i=1}^{N} X_i$, with N denoting the number of customer batches that arrive to the system during a customer service time. Due to the compound Poisson structure of the batch arrival process, N has a mixed-Poisson (MP) probability mass function with rate λ and mixing distribution function $A(\cdot)$,

$$\alpha_j(A) \stackrel{\text{def}}{=} P(N = j) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} A(\mathrm{d}t), \quad j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

In addition, conditioning on the number of batches that arrive during an interval of length t, the probability that a total of l customers arrive during such period is given by

$$\begin{aligned}\beta_l(t) = P(C_A = l|S_A = t) &= \sum_{j=0}^{l} P(C_A = l|N = j)P(N = j|S_A = t)\\ &= \sum_{j=0}^{l} e^{-\lambda t}\frac{(\lambda t)^j}{j!}\, f_l^{(j)} \end{aligned} \tag{10}$$

where $f_l^{(j)}$ denotes the probability that the total number of customers in j customer batches is equal to l, i.e.,

$$f_l^{(0)} = \delta_{0l}, \quad \text{and} \quad f_l^{(j)} = \sum_{i=j-1}^{l-1} f_i^{(j-1)} f_{l-i} \tag{11}$$

for $j \in \mathbb{N}_+$ and $l = j, j+1, \ldots$, where δ_{il} is the Kronecker delta function, i.e., $\delta_{il} = 1$ if $i = l$ and $\delta_{il} = 0$ otherwise. Therefore, the probability that exactly l customers arrive during a customer service time is given by the compound mixed-Poisson (CMP) probabilities

$$r_l(A) = P(C_A = l) = \int_0^\infty \beta_l(t)\, A(\mathrm{d}t) = \sum_{j=0}^{l} \alpha_j(A)\, f_l^{(j)}, \quad l \in \mathbb{N}_0. \tag{12}$$

In general, it is not possible to get simple closed form expressions for the MP and CMP probabilities ($\alpha_l(A)$ and $r_l(A)$). Nevertheless, for a large range of mixing distributions (e.g., gamma, beta, Pareto, shifted Pareto, and generalized Pareto mixing distributions, among others) this task constitutes no problem as such probabilities can be computed in a fast stable recursive way. In this respect, Willmot (1993) proposed a general method for the derivation of simple recursive formulas for the evaluation of MP probabilities, $\int_{t_L}^{t_U} \frac{e^{-t}t^l}{l!} A(\mathrm{d}t), l \in \mathbb{N}_0$, provided that the probability density function $a(\cdot)$ associated with $A(\cdot)$, defined on (t_L, t_U), $0 \le t_L < t_U \le \infty$, belongs to the class $W(m)$ of absolutely continuous distributions, i.e., verifies:

(i) for $0 \le t_L < t_U \le \infty$,

$$A(t_L) = 0 \text{ and } A(t_U) = 1;$$

(ii) for $t_L \le t \le t_U$,

$$\frac{\mathrm{d}\log a(t)}{\mathrm{d}t} = \frac{\sum_{j=0}^{m} \eta_j\, t^j}{\sum_{j=0}^{m} \psi_j\, t^j} = \frac{\eta(t)}{\psi(t)} \tag{13}$$

for some polynomials $\eta(t)$ and $\psi(t)$ with coefficients η_j and ψ_j, $j = 0, 1, \ldots, m$, respectively, where at least one of these two polynomials has degree $m > 0$. It is understood that at least one of η_m and ψ_m is different from zero, and the numerator and the denominator in (13) are allowed to have common factors. General recursive procedures for the evaluation of CMP probabilities can be found in Willmot (1993) for the case of $A \in W(1)$, and in Wang and Sobrero (1994) and Hesselager (1996) for the $A \in W(m)$ general case. We note that there is an abundant literature on MP and CMP topics, for which Grandell (1997) offers a good synthesis and references.

Next, based on the Chadjiconstantinidis and Antzoulakos (2002) results for the computation of moments of CMP distributions associated to mixing distributions on $W(m)$, we address in Lemma 1 the computation of integer moments of C. Borrowing their notation, we consider the auxiliary functions

$$H_j(k) = \int_{t_L}^{t_U} t^j \mathbb{E}\left[C_A^k | S_A = t\right] A(\mathrm{d}t), \quad j = 0, 1, \ldots, m \quad \text{and} \quad k \geq 0 \tag{14}$$

recalling that,

$$\mathbb{E}\left[C_A^k | S_A = t\right] = \sum_{l=0}^{\infty} l^k \beta_l(t). \tag{15}$$

Noting that, for $k \geq 1$, the conditional moment of order k of the CMP distribution of C satisfies the recursion (c.f. De Pril 1986)

$$\mathbb{E}\left[C_A^k | S_A = t\right] = \sum_{s=0}^{k-1} \binom{k-1}{s} \lambda t \, \mathbb{E}\left[X^{s+1}\right] \mathbb{E}\left[C_A^{k-s-1} | S_A = t\right] \tag{16}$$

we can now assert the following recursive scheme to compute

$$\mathbb{E}\left[C_A^k\right] = H_0(k), \quad k \in \mathbb{N}_0. \tag{17}$$

Lemma 1 *For $m \geq 0$, the functions $\{H_0(k), H_1(k), \ldots, H_m(k)\}, k \geq 0$, and hence the moments $\mathbb{E}\left[C_A^k\right] = H_0(k)$ of the CMP distribution C_A, in the case where the mixing density $a(\cdot)$ belongs to $W(m)$, can be evaluated recursively as follows.*

As

$$H_j(0) = \mathbb{E}\left[S_A^j\right], \quad j = 0, 1, \ldots, m$$

it holds that, for $j = 0, 1, \ldots, m-1$ and $k \geq 1$,

$$H_j(k) = \lambda \sum_{s=0}^{k-1} \binom{k-1}{s} \mathbb{E}\left[X^{s+1}\right] H_{j+1}(k-s-1) \tag{18}$$

and, for $\eta_m \neq 0$ and $k \geq 0$,

$$\eta_m H_m(k) = V_{t_U}(k) - V_{t_L}(k) - \sum_{j=0}^{m-1} \left[(j+1)\psi_{j+1} + \eta_j\right] H_j(k) - \sum_{j=0}^{m} \lambda \psi_j \sum_{s=1}^{k} \binom{k}{s} \mathbb{E}\left[X^s\right] H_j(k-s) \tag{19}$$

whereas, for $\eta_m = 0$ and $k \geq 0$,

$$\begin{aligned} \mathbb{E}[X]\left[\eta_{m-1} + (m+k+1)\psi_m\right] \lambda H_m(k) &= V_{t_U}(k+1) - V_{t_L}(k+1) \\ &- \sum_{j=0}^{m-2} \left[(j+1)\psi_{j+1} + \eta_j\right] H_j(k+1) - \sum_{j=0}^{m-1} \lambda \psi_j \sum_{s=1}^{k+1} \binom{k+1}{s} \mathbb{E}\left[X^s\right] H_j(k-s+1) \\ &- \sum_{s=2}^{k+1} \mathbb{E}\left[X^s\right] H_m(k-s+1)\, \lambda \left[(\eta_{m-1} + m\psi_m)\binom{k}{s-1} + \psi_m \binom{k+1}{s}\right] \end{aligned} \tag{20}$$

where, for $k \geq 0$,

$$V_t(k) = \mathbb{E}\left[C_A^k | S_A = t\right] a(t) \sum_{j=0}^{m} \psi_j t^j. \tag{21}$$

ALGORITHM 1
Input: n, λ, m, K, $(f_l)_{1\le l\le n-1}$, $(\mathbb{E}[X^s])_{1\le s\le K+1}$, t_L, t_U, $(a(t))_{t=t_L,t_U}$, $(\eta_j,\psi_j)_{0\le j\le m}$, $(\mathbb{E}\left[S_A^j\right])_{0\le j\le m}$, $(\alpha_l(A))_{0\le l\le n-1}$
For $l=0,1,\ldots,n-1$
  For $j=0,1,\ldots,l$
    Compute $f_l^{(j)}$ using (11)
  Compute $r_l(A)$ using (12)
For $j=0,1,\ldots,m$
  $H_j(0)=\mathbb{E}\left[S_A^j\right]$
Compute $V_{t_L}(1)$ and $V_{t_U}(1)$ using (16) and (21)
For $k=1,\ldots,K$
  For $j=0,1,\ldots,m-1$
    Compute $H_j(k)$ using (18)
  Compute $V_{t_L}(k+1)$ and $V_{t_U}(k+1)$ using (16) and (21)
  Compute $H_m(k)$ from (19) or (20) depending wether $\eta_m\neq 0$ or $\eta_m=0$, respectively
  $\mathbb{E}\left[(L_1^{(1)})^k\right]:=H_0(k)$ (from (4) and (17))
  For $p=2,3,\ldots,n$
    Compute $\Upsilon_{p-1}^{(k)}(A)$ using (8)
    For $l=1,\ldots,p-1$
      Compute $\Phi_l^{(p,k)}$ using (7)
    Compute $\mathbb{E}\left[(L_1^{(p)})^k\right]$ using (5)
    For $i=2,\ldots,p$
      Compute $\mathbb{E}\left[(L_i^{(p)})^k\right]$ using (6)
Output: $\left(\mathbb{E}\left[(L_i^{(n)})^k\right]\right)_{1\le i\le n,\ 1\le k\le K}$

Fig. 1 Algorithm to compute the moments of the number of customers lost in busy-periods of $M^X/G/1/n$ systems

The proof follows as that of Theorem 2.1 from Chadjiconstantinidis and Antzoulakos (2002) and will thus be omitted. Capitalizing on the previous recursive schemes, Algorithm 1 in Fig. 1 can be used to compute the kth moments of the number of customers lost in i-busy-periods of $M^X/G/1/n$ systems with service time S_A having distribution $A\in W(m)$,

$$\mathbb{E}\left[(L_i^{(n)})^k\right]$$

for $1\le i\le n$ and $1\le k\le K$, given a fixed $K\in\mathbb{N}_+$.

3 Moments of losses in a busy-period of oscillating systems

In this section, we address the recursive evaluation of the moments of the number of customers lost in multi busy-periods of M^X/G_1–$G_2/1/\mathbf{n}$ systems with $\mathbf{n}=(n,a,b)$.

As mentioned, in the oscillating systems with lower barrier a and upper barrier b, service times oscillate between two forms depending on the evolution of the number of customers in the system. In particular, if the system is empty it is under (operating) phase 1, and it remains in phase 1 until the subsequent moment at which the number of customers in the system reaches or goes above the upper barrier b. After that instant, the service is upgraded in a nonpreemptive manner, changing to operating phase 2. The system then remains in phase 2 until the subsequent moment at which the number of customers in the system falls to or below the lower barrier a. After that instant, the service is downgraded in a nonpreemptive manner, changing to operating phase 1. Services initiated in phase 1 have duration with distribution function A_1, and services initiated in phase 2 have duration with distribution function A_2.

In what follows we admit that the distribution functions $A_q, q = 1, 2$, belong to the class $W(m_q)$, so that for some $0 \leq t_{Lq} < t_{Uq} \leq \infty$, $A_q(t_{Lq}) = 0$ and $A_q(t_{Uq}) = 1$, and, for $t_{Lq} \leq t \leq t_{Uq}$,

$$\frac{\mathrm{d}\log a_q(t)}{\mathrm{d}t} = \frac{\sum_{j=0}^{m_q} \eta_{q,j}\, t^j}{\sum_{j=0}^{m_q} \psi_{q,j}\, t^j}.$$

The M^X/G_1–$G_2/1/\mathbf{n}$ state process is modelled by the continuous-time bivariate process $X = (X(t))_{t\geq 0} = (X_1(t), X_2(t))_{t\geq 0}$, where $X_1(t)$ denotes the number of customers in the system at time t and $X_2(t)$ denotes the phase of the system (i.e., the phase the system is operating under) at time t. We note that X is a Markov regenerative process with state space

$$E^{\mathbf{n}} = \{(i, 1) : 0 \leq i \leq b-1\} \cup \{(i, 2) : a+1 \leq i \leq n\}$$

associated to the renewal sequence of customer post-departure epochs.

Similarly to the regular systems, we let $L^{\mathbf{n}}_{(i,j)}$ denote the number of customers lost in an (i, j)-busy-period of the M^X/G_1–$G_2/1/\mathbf{n}$ system, i.e., a busy-period initiated in state (i, j) with a customer initiating service at that instant and finishing at the subsequent instant at which the system becomes empty.

Next, we address the computation of moments of the number of customers lost in busy-periods of M^X/G_1–$G_2/1/(n, a, b)$ systems, starting by the cases where the lower barrier a is equal to zero, M^X/G_1–$G_2/1/(n, 0, b)$ systems.

We start by noting that, in a M^X/G_1–$G_2/1/(n, 0, b)$ system, when a busy-period starts with the system operating in phase 2, the system remains in phase 2 during the entire busy-period and its duration does not depend on the upper barrier, b. As a consequence, and taking into account that an M^X/G_1–$G_2/1/(n, 0, 1)$ system behaves as a regular $M^X/G_2/1/n$ system,

$$L^{(n,0,b)}_{(i,2)} \overset{\mathrm{d}}{=} L^{(n,0,1)}_{(i,2)} \overset{\mathrm{d}}{=} L^{(n)}_i \tag{22}$$

with $L^{(n)}_i$ denoting the number of customers lost during an i-busy-period of a regular system with service time distribution A_2.

In addition, when a busy-period starts with the system operating in phase 1, the distribution of the number of customers lost in an $(i, 1)$-busy-period of the M^X/G_1–$G_2/1/(n, 0, b)$ system, with $1 \leq i \leq b-1$, given the number of customer arrivals during the service time of the customer that initiates the busy-period, C_{A_1}, verifies:

$$[L^{(n,0,b)}_{(i,1)} | C_{A_1} = l] \overset{\mathrm{d}}{=} \begin{cases} L^{(n,0,b)}_{(l+i-1,1)} & 0 \leq l < b-i \\ L^{(n,0,b)}_{(l+i-1,2)} & b-i \leq l \leq n-i \\ l-(n-i) + L^{(n,0,b)}_{(n-1,2)} & l > n-i \end{cases}. \tag{23}$$

We are now in position to compute the moments of the number of customers lost in (i, j)-busy-periods of M^X/G_1–$G_2/1/(n, 0, b)$, with $b > 1$.

Theorem 2 *The moments of the number of customers lost in (i, j)-busy-periods of M^X/G_1–$G_2/1/(n, 0, b)$ systems with $b > 1$, are such that, for $k \in \mathbb{N}_+$,*

$$\mathbb{E}\left[\left(L^{(n,0,b)}_{(i,2)}\right)^k\right] = \mathbb{E}\left[\left(L^{(n,0,1)}_{(i,2)}\right)^k\right] \tag{24}$$

for $1 \le i \le n$*, and*

$$\mathbb{E}\left[\left(L_{(i,1)}^{(n,0,b)}\right)^k\right] = \xi_i^{(k)} - \tau_i^{(k)} \frac{\xi_0^{(k)}}{\tau_0^{(k)}} \tag{25}$$

for $1 \le i \le b-1$*, with* $\xi_{b-1}^{(k)} = 0$ *and* $\tau_{b-1}^{(k)} = 1$*, and*

$$\xi_{i-1}^{(k)} = \frac{\xi_i^{(k)} - \sum_{l=1}^{b-i-1} r_l(A_1)\xi_{l+i-1}^{(k)} - \zeta(i,k)}{r_0(A_1)} \tag{26}$$

$$\tau_{i-1}^{(k)} = \frac{\tau_i^{(k)} - \sum_{l=1}^{b-i-1} r_l(A_1)\tau_{l+i-1}^{(k)}}{r_0(A_1)} \tag{27}$$

where

$$\zeta(i,k) = \sum_{l=b-i}^{n-i} r_l(A_1)\mathbb{E}\left[\left(L_{(l+i-1,2)}^{(n,0,b)}\right)^k\right] + \sum_{s=0}^{k}\binom{k}{s}\Upsilon_{n-i}^{(k-s)}(A_1)\mathbb{E}\left[\left(L_{(n-1,2)}^{(n,0,b)}\right)^s\right] \tag{28}$$

with $\Upsilon_p^{(q)}(A_1)$ *as defined in (8).*

Proof We begin the proof by noting that statement (24) trivially follows from (22). From (23), the total probability law leads to

$$\begin{aligned}\mathbb{E}\left[\left(L_{(i,1)}^{(n,0,b)}\right)^k\right] &= r_0(A_1)\mathbb{E}\left[\left(L_{(i-1,1)}^{(n,0,b)}\right)^k\right] + \sum_{l=1}^{b-i-1} r_l(A_1)\mathbb{E}\left[\left(L_{(l+i-1,1)}^{(n,0,b)}\right)^k\right] \\ &\quad + \sum_{l=b-i}^{n-i} r_l(A_1)\mathbb{E}\left[\left(L_{(l+i-1,2)}^{(n,0,b)}\right)^k\right] \\ &\quad + \sum_{l\ge n-i+1} r_l(A_1)\mathbb{E}\left[(l-(n-i)+L_{(n-1,2)}^{(n,0,b)})^k\right]\end{aligned}$$

for $1 \le i \le b-1$. Thus, using (8) and (9), we get

$$\begin{aligned}r_0(A_1)\mathbb{E}\left[\left(L_{(i-1,1)}^{(n,0,b)}\right)^k\right] &= \mathbb{E}\left[\left(L_{(i,1)}^{(n,0,b)}\right)^k\right] - \sum_{l=1}^{b-i-1} r_l(A_1)\mathbb{E}\left[\left(L_{(l+i-1,1)}^{(n,0,b)}\right)^k\right] \\ &\quad - \sum_{l=b-i}^{n-i} r_l(A_1)\mathbb{E}\left[\left(L_{(l+i-1,2)}^{(n,0,b)}\right)^k\right] \\ &\quad - \sum_{s=0}^{k}\binom{k}{s}\Upsilon_{n-i}^{(k-s)}(A_1)\mathbb{E}\left[\left(L_{(n-1,2)}^{(n,0,b)}\right)^s\right]\end{aligned}$$

which asserts that the moments of the number of customers lost in $(i-1,1)$-busy-periods can be computed recursively from moments of the number of customers lost in busy-periods initiated with i or more customers in system. After some algebraic manipulation, this implies that we may write $\mathbb{E}\left[\left(L_{(i,1)}^{(n,0,b)}\right)^k\right]$ in the form

$$\mathbb{E}\left[\left(L_{(i,1)}^{(n,0,b)}\right)^k\right] = \xi_i^{(k)} + \tau_i^{(k)}\mathbb{E}\left[\left(L_{(b-1,1)}^{(n,0,b)}\right)^k\right] \tag{29}$$

ALGORITHM 2
Input: n, λ, K, b, $(f_l)_{1\le l\le n-1}$, $(\mathbb{E}[X^s])_{1\le s\le K+1}$
Input (for q = 1,2): m_q, t_{Lq}, t_{Uq}, $a_q(t_{Lq})$, $a_q(t_{Uq})$, $(\eta_{q,j},\psi_{q,j})_{0\le j\le m_q}$, $(\mathbb{E}[S^j_{A_q}])_{0\le j\le m_q}$, $(\alpha_l(A_q))_{0\le l\le n-1}$
For $k=1,\dots,K$
  For $i=1,\dots,n$
    Compute $\mathbb{E}[(L^{(n,0,1)}_{(i,2)})^k]$ using Algorithm 1 with service time distribution A_2
    $\mathbb{E}[(L^{(n,0,b)}_{(i,2)})^k] := \mathbb{E}[(L^{(n,0,1)}_{(i,2)})^k]$
    $(\xi^{(k)}_{b-1},\tau^{(k)}_{b-1}) := (0,1)$
    For $j=b-1,b-2,\dots,1$
      Compute $(\xi^{(k)}_{j-1},\tau^{(k)}_{j-1})$ using (26)–(28)
    Compute $\mathbb{E}[(L^{(n,0,b)}_{(i,1)})^k]$ using (25)
Output: $\left(\mathbb{E}\left[(L^{(n,0,b)}_{(i,j)})^k\right]\right)_{(i,j)\in E^{(n,0,b)},\ 1\le k\le K}$

Fig. 2 Algorithm to compute moments of the number of customers lost in busy-periods of M^X/G_1–$G_2/1/(n,0,b)$ systems

satisfying the backward Eqs. (26)–(27). Choosing $\xi^{(k)}_{b-1}=0$ and $\tau^{(k)}_{b-1}=1$, we get $\mathbb{E}\left[(L^{(n,0,b)}_{(b-1,1)})^k\right]=-\frac{\xi^{(k)}_0}{\tau^{(k)}_0}$ using the fact that $0=\mathbb{E}\left[(L^{(n,0,b)}_{(0,1)})^k\right]=\xi^{(k)}_0+\tau^{(k)}_0\mathbb{E}\left[(L^{(n,0,b)}_{(b-1,1)})^k\right]$. This, in turn, leads to (25) in view of (29). □

Based in Theorem 2, we propose the use of Algorithm 2 in Fig. 2 to recursively compute the moments of the number of customers lost in busy-periods of M^X/G_1–$G_2/1/(n,0,b)$ systems.

For the computation of the moments of the number of customers lost in (i,j)-busy-periods of M^X/G_1–$G_2/1/(n,a,b)$ systems, with $a>0$, we start by noting that an M^X/G_1–$G_2/1/(n,n-1,n)$ system always operates in phase 1, thus behaving as a regular $M^X/G_1/1/n$ system with customer service time distribution A_1. In such case,

$$\mathbb{E}\left[\left(L^{(n,n-1,n)}_{(i,1)}\right)^k\right]=\mathbb{E}\left[\left(L^{(n)}_i\right)^k\right],\quad k\in\mathbb{N}_+$$

as $L^{(n,n-1,n)}_{(i,1)}\overset{\mathrm{d}}{=}L^{(n)}_i$, with $L^{(n)}_i$ denoting the number of customers lost during an i-busy-period of an $M^X/G_1/1/n$ regular system.

Now, to address the computation of $(\mathbb{E}[(L^{(n,a,b)}_{(i,j)})^k])^{k\in\mathbb{N}_+}_{(i,j)\in E^n}$ for M^X/G_1–$G_2/1/(n,a,b)$ systems, with $0<a<n-1$, we resort to the Markov regenerative structure of oscillating systems in order to relate the number of customers lost during an (i,j)-busy-period, in systems with fixed capacity and barriers, with busy-periods of systems with smaller or equal capacity and barriers, and initiated with fewer customers.

In fact, supposing that we take out of consideration one of the customers initially present in the M^X/G_1–$G_2/1/\mathbf{n}$ system and assuming that such a customer will start being served only when being alone in the system, the Markov regenerative property guarantees that in any (i,j)-busy-period starting at time zero ($\{X(0)=(i,j), X(0^-)\neq(i,j)\}$, with $i>1$) the time the system takes to reach state $(1,1)$—from state (i,j)—has the same distribution as the duration of an $(i-1,j)$-busy-period of an M^X/G_1–$G_2/1/\mathbf{n-1}$, where $\mathbf{n-1}=(n-1,a-1,b-1)$. As a consequence, for $(i,j)\in E^{\mathbf{n}}$ and $0<a<n-1$,

$$L^{\mathbf{n}}_{(i,j)}\overset{\mathrm{d}}{=}L^{\mathbf{n-1}}_{(i-1,j)}\oplus L^{\mathbf{n}}_{(1,1)} \tag{30}$$

with $L^{\mathbf{n}}_{(0,1)} = 0$. Thus, letting $\mathbf{1} = (1, 1, 1)$, it simply follows by induction that, for $l \leq \min(i, a)$,

$$L^{\mathbf{n}}_{(i,j)} \stackrel{\text{d}}{=} L^{\mathbf{n}-l\mathbf{1}}_{(i-l,j)} \bigoplus \left[\bigoplus_{m=0}^{l-1} L^{\mathbf{n}-m\mathbf{1}}_{(1,1)} \right]. \tag{31}$$

This asserts that the number of customers lost in an (i, j)-busy-period of an $M^X/G_1–G_2/1/\mathbf{n}$ system, $1 \leq a \leq i$, may be expressed as the sum of the number of customers lost in a $(i-a, j)$-busy-period of an $M^X/G_1–G_2/1/(n-a, 0, b-a)$ system and the number of customers lost in multiple $(1, 1)$-busy-periods of $M^X/G_1–G_2/1/\mathbf{n} - l\mathbf{1}$ systems, $0 \leq l < a$, with smaller capacities and barriers than the original system, but otherwise with the same parameters.

Denoting by $\mathbf{1}_{\{z\}}$ the indicator function of the statement z, the distribution of the number of customers lost during an $(1, 1)$-busy-period of the $M^X/G_1–G_2/1/\mathbf{n}$ system, with $0 < a < n-1$, conditional to the number of customer arrivals during the service time of the customer that initiates the $(1, 1)$-busy-period, C_{A_1}, verifies

$$[L^{\mathbf{n}}_{(1,1)} | C_{A_1} = l] \stackrel{\text{d}}{=} \begin{cases} 0 & l = 0 \\ L^{\mathbf{n}}_{(l,1)} & 1 \leq l \leq b-2 \\ L^{\mathbf{n}}_{(l,1+\mathbf{1}_{\{a<b-1\}})} & l = b-1 \\ L^{\mathbf{n}}_{(l,2)} & b \leq l \leq n-1 \\ l-(n-1) + L^{\mathbf{n}}_{(n-1,2)} & l \geq n \end{cases}, \tag{32}$$

which enables us to compute moments of the number of customers lost in (i, j)-busy-periods of $M^X/G_1–G_2/1/\mathbf{n}$ systems, with $0 < a < n-1$.

Theorem 3 *Let, for* $(i, j) \in E^{\boldsymbol{n}} \backslash \{(0, 1)\}$ *and* $u \in \mathbb{N}_0$,

$$\Psi^{(\boldsymbol{n},u)}_{(i,j)} = \sum_{s=0}^{u-1} \binom{u}{s} \mathbb{E}\left[\left(L^{\boldsymbol{n}-\boldsymbol{1}}_{(i-1,j)} \right)^{u-s} \right] \mathbb{E}\left[\left(L^{\boldsymbol{n}}_{(1,1)} \right)^{s} \right] \tag{33}$$

and $\Upsilon^{(q)}_p(A_1)$ *as defined in* (8). *Then, the integer moments of the number of customers lost in* (i, j)*-busy-periods of* $M^X/G_1–G_2/1/\boldsymbol{n}$ *systems, with lower barrier* a, $0 < a < n-1$, *are such that, for* $k \in \mathbb{N}_+$,

$$\mathbb{E}\left[\left(L^{\boldsymbol{n}}_{(1,1)} \right)^{k} \right] = \frac{1}{r_0(A_1)} \left(\sum_{l=2}^{b-2} r_l(A_1) \Psi^{(\boldsymbol{n},k)}_{(l,1)} + r_{b-1}(A_1) \Psi^{(\boldsymbol{n},k)}_{(b-1,1+\boldsymbol{1}_{\{a<b-1\}})} \right.$$
$$\left. + \sum_{l \geq b} r_l(A_1) \Psi^{(\boldsymbol{n},k)}_{(\min(l,n-1),2)} + \sum_{s=0}^{k-1} \binom{k}{s} \Upsilon^{(k-s)}_{n-1}(A_1) \mathbb{E}\left[\left(L^{\boldsymbol{n}}_{(n-1,2)} \right)^{s} \right] \right). \tag{34}$$

Moreover, if $(i, j) \in E^{\boldsymbol{n}}$, *then*

$$\mathbb{E}\left[\left(L^{\boldsymbol{n}}_{(i,j)} \right)^{k} \right] = \sum_{s=0}^{k} \binom{k}{s} \mathbb{E}\left[\left(L^{\boldsymbol{n}-\boldsymbol{1}}_{(i-1,j)} \right)^{k-s} \right] \mathbb{E}\left[\left(L^{\boldsymbol{n}}_{(1,1)} \right)^{s} \right]. \tag{35}$$

Proof We first note that, for $(i, j) \in E^{\mathbf{n}}$, (35) follows trivially from (30) and (9)

$$\mathbb{E}\left[\left(L^{\mathbf{n}}_{(i,j)}\right)^k\right] = \mathbb{E}\left[\left(L^{\mathbf{n-1}}_{(i-1,j)} \oplus L^{\mathbf{n}}_{(1,1)}\right)^k\right] = \sum_{s=0}^{k}\binom{k}{s}\mathbb{E}\left[\left(L^{\mathbf{n-1}}_{(i-1,j)}\right)^{k-s}\right]\mathbb{E}\left[\left(L^{\mathbf{n}}_{(1,1)}\right)^s\right]. \tag{36}$$

By separating in (36) the term for which $s = k$ from the remaining terms we obtain

$$\mathbb{E}\left[\left(L^{\mathbf{n}}_{(i,j)}\right)^k\right] = \mathbb{E}\left[\left(L^{\mathbf{n}}_{(1,1)}\right)^k\right] + \Psi^{(\mathbf{n},k)}_{(i,j)} \tag{37}$$

with $\Psi^{(\mathbf{n},k)}_{(i,j)}$ as defined in (33). Then, as from (32), (9), and (37),

$$\begin{aligned}\mathbb{E}\left[\left(L^{\mathbf{n}}_{(1,1)}\right)^k\right] &= r_1(A_1)\mathbb{E}\left[\left(L^{\mathbf{n}}_{(1,1)}\right)^k\right] + \sum_{l=2}^{b-2} r_l(A_1)\left[\mathbb{E}\left[\left(L^{\mathbf{n}}_{(1,1)}\right)^k\right] + \Psi^{(\mathbf{n},k)}_{(l,1)}\right]\\ &\quad + r_{b-1}(A_1)\left[\mathbb{E}\left[\left(L^{\mathbf{n}}_{(1,1)}\right)^k\right] + \Psi^{(\mathbf{n},k)}_{(b-1,1+\mathbf{1}_{\{a<b-1\}})}\right]\\ &\quad + \sum_{l=b}^{n-1} r_l(A_1)\left[\mathbb{E}\left[\left(L^{\mathbf{n}}_{(1,1)}\right)^k\right] + \Psi^{(\mathbf{n},k)}_{(l,2)}\right]\\ &\quad + \sum_{l\geq n} r_l(A_1)\sum_{s=0}^{k}\binom{k}{s}(l-(n-1))^{k-s}\left[\mathbb{E}\left[\left(L^{\mathbf{n}}_{(1,1)}\right)^s\right] + \Psi^{(\mathbf{n},s)}_{(n-1,2)}\right], \end{aligned} \tag{38}$$

(34) is obtained by separating in (38) the term for which $s = k$ from the remaining terms and taking into account (8). □

Based on the previous results, we present, in Fig. 3, an algorithm to compute integer moments of the number of customers lost in (i, j)-busy-periods of $M^X/G_1–G_2/1/\mathbf{n}$ systems with $0 \leq a < b \leq n$. We start with the computation of the moments of the number of losses in busy-periods of systems with capacity $n - a$, lower barrier 0 and upper barrier $b - a$ according to Algorithm 2. Then, by successively adding one unit to the system capacity and barriers, we compute for such systems the moments of the number of customers lost in (i, j)-busy-periods, until the system achieves the desired capacity and barriers a and b.

4 Numerical illustration

In this section, using the above-derived results, we compute moments of the number of customers lost in busy-periods of regular and nonpreemptive oscillating systems that will be denoted here by $M^X/G(\mu)/1/n$ and $M^X/G(\mu_1) - G(\mu_2)/1/\mathbf{n}$, respectively. We analyze their sensitivity with respect to the traffic intensity, the batch size and service time distributions, and the number of customers that start the busy-periods. To this purpose, several different batch size and service time distributions have been considered. Specifically, it is assumed that customers arrive in batches with the following size distributions, with common mean $\bar{f}$: $D(\bar{f})$—deterministic with constant value $\bar{f}$; $Geo(1/\bar{f})$—geometric with success probability $1/\bar{f}$; $1 + B(c, (\bar{f} - 1)/c)$—shifted binomial, a binomial with c trials and success probability $(\bar{f} - 1)/c$, added of one unit; and $U\{1, \ldots, 2\bar{f} - 1\}$—uniform discrete on

ALGORITHM 3
Input: n, λ, K, a, b, $(f_l)_{1\leq l\leq n-1}$, $(\mathbb{E}[X^s])_{1\leq s\leq K+1}$
Input (for $\mathbf{q=1,2}$): m_q, t_{Lq}, t_{Uq}, $a_q(t_{Lq})$, $a_q(t_{Uq})$, $(\eta_{q,j},\psi_{q,j})_{0\leq j\leq m_q}$, $(\mathbb{E}[S_{A_q}^j])_{0\leq j\leq m_q}$, $(\alpha_l(A_q))_{0\leq l\leq n-1}$
For $k=1,2,\ldots,K$
  For $(i,j)\in E^{\mathbf{n}-a\mathbf{1}}$
    Compute $\mathbb{E}[(L_{(i,j)}^{(\mathbf{n}-a\mathbf{1})})^k]$, using Algorithm 2
For $l=1,2,\ldots,a$
  For $k=1,2,\ldots,K$
    Compute $\mathbb{E}[(L_{(1,1)}^{\mathbf{n}-(a-l)\mathbf{1}})^k]$ using (34), (33) and (8)
    For $i=1,2,\ldots,b-a+l-1$
      Compute $\mathbb{E}[(L_{(i,1)}^{\mathbf{n}-(a-l)\mathbf{1}})^k]$ using (35)
    For $i=l+1,l+2,\ldots,n-a+l$
      Compute $\mathbb{E}[(L_{(i,2)}^{\mathbf{n}-(a-l)\mathbf{1}})^k]$ using (35)
Output: $\mathbb{E}\left[(L_{(i,j)}^{\mathbf{n}-(a-l)\mathbf{1}})^k\right]$ for $0\leq l\leq a$, $(i,j)\in E^{\mathbf{n}-(a-l)\mathbf{1}}$ and $1\leq k\leq K$

Fig. 3 Algorithm to compute moments of the number of customers lost in busy-periods of M^X/G_1–$G_2/1/\mathbf{n}$ systems

the set $\{1, 2, \ldots, 2\bar{f} - 1\}$. We also consider the following service time distributions with mean μ^{-1}: $M(\mu)$—exponential with rate μ; $U(0, 2/\mu)$—uniform on the interval $(0, 2/\mu)$; $GP(\kappa, \theta, \beta)$—generalized Pareto with parameters κ, θ and β, with $\theta = (\kappa - 1)/(\mu\beta)$; and $SP(\kappa, \theta)$—shifted Pareto with parameters κ and θ, with $\theta = (\kappa - 1)/\kappa\mu$.

At this point it should be noted that the mean duration of busy periods in different systems could vary significantly, and therefore the results on busy periods can not be extended to the long-run system behavior without additional arguments. The figures presented in this section have markers identifying a particular system placed at points calculated by the algorithms derived in the paper, and include curves that are drawn by connecting successive points with a given marker using linear interpolation.

Figure 4 depicts the sensitivity of the number of customers lost during a busy-period with respect to the traffic intensity and the batch size and service time distributions. As expected, the results show that, among the service times and batch size distributions considered, the mean and standard deviation of the number of customers lost during a busy-period increase with the traffic intensity. Small mean number of customers lost during a busy-period are observed for small traffic intensities, with small differences between the studied systems being registered. As the traffic load increases, the mean number of customer losses in a busy-period rapidly increases and large differences arise among systems with different service time or batch size distributions. In such load traffic situations, the numbers of customers lost in busy-periods of systems with (heavy tailed) generalized Pareto service time distributions have smaller mean and standard deviation than those of systems with any of the other considered service time distributions. In addition, among the studied systems, we observe that the oscillating systems experience smaller mean and standard deviation of the number of customer losses during a busy-period than regular systems having for service time distribution the phase 1 service time of the oscillating systems. This was expected as the oscillating systems were designed to reduce the system overflow by forcing the server to increase temporarily its service rate whenever the number of customers in the queue is high.

Figure 5 exhibits the mean and standard deviation of the number of customers lost during a (1,1)-busy-period, as a function of the mean batch size for $M^X/G(0.9)$–$G(1.1)/1/(10, 6, 8)$

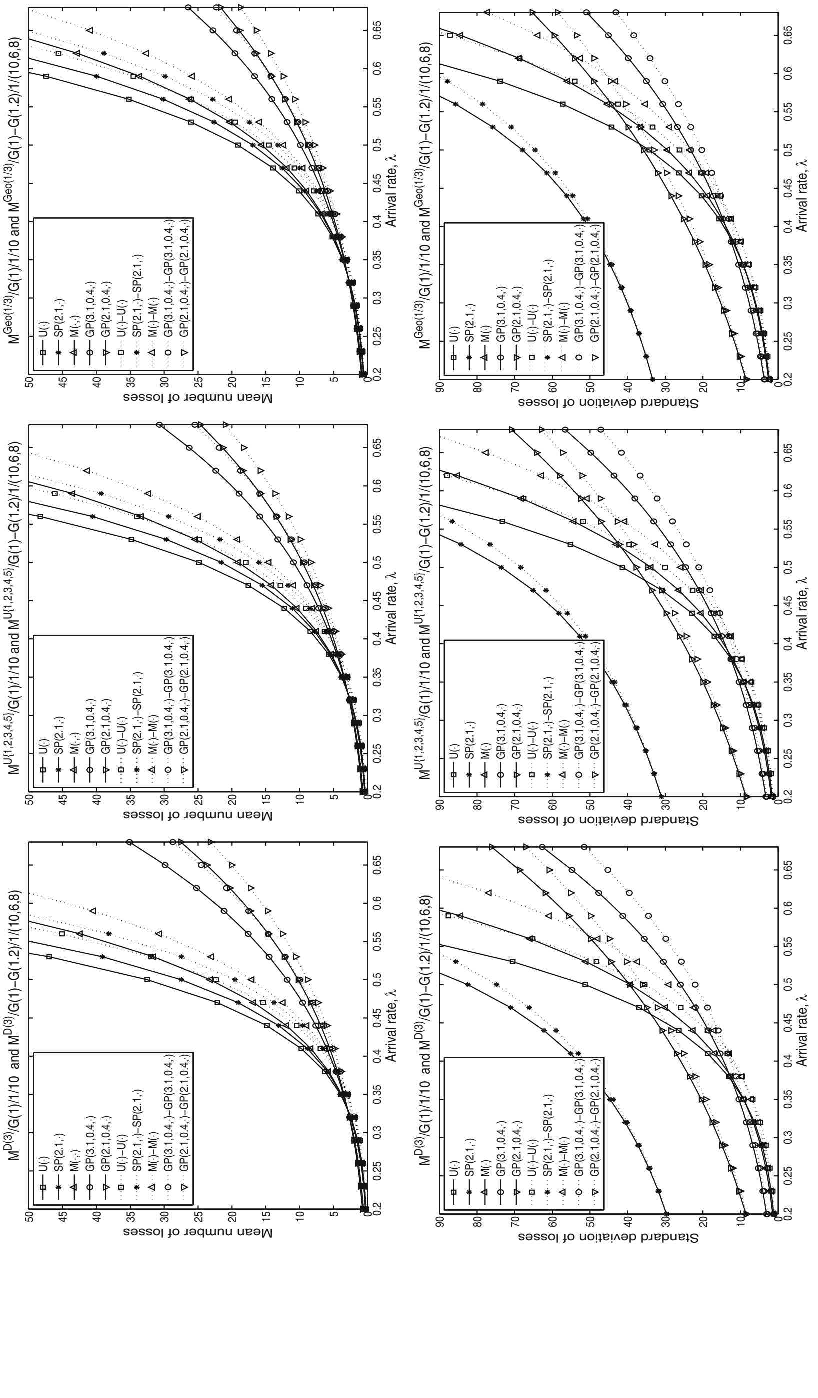

Fig. 4 Mean and standard deviation of the number of customers lost in 3-busy-periods of $M^X/G(1)/1/10$ and $M^X/G(1)–G(1.2)/1/(10, 6, 8)$ systems as a function of the arrival rate, for deterministic, uniform discrete, and geometric batch size distributions with mean 3

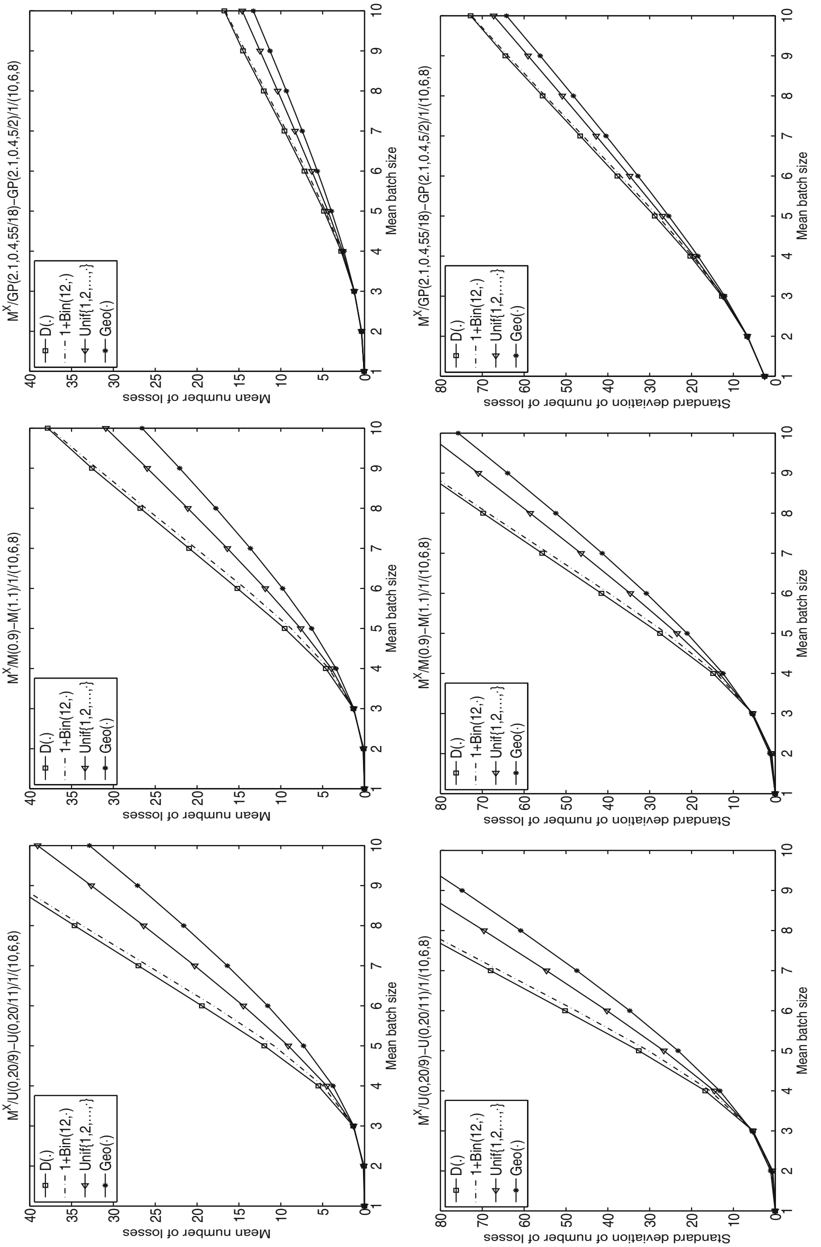

Fig. 5 Mean and standard deviation of the number of customers lost in a single-busy-period of $M^X/G(0.9)–G(1.1)/1/(10, 6, 8)$ systems as a function of the mean batch size, for deterministic, shifted binomial, uniform discrete, and geometric batch size distributions

systems with: uniform, exponential, and generalized Pareto service time distributions; and deterministic, shifted binomial, uniform discrete, and geometric batch size distributions with arrival rate $\lambda = 1/3$. We observe that the mean and standard deviation of the number of customers lost during a busy-period increase with the mean batch size, which corresponds to an increase of the traffic intensity. For small traffic intensities, corresponding to mean batch size smaller than λ^{-1} in the systems presented in Fig. 5, we observe a weak effect of the batch size distribution, with all systems exhibiting small mean and standard deviation of the number of customer losses in a busy-period. Such measures shows a tendency to rapidly increase for larger traffic intensities and batch size distributions with smaller variability. In fact, among the service times and the batch size distributions considered, we observe that, for systems sharing the same mean batch size, the ones with deterministic batch size distributions experience a larger mean number of customers lost in a busy-period. By contrast, systems with geometric batch size distribution present the smallest mean number of customer losses during a busy-period. According to the observed in Fig. 4 and independently of the batch size distribution, the systems with Pareto service time distributions, which are heavy tailed, experience smaller mean number of customers lost during a busy-period.

Figures 6 and 7 address the sensitivity of the mean number of customers lost during a multi busy-period on the number of customers that initiated the busy-period. Figure 6 gives the mean number of customers lost in $(i, 1)$-busy-periods as a function of the arrival rate for $M^{D(3)}/G(1)$–$G(1.2)/1/(10, 6, 8)$ systems with uniform, exponential, and generalized Pareto service time distributions. Figure 7 presents the mean number of customers lost in $(i, 1)$-busy-periods as a function of the mean batch size for $M^X/G(0.9){-}G(1.1)/1/(10, 6, 8)$ systems with uniform and generalized Pareto service time distributions, deterministic, uniform discrete, and geometric batch size distributions, and arrival rate $\lambda = 1/3$. As expected, the mean number of customers lost in $(i, 1)$-busy-periods increases with the number of customers that start the busy-period, i, in a concave fashion. That is, the increment on the mean number of customer lost during an $(i, 1)$-busy-period obtained from increasing by one unit the value of i decreases as i increases.

5 Conclusions

This paper analyzes regular and nonpreemptive oscillating $M^X/G/1/n$ systems. A recursive scheme on the queue capacity was presented to compute the moments of the number of losses in busy-periods starting with an arbitrary number of customers in the system. The recursive scheme takes advantage of the Markov regenerative structure of these systems, expressing the number of customer losses in busy-periods initiated by multiple customers as the convolution of the number of customers lost in busy-periods initiated with fewer customers in systems with capacities not exceeding that of the original system and the number of customers lost in busy-periods initiated by a single customer in the original system.

The derived computational procedure is fast and easy to implement, requiring only as a starting point mixed-Poisson probabilities and moments associated with the service time distributions of the oscillating systems under consideration. Numerical examples have been carried out to study the influence of the traffic intensity, batch size and service time distributions, and the number of customers initiating the busy-period on the number of customers lost during a busy-period.

By changing the service time distribution according to the evolution of the number of customers in the system, the oscillating systems considered may be used to achieve high

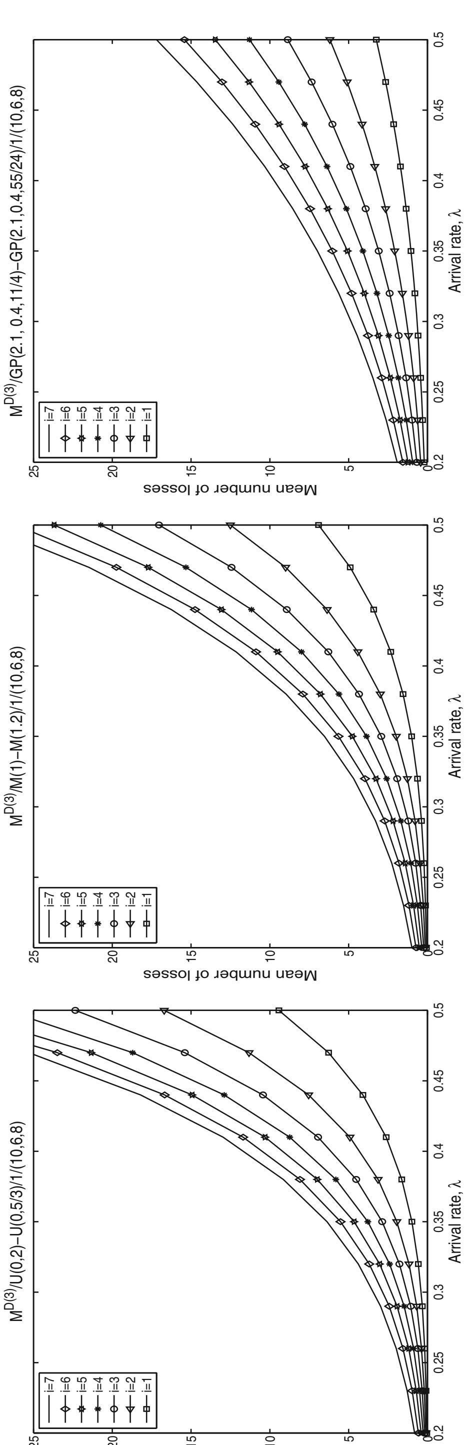

Fig. 6 Mean number of customers lost in $(i, 1)$-busy-periods of $M^{D(3)}/G(1)–G(1.2)/1/(10, 6, 8)$ systems as a function of the arrival rate

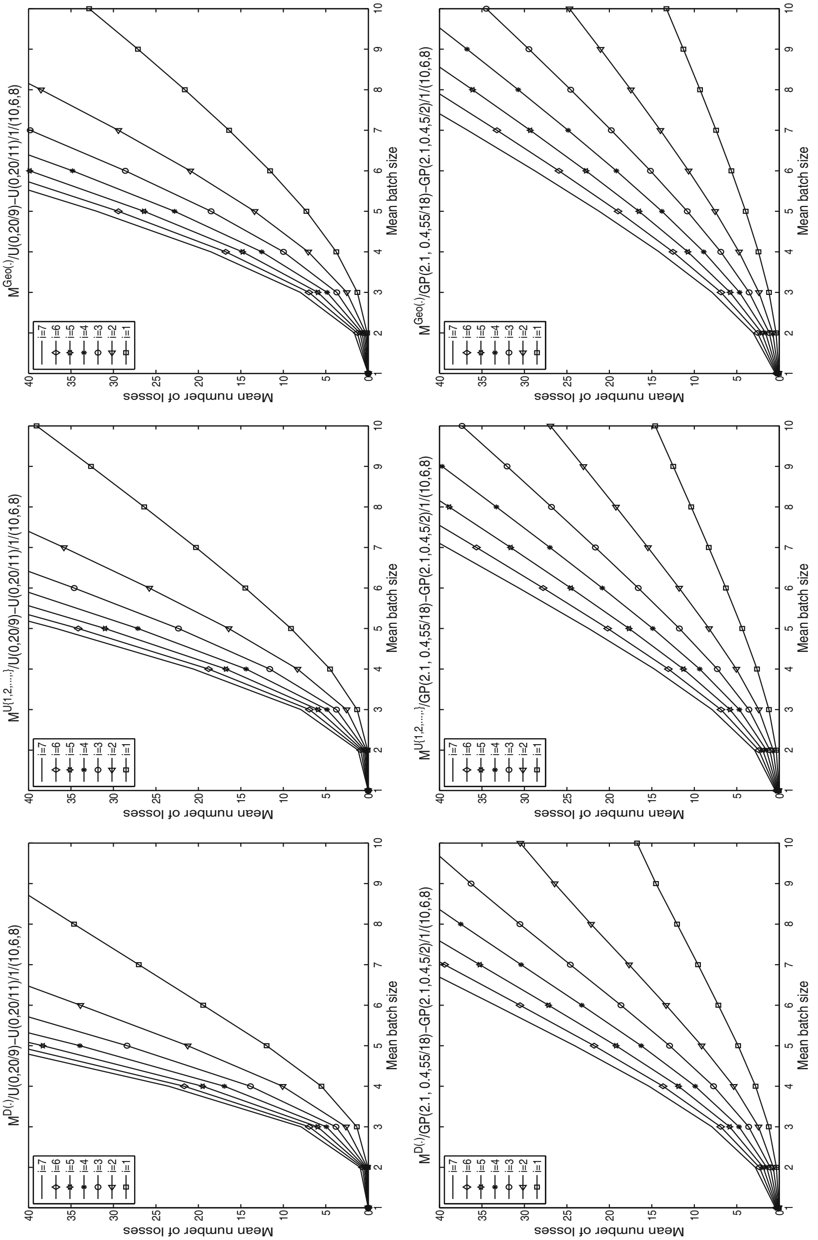

Fig. 7 Mean number of customers lost in $(i, 1)$-busy-periods of $M^X/G(0.9)$–$G(1.1)/1/(10, 6, 8)$ systems as a function of the mean batch size, for deterministic, uniform discrete, and geometric batch size distributions

throughput with a small numbers of customer losses in busy-periods. Such results may be used by system designers to control the quality of service with reduced costs. Moreover, the algorithms developed in the paper allow us to test and compare different scenarios.

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