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Advances in Applied Clifford Algebras

Fundamental Solution for Natural Powers of the Fractional Laplace and Dirac Operators in the Riemann–Liouville Sense

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Dedicated to Professor Sirkka-Liisa Eriksson on occasion of her 60th birthday.

Abstract. In this paper, we study the fundamental solution of natural powers of the n-parameter fractional Laplace and Dirac operators defined via Riemann–Liouville fractional derivatives. To do this we use iteration through the fractional Poisson equation starting from the fundamental solutions of the fractional Laplace $\Delta^{\alpha}_{a^+}$ and Dirac $D^{\alpha}_{a^+}$ operators, admitting a summable fractional derivative. The family of fundamental solutions of the corresponding natural powers of fractional Laplace and Dirac operators are expressed in operator form using the Mittag–Leffler function.

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1. Introduction

During the last decades, the study of the so-called fractional Laplace operator has received the attention of several authors (see for example [1,14] and references therein indicated). This operator is defined as a singular integral operator or as a Fourier multiplier in Fourier domain and has the purpose of extending the harmonic function theory of the Laplace operator by taking

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into account the long-range interactions that occur in a number of applications. Motivated by fractional calculus and fractional derivatives it appeared recently new definitions for fractional Laplace operators (see [5,6]).

In this paper we consider a n-parameter fractional Laplace operator defined in n-dimensional space and the associated n-parameter fractional Dirac operator over a Clifford algebra, both defined via Riemann–Liouville fractional derivatives with different fractional order of differentiation for each direction. Previous approaches for this type of operators can be found in [5,6]. There the authors studied eigenfunctions and fundamental solutions for the three-parameter fractional Laplace operator defined with Caputo and Riemann–Liouville fractional derivatives, and derived also fundamental solutions for the corresponding fractional Dirac operator which factorizes the fractional Laplace operator. In both cases, the authors applied an operational approach via Laplace transform to construct general families of fundamental solutions.

The aim of this paper is to present an expression for the family of fundamental solutions of natural powers of the n-parameter fractional Laplace operator, as well as a family of fundamental solutions of natural powers of the fractional Dirac operator. To do this, we use the fundamental solution of the Laplace operator Δ^{α}_{a+} and the fundamental solution of the Dirac operator D^{α}_{a+} and the iteration process using the fractional Poisson equation in order to get the families of fundamental solutions expressed in operator form using the Mittag–Leffler function.

We explain now how this paper is organized. In Sect. 2 we recall some basic knowledge about fractional calculus and Clifford analysis. In Sect. 3 we solve the Poisson equation for the n-parameter fractional Laplace operator, being this the key to obtain the results, because it connects the fundamental solution of previous order of the powers of the operator with the next order. In Sect. 4 are presented the fundamental solutions of natural powers of the n-parameter fractional Laplace operator together with a detailed discussion for the integer case. To finish, in Sect. 5 we present the fundamental solutions of natural powers of the n-parameter fractional Dirac operator.

2. Preliminaries

2.1. Fractional Calculus

Let $(D^{\alpha}_{a+} f)(x)$ denote the fractional Riemann–Liouville derivative of order $\alpha > 0$ (see [11])

$$\left(D^{\alpha}_{a+} f\right)(x) = \left(\frac{d}{dx}\right)^{m} \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}}\, dt, \tag{2.1}$$

where $m = [\alpha] + 1$, $x > a$, and $[\alpha]$ means the integer part of α. When $0 < \alpha < 1$ then (2.1) takes the form

$$\left(D^{\alpha}_{a+} f\right)(x) = \frac{d}{dx} \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha}}\, dt. \tag{2.2}$$

The Riemann–Liouville fractional integral of order $\alpha > 0$ is given by (see [11])

$$\left(I^{\alpha}_{a+} f\right)(x) \; = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > a. \tag{2.3}$$

We denote by $I^{\alpha}_{a+}(L_1)$ the class of functions f represented by the fractional integral (2.3) of a summable function, that is $f = I^{\alpha}_{a+}\varphi$, $\varphi \in L_1(a,b)$. A description of this class of functions was given in [13].

Theorem 2.1. *A function* $f \in I^{\alpha}_{a+}(L_1), \alpha > 0$ *if and only if* $I^{m-\alpha}_{a+} f \in AC^m$ $([a,b])$, $m = [\alpha]+1$ *and* $(I^{m-\alpha}_{a+} f)^{(k)}(a) = 0$, $k = 0, \ldots, m-1$.

In Theorem 2.1 $AC^m([a,b])$ denotes the class of functions f, which are continuously differentiable on the segment $[a,b]$ up to order $m-1$ and $f^{(m-1)}$ is absolutely continuous on $[a,b]$. Removing the last condition in Theorem 2.1 we obtain the class of functions that admits a summable fractional derivative.

Definition 2.2. (*See* [13]) A function $f \in L_1(a,b)$ has a summable fractional derivative $\left(D^{\alpha}_{a+} f\right)(x)$ if $\left(I^{m-\alpha}_{a+}\right)(x) \in AC^m([a,b])$, where $m = [\alpha]+1$.

If a function f admits a summable fractional derivative, then the composition of (2.1) and (2.3) can be written in the form (see [13, Thm. 2.4])

$$\left(I^{\alpha}_{a+} D^{\alpha}_{a+} f\right)(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \; \left(I^{m-\alpha}_{a+} f\right)^{(m-k-1)}(a) \tag{2.4}$$

with $m = [\alpha]+1$. We remark that if $f \in I^{\alpha}_{a+}(L_1)$ then (2.4) reduces to $(I^{\alpha}_{a+} D^{\alpha}_{a+} f)(x) = f(x)$. Nevertheless we note that $D^{\alpha}_{a+}\ I^{\alpha}_{a+} f = f$ in both cases. This is a particular case of a more general property (cf. [12, (2.114)])

$$D^{\alpha}_{a+}\left(I^{\gamma}_{a+} f\right) = D^{\alpha-\gamma}_{a+} f, \quad \alpha \geq \gamma > 0. \tag{2.5}$$

It is important to remark that the semigroup property for the composition of fractional derivatives does not hold in general (see [12, Sect. 2.3.6]). In fact, the property

$$D^{\beta}_{a+}\left(D^{\alpha}_{a+} f\right) = D^{\beta+\alpha}_{a+} f \tag{2.6}$$

holds whenever

$$f^{(j)}(a^+) = 0, \quad j = 0, 1, \ldots, m-1, \tag{2.7}$$

and $f \in AC^{m-1}([a,b])$, $f^{(m)} \in L_1(a,b)$ and $m = [\alpha]+1$. There are other sufficient conditions that ensure the semigroup property (see [5]).

One important function used in this paper is the two-parameter Mittag–Leffler function $E_{\mu,\nu}(z)$ [8], which is defined in terms of the power series by

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \nu)}, \quad \mu > 0,\ \nu > 0,\ z \in \mathbb{C}. \tag{2.8}$$

In particular, the function $E_{\mu,\nu}(z)$ is entire of order $\rho = \frac{1}{\mu}$ and type $\sigma = 1$. Two important fractional integral and differential formulae involving the two-parametric Mittag–Leffler function are the following (see [8, pp. 87–88]

$$I^{\alpha}_{a^+}\left((x-a)^{\nu-1}E_{\mu,\nu}\left(k(x-a)^{\mu}\right)\right)=(x-a)^{\alpha+\nu-1}E_{\mu,\nu+\alpha}\left(k(x-a)^{\mu}\right) \tag{2.9}$$

for all $\alpha>0, k\in\mathbb{C}, x>a, \mu>0, \nu>0$, and

$$D^{\alpha}_{a^+}\left((x-a)^{\nu-1}E_{\mu,\nu}\left(k(x-a)^{\mu}\right)\right)=(x-a)^{\nu-\alpha-1}E_{\mu,\nu-\alpha}\left(k(x-a)^{\mu}\right) \tag{2.10}$$

for all $\alpha>0, k\in\mathbb{C}, x>a, \mu>0, \nu>0, \nu\neq\alpha-p$, with $p=0,\ldots,m-1$, and $m=[\alpha]+1$.

Remark 2.3. For $\nu=\alpha-p$, with $p=0,\ldots,m-1$ and $m=[\alpha]+1$ we have that $D^{\alpha}_{a^+}((x-a)^{\alpha-p-1})=0$, which implies that the first term in the series expansion of $(x-a)^{\nu-1}E_{\mu,\nu}(k(x-a)^{\mu})$ vanishes. Therefore, the derivation rule (2.10) must be replaced in these cases by the following derivation rule:

$$\begin{aligned} &D^{\alpha}_{a^+}\left((x-a)^{\alpha-p-1}E_{\mu,\alpha-p}\left(k(x-a)^{\mu}\right)\right)\\ &\quad=(x-a)^{\mu-p-1}k\,E_{\mu,\mu-p}\left(k(x-a)^{\mu}\right), \end{aligned} \tag{2.11}$$

with $p=0,\ldots,n-1$.

The approach presented in this paper is based on the Laplace transform and leads to the solution of a linear Abel integral equation of the second kind.

Theorem 2.4. [8, Thm. 4.2] *Let $f\in L_1[a,b], \alpha>0$ and $\lambda\in\mathbb{C}$. Then the integral equation*

$$u(x)=f(x)+\frac{\lambda}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}u(t)\;dt,\quad x\in[a,b]$$

has a unique solution

$$u(x)=f(x)+\lambda\int_a^x(x-t)^{\alpha-1}E_{\alpha,\alpha}(\lambda(x-t)^{\alpha})f(t)\;dt. \tag{2.12}$$

2.2. Clifford Analysis

Let $\{e_1,\ldots,e_n\}$ be the standard basis of the Euclidean vector space in $\mathbb{R}^n$. The associated Clifford algebra $\mathbb{R}_{0,n}$ is the free algebra generated by $\mathbb{R}^n$ modulo $x^2=-||x||^2\,e_0$, where $x\in\mathbb{R}^n$ and e_0 is the neutral element with respect to the multiplication operation in the Clifford algebra $\mathbb{R}_{0,n}$. The defining relation induces the multiplication rules

$$e_ie_j+e_je_i=-2\delta_{ij}, \tag{2.13}$$

where δ_{ij} denotes the Kronecker's delta. In particular, $e_i^2=-1$ for all $i=1,\ldots,n$. The standard basis vectors thus operate as imaginary units. A vector space basis for $\mathbb{R}_{0,n}$ is given by the set $\{e_A:\ A\subseteq\{1,\ldots,n\}\}$ with $e_A=e_{l_1}e_{l_2}\ldots e_{l_r}$, where $1\leq l_1<\cdots<l_r\leq n$, $0\leq r\leq n$, $e_\emptyset:=e_0:=1$. Each $a\in\mathbb{R}_{0,n}$ can be written in the form $a=\sum_A a_A\,e_A$, with $a_A\in\mathbb{R}$. The conjugation in the Clifford algebra $\mathbb{R}_{0,n}$ is defined by $\overline{a}=\sum_A a_A\,\overline{e}_A$, where $\overline{e}_A=\overline{e}_{l_r}\,\overline{e}_{l_{r-1}}\ldots\overline{e}_{l_1}$, and $\overline{e}_j=-e_j$ for $j=1,\ldots,n$, $\overline{e}_0=e_0=1$.

Clifford analysis can be regarded as a higher-dimensional generalization of complex function theory in the sense of the Riemann approach. An $\mathbb{R}_{0,n}$-valued function f over $\Omega\subset\mathbb{R}^n$ has the representation $f=\sum_A e_Af_A$, with components $f_A : \Omega \to \mathbb{R}_{0,n}$. Properties such as continuity or differentiability have to be understood componentwise. Next, we recall the Euclidean Dirac operator $D = \sum_{j=1}^{n} e_j \ \partial_{x_j}$, which factorizes the n-dimensional Euclidean Laplace, i.e., $D^2 = -\Delta = -\sum_{j=1}^{n} \partial x_j^2$. An $\mathbb{R}_{0,n}$-valued function f is called *left-monogenic* if it satisfies $Du = 0$ on Ω (resp. *right-monogenic* if it satisfies $uD = 0$ on Ω).

For more details about Clifford algebras and basic concepts of its associated function theory we refer the interested reader for example to [3,9]. Connections between Clifford analysis and fractional calculus were studied in [5,6,10,15].

3. The Poisson Problem

Let $\Omega = \prod_{j=1}^{n} [a_j, b_j]$ be any bounded open rectangular domain, let $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in]0, 1]$, $i = 1, \ldots, n$, and let us consider the n-parameter fractional Laplace operator $\Delta_{a^+}^{\alpha}$ defined over Ω by means of the Riemann–Liouville fractional derivative given by

$$\Delta_{a^+}^{\alpha} = \sum_{j=1}^{n} \partial_{x_j^+}^{1+\alpha_j}. \tag{3.1}$$

The previous fractional operator is associated to the corresponding fractional Dirac operator defined by

$$D_{a^+}^{\alpha} = \sum_{j=1}^{n} e_j \, \partial_{x_j^+}^{\frac{1+\alpha_j}{2}}. \tag{3.2}$$

For $j = 1, \ldots, n$ the partial derivatives $\partial_{x_j^+}^{1+\alpha_j}$ and $\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}$ are the Riemann–Liouville fractional derivatives (2.2) of orders $1 + \alpha_j$ and $\frac{1+\alpha_j}{2}$, with respect to the variable $x_j \in [a_j, b_j]$. Like in the three-dimensional case (see [5]), under certain conditions we have $\Delta_{a^+}^{\alpha} = -D_{a^+}^{\alpha} \, D_{a^+}^{\alpha}$. Due to the nature of the fundamental solutions of these operators we additionally need to consider the variable $\widehat{x} = (x_2, \ldots, x_n) \in \widehat{\Omega} = \prod_{j=2}^{n} [a_j, b_j]$, and the fractional Laplace and Dirac operators acting on $\widehat{x}$ defined by:

$$\widehat{\Delta}_{a^+}^{\alpha} = \sum_{j=2}^{n} \partial_{x_j^+}^{1+\alpha_j}, \quad \widehat{D}_{a^+}^{\alpha} = \sum_{j=2}^{n} e_j \, \partial_{x_j^+}^{\frac{1+\alpha_j}{2}}. \tag{3.3}$$

Consider the following Poisson problem

$$\Delta_{a^+}^{\alpha} v(x) = u(x), \tag{3.4}$$

where we suppose that $v(x) = v(x_1, \ldots, x_n)$ admits summable fractional derivative $\partial_{x_1^+}^{1+\alpha_1} v(x)$ in the variable x_1 and belongs to $I_{a_j^+}^{1+\alpha_j}(L_1)$ in the variables x_j, for $j = 2, \ldots, n$. Starting to apply the fractional integral $I_{a_1^+}^{1+\alpha_1}$ to both sides of the previous equation and taking into account (2.4) we get

$$v(x) - \frac{(x_1-a_1)^{\alpha_1}}{\Gamma(\alpha_1+1)} \left(\partial^{\alpha_1}_{x_1^+} v\right)(a_1,\widehat{x}) - \frac{(x_1-a_1)^{\alpha_1-1}}{\Gamma(\alpha_1)} \left(I^{1-\alpha_1}_{a_1^+} v\right)(a_1,\widehat{x})$$
$$+ \sum_{k=2}^{n} \left(I^{1+\alpha_1}_{a_1^+}\, \partial^{1+\alpha_k}_{x_k^+} v\right)(x) = \left(I^{1+\alpha_1}_{a_1^+} u\right)(x).$$

Applying successively the fractional integrals $I^{1+\alpha_j}_{a_j^+}$, with $j = 2,\ldots,n$, to both sides of the previous equation, recalling that we supposed that v belongs to $I^{1+\alpha_j}_{a_j^+}(L_1)$ in the variables x_j, applying Fubini's theorem, and rearranging the terms, we obtain:

$$\left(I^{1+\alpha_1}_{a_1^+} \sum_{k=2}^{n} \prod_{\substack{j=2\\ j\neq k}}^{n} I^{1+\alpha_j}_{a_j^+} v\right)(x) + \left(\prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} v\right)(x) - \left(\prod_{j=1}^{n} I^{1+\alpha_j}_{a_j^+} u\right)(x)$$
$$= \frac{(x_1-a_1)^{\alpha_1}}{\Gamma(\alpha_1+1)} \left(\prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} f_1\right)(\widehat{x}) + \frac{(x_1-a_1)^{\alpha_1-1}}{\Gamma(\alpha_1)} \left(\prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} f_0\right)(\widehat{x}), \tag{3.5}$$

where f_0 and f_1 are fractional initial conditions given by

$$f_0(\widehat{x}) = \left(I^{1-\alpha_1}_{a_1^+} v\right)(a_1,\widehat{x}), \quad f_1(\widehat{x}) = \left(\partial^{\alpha_1}_{x_1^+} v\right)(a_1,\widehat{x}). \tag{3.6}$$

We observe that the fractional integrals in (3.5) are Laplace-transformable functions. Therefore, we may apply the n-dimensional Laplace transform with respect to $x_2,\ldots,x_n$, which we define by

$$\begin{aligned}\mathcal{F}(\widehat{s}) &= \mathcal{F}(s_2,\ldots,s_n)\\ &= \mathcal{L}\{f\}(s_2,\ldots,s_n)\\ &= \int_{a_2}^{+\infty}\cdots\int_{a_n}^{+\infty} \exp\left(-\sum_{p=2}^{n} s_p x_p\right) f(x_2,\ldots,x_n)\,dx_n\ldots dx_2.\end{aligned}$$

Taking into account its convolution and operational properties (see [4,11]), we obtain the following relations for each term in (3.5) (with $k = 2,\ldots,n$):

$$\mathcal{L}\left\{I^{1+\alpha_1}_{a_1^+} \sum_{k=2}^{n} \prod_{\substack{j=2\\ j\neq k}}^{n} I^{1+\alpha_j}_{a_j^+} v\right\}(x_1,\widehat{s}) = \sum_{k=2}^{n} \prod_{\substack{p=2\\ p\neq k}}^{n} s_p^{-1-\alpha_p} \left(I^{1+\alpha_1}_{a_1^+} \mathcal{V}\right)(x_1,\widehat{s}),$$
$$\mathcal{L}\left\{\prod_{j=2}^{n} I^{1+\alpha_j}_{a_j^+} v\right\}(x_1,\widehat{s}) = \prod_{p=2}^{n} s_p^{-1-\alpha_p} \mathcal{V}(x_1,\widehat{s}),$$
$$\mathcal{L}\left\{\prod_{j=1}^{n} I^{1+\alpha_j}_{a_j^+} u\right\}(x_1,\widehat{s}) = \prod_{p=2}^{n} s_p^{-1-\alpha_p} \left(I^{1+\alpha_1}_{a_1^+} \mathcal{U}\right)(x_1,\widehat{s}),$$

$$\mathcal{L}\left\{\frac{(x_1-a_1)^{\alpha_1-1}}{\Gamma(\alpha_1)}\left(\prod_{j=2}^{n} I_{a_j^+}^{1+\alpha_j} f_0\right)\right\}(x_1,\widehat{s}) = \frac{(x_1-a_1)^{\alpha_1-1}}{\Gamma(\alpha_1)}\prod_{p=2}^{n} s_p^{-1-\alpha_p}\mathcal{F}_0(\widehat{s}),$$
$$\mathcal{L}\left\{\frac{(x_1-a_1)^{\alpha_1}}{\Gamma(\alpha_1+1)}\left(\prod_{j=2}^{n} I_{a_j^+}^{1+\alpha_j} f_1\right)\right\}(x_1,\widehat{s}) = \frac{(x_1-a_1)^{\alpha_1}}{\Gamma(\alpha_1+1)}\prod_{p=2}^{n} s_p^{-1-\alpha_p}\mathcal{F}_1(\widehat{s}). \tag{3.7}$$

Combining all the resulting terms and multiplying by $\prod_{p=2}^{n} s_p^{1+\alpha_p}$ we obtain the following second kind homogeneous integral equation of Volterra type:

$$\mathcal{V}(x_1,\widehat{s}) + \frac{1}{\Gamma(\alpha_1+1)}\sum_{p=2}^{n} s_p^{1+\alpha_p}\int_{a_1}^{x_1}(x_1-t)^{\alpha_1}\,\mathcal{V}(t,\widehat{s})\,dt$$
$$= \left(F(x_1,\widehat{s}) + \left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s})\right), \tag{3.8}$$

where

$$F(x_1,\widehat{s}) = \frac{(x_1-a_1)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathcal{F}_0(\widehat{s}) + \frac{(x_1-a_1)^{\alpha_1}}{\Gamma(\alpha_1+1)}\mathcal{F}_1(\widehat{s})$$

and $\mathcal{F}_k(\widehat{s}) = \mathcal{L}\{f_k\}(s)$, with $k=0,1$. Using (2.12), we have that the unique solution of (3.8) in the class of summable functions is:

$$\mathcal{V}(x_1,\widehat{s}) = F(x_1,\widehat{s}) + \left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s})$$
$$-\sum_{p=2}^{n} s_p^{1+\alpha_p}\int_{a_1}^{x_1}(x_1-t)^{\alpha}E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-t)^{\alpha_1+1}\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)$$
$$\times\left(F(t,\widehat{s}) + \left(I_{a_1^+}^{1+\alpha_1}\mathcal{U}\right)(x_1,\widehat{s})\right)\,dt, \tag{3.9}$$

which involves the two-parameter Mittag–Leffler function. Due the convergence of the integrals and the series that appear in (3.9), we can interchange them and rewrite (3.9) in the following way:

$$\mathcal{V}(x_1,\widehat{s}) = (x_1-a_1)^{\alpha_1-1}\,E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{\alpha_1+1}\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)F_0(s)$$
$$+(x_1-a_1)^{\alpha_1}\,E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{\alpha_1+1}\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)F_1(s)$$
$$+\sum_{n=0}^{+\infty}\left(-\left(\sum_{p=2}^{n} s_p^{1+\alpha_p}\right)\right)^{n} I_{a_1^+}^{(1+\alpha_1)(n+1)}\mathcal{U}(x_1,\widehat{s}). \tag{3.10}$$

In order to cancel the Laplace transform, we need to take into account its distributional form in Zemanian's space (for more details about generalized integral transforms see [16]) and the following relation:

$$\lim_{r_2,\ldots,r_n\to+\infty}\int_{\sigma_1-ir_2}^{\sigma_1+ir_2}\cdots\int_{\sigma_n-ir_n}^{\sigma_n+ir_n}\prod_{p=2}^{n}s_p^{n(1+\alpha_p)}F_k(s)\exp\left(\sum_{p=2}^{n}s_px_p\right)ds_n\ldots ds_2$$
$$=\left(\prod_{j=2}^{n}\partial_{x_j^+}^{n(1+\alpha_j)}f_k\right)(\widehat{x}),$$

where $k=0,1$. Therefore, applying the multinomial theorem and after straightforward calculations we get the following solution of (3.4):

$$v(x)=\sum_{k=0}^{+\infty}(-1)^k\frac{(x_1-a_1)^{k(1+\alpha_1)+\alpha_1-1}}{\Gamma((1+\alpha_1)k+\alpha_1)}\left(\widehat{\Delta}_{a^+}^{\alpha}\right)^k f_0(\widehat{x})$$
$$+\sum_{k=0}^{+\infty}(-1)^k\frac{(x_1-a_1)^{k(1+\alpha_1)+\alpha_1}}{\Gamma((1+\alpha_1)k+(1+\alpha_1))}\left(\widehat{\Delta}_{a^+}^{\alpha}\right)^k f_1(\widehat{x})$$
$$+\sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)}u(x). \tag{3.11}$$

From the previous calculations we obtain the following theorem, where we describe the solution of the Poisson equation in a operator form using the Mittag–Leffler function (2.8).

Theorem 3.1. *The solution $v(x)$ of the Poisson equation (3.4) is given in the operator form by*

$$v(x)=(x_1-a_1)^{\alpha_1-1}E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{\alpha_1+1}\widehat{\Delta}_{a^+}^{\alpha}\right)f_0(\widehat{x})$$
$$+(x_1-a_1)^{\alpha_1}E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{\alpha_1+1}\widehat{\Delta}_{a^+}^{\alpha}\right)f_1(\widehat{x}),$$
$$+\sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)}u(x), \tag{3.12}$$

where the functions f_0 and f_1 are the Cauchy's fractional conditions given by (3.6)*.*

Proof. We give a direct proof of the theorem. Since $\partial_{x_1^+}^{1+\alpha_1}(x_1-a_1)^{\alpha_1-1}=0$ and $\partial_{x_1^+}^{1+\alpha_1}(x_1-a_1)^{\alpha_1}=0$ we need to use the derivation rule (2.11) with respect to x_1 for the first two terms of (3.12). Concerning the third term in (3.12) we take into account that $\partial_{x_1^+}^{1+\alpha_1}I_{a_1^+}^{1+\alpha_1}=I$. Therefore, applying the operator $\Delta_{a^+}^{\alpha}=\partial_{x_1^+}^{1+\alpha_1}+\widehat{\Delta}_{a^+}^{\alpha}$ to (3.12) we obtain

$$\Delta_{a^+}^{\alpha}v(x)=-\widehat{\Delta}_{a^+}^{\alpha}(x_1-a_1)^{\alpha_1-1}E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{\alpha_1+1}\widehat{\Delta}_{a^+}^{\alpha}\right)f_0(\widehat{x})$$
$$+\widehat{\Delta}_{a^+}^{\alpha}(x_1-a_1)^{\alpha_1-1}E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{\alpha_1+1}\widehat{\Delta}_{a^+}^{\alpha}\right)f_0(\widehat{x})$$
$$-\widehat{\Delta}_{a^+}^{\alpha}(x_1-a_1)^{\alpha_1}E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{\alpha_1+1}\widehat{\Delta}_{a^+}^{\alpha}\right)f_1(\widehat{x})$$
$$+\widehat{\Delta}_{a^+}^{\alpha}(x_1-a_1)^{\alpha_1}E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{\alpha_1+1}\widehat{\Delta}_{a^+}^{\alpha}\right)f_1(\widehat{x})$$

$$
\begin{aligned}
&+u(x) - \widehat{\Delta}^{\alpha}_{a^+} \sum_{k=0}^{+\infty} \left(-\widehat{\Delta}^{\alpha}_{a^+}\right)^k I^{(1+\alpha_1)(k+1)}_{a_1^+} u(x) \\
&+\widehat{\Delta}^{\alpha}_{a^+} \sum_{k=0}^{+\infty} \left(-\widehat{\Delta}^{\alpha}_{a^+}\right)^k I^{(1+\alpha_1)(k+1)}_{a_1^+} u(x) \\
&= u(x).
\end{aligned}
$$

□

We remark that if $u(x) = 0$ then (3.12) reduces to the fundamental solution of the fractional Laplace operator. Moreover, in the special case $\alpha = (1, \ldots, 1)$, considering $f_0(\widehat{x}) = \|\widehat{x} - \widehat{a}\|^{2-n}$ and $f_1(\widehat{x}) = 0$, the solution of the Poisson problem $\Delta u = v$ has the following representation

$$
v(x) = \|x - a\|^{2-n} + \sum_{n=0}^{+\infty} \left(-\widehat{\Delta}\right)^n I^{2(n+1)}_{a_1^+} u(x), \tag{3.13}
$$

where $\widehat{\Delta} = \sum_{j=2}^{n} \partial^2_{x_j}$.

4. Fundamental Solution for Natural Powers of the Fractional Laplace Operator

Consider a function G_1^α be a fundamental solution of $\Delta^\alpha_{a^+}$ and a function G_2^α such that $\Delta^\alpha_{a^+} G_2^\alpha = G_1^\alpha$. Then G_2^α is a fundamental solution of $(\Delta^\alpha_{a^+})^2$, since $(\Delta^\alpha_{a^+})^2 G_2^\alpha = \Delta^\alpha_{a^+}(\Delta^\alpha_{a^+} G_2^\alpha) = \Delta^\alpha_{a^+} G_1^\alpha = \delta$. In a similar way, if G_3^α is such that $\Delta^\alpha_{a^+} G_3^\alpha = G_2^\alpha$ then G_3^α is a fundamental solution of $(\Delta^\alpha_{a^+})^3$. Hence we can deduce by induction the following theorem.

Theorem 4.1. *Let G_i^α, with $i \in \mathbb{N}$, be a fundamental solution of $(\Delta^\alpha_{a^+})^i$. Then the function G_{i+1}^α such that $\Delta^\alpha_{a^+} G_{i+1}^\alpha = G_i^\alpha$ is a fundamental solution of $(\Delta^\alpha_{a^+})^{i+1}$.*

Using Theorem 4.1 we can deduce an expression for G_i^α. First we need the expression for the fundamental solution of $\Delta^\alpha_{a^+}$, which can be obtained putting $u = 0$ in Theorem 3.1.

Theorem 4.2. *A family of fundamental solutions for the fractional Laplace operator $\Delta^\alpha_{a^+}$ is given by*

$$
\begin{aligned}
G_1^\alpha(x) = {} & (x_1 - a_1)^{\alpha_1 - 1} E_{1+\alpha_1, \alpha_1}\left(-(x_1 - a_1)^{1+\alpha_1} \widehat{\Delta}^\alpha_{a^+}\right) f_{1,0}(\widehat{x}) \\
& + (x_1 - a_1)^{\alpha_1} E_{1+\alpha_1, 1+\alpha_1}\left(-(x_1 - a_1)^{1+\alpha_1} \widehat{\Delta}^\alpha_{a^+}\right) f_{1,1}(\widehat{x}),
\end{aligned} \tag{4.1}
$$

where

$$
f_{1,0}(\widehat{x}) = \left(I^{1-\alpha_1}_{a_1^+} G_1^\alpha\right)(a_1, \widehat{x}), \quad f_{1,1}(\widehat{x}) = \left(\partial^{\alpha_1}_{x_1^+} G_1^\alpha\right)(a_1, \widehat{x}). \tag{4.2}
$$

Taking into account Theorem 3.1, Theorem 4.2, and the transition relation $\Delta^\alpha_{a^+} G_2^\alpha = G_1^\alpha$, we can deduce the following expression for G_2^α, which is the fundamental solution of $(\Delta^\alpha_{a^+})^2$, in terms of G_1^α:

$$
\begin{aligned}
G_2^\alpha(x) &= (x_1-a_1)^{\alpha_1-1}\, E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{2,0}(\widehat{x}) \\
&\quad + (x_1-a_1)^{\alpha_1}\, E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{2,1}(\widehat{x}) \\
&\quad + \sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^\alpha\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)} G_1^\alpha(x),
\end{aligned}
\tag{4.3}
$$

where $f_{2,0}(\widehat{x}) = (I_{a_1^+}^{1-\alpha_1} G_2^\alpha)(a_1,\widehat{x})$ and $f_{2,1}(\widehat{x}) = (\partial_{x_1^+}^{\alpha_1} G_2^\alpha)(a_1,\widehat{x})$. By induction and using the transition relation $\Delta_{a^+}^\alpha G_{i+1}^\alpha = G_i^\alpha$ we obtain the following result.

Theorem 4.3. *For $i\in\mathbb{N}$, a family of fundamental solutions G_{i+1}^α for the operator $(\Delta_{a^+}^\alpha)^{i+1}$ is given by*

$$
\begin{aligned}
G_{i+1}^\alpha(x) &= (x_1-a_1)^{\alpha_1-1}\, E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{i+1,0}(\widehat{x}) \\
&\quad + (x_1-a_1)^{\alpha_1}\, E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{i+1,1}(\widehat{x}) \\
&\quad + \sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^\alpha\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)} G_i^\alpha(x)
\end{aligned}
\tag{4.4}
$$

where $f_{i+1,0}(\widehat{x}) = (I_{a_1^+}^{1-\alpha_1} G_{i+1}^\alpha)(a_1,\widehat{x})$, $f_{i+1,1}(\widehat{x}) = (\partial_{x_1^+}^{\alpha_1} G_{i+1}^\alpha)(a_1,\widehat{x})$, and G_i^α is a fundamental solution of $(\Delta_{a^+}^\alpha)^i$.

Example 4.4. Here we present another expression for G_2^α. Substituting (4.1) into (4.3) we have

$$
\begin{aligned}
G_2^\alpha(x) &= (x_1-a_1)^{\alpha_1-1}\, E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{2,0}(\widehat{x}) \\
&\quad +(x_1-a_1)^{\alpha_1}\, E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{2,1}(\widehat{x}) \\
&\quad +\sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^\alpha\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)} \Big[(x_1-a_1)^{\alpha_1-1} \\
&\quad \times E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{1,0}(\widehat{x})\Big] \\
&\quad +\sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^\alpha\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)} \Big[(x_1-a_1)^{\alpha_1} \\
&\quad \times E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{1,1}(\widehat{x})\Big].
\end{aligned}
$$

Making use of the integral formula (2.9) to calculate the fractional integrals that appear in the last two terms, we obtain

$$
\begin{aligned}
G_2^\alpha(x) &= (x_1-a_1)^{\alpha_1-1}\, E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{2,0}(\widehat{x}) \\
&\quad +(x_1-a_1)^{\alpha_1}\, E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^\alpha\right) f_{2,1}(\widehat{x}) \\
&\quad +(x_1-a_1)^{2\alpha_1}\sum_{k=0}^{+\infty}(x_1-a_1)^{(1+\alpha_1)k}\left(-\widehat{\Delta}_{a^+}^\alpha\right)^k
\end{aligned}
$$

$$\times E_{1+\alpha_1,1+2\alpha_1+(1+\alpha_1)k}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}^{\alpha}_{a^+}\right) f_{1,0}(\widehat{x})$$
$$+(x_1-a_1)^{1+2\alpha_1}\sum_{k=0}^{+\infty}(x_1-a_1)^{(1+\alpha_1)k}\left(-\widehat{\Delta}^{\alpha}_{a^+}\right)^k$$
$$\times E_{1+\alpha_1,(1+\alpha_1)(k+2)}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}^{\alpha}_{a^+}\right) f_{1,1}(\widehat{x}). \qquad (4.5)$$

Concerning the integer case, i.e. $\alpha=(1,\ldots,1)$, the fundamental solution of the equation $\Delta^l u=0$ which is polyharmonic of least degree l in $\mathbb{R}^n\backslash\{a\}$ is given by (cf. [2]):

$$u_l(x)=\frac{\|x-a\|^{2l-n}}{\gamma_{l-1}}\quad \text{for odd } n \text{ and } l=1,2,\ldots \qquad (4.6)$$

$$u_l(x)=\begin{cases}\dfrac{\|x-a\|^{2l-n}}{\gamma'_{l-1}}, & \text{for even } n \text{ and } l=1,2,\ldots,\dfrac{n}{2}-1\\ -\dfrac{\|x-a\|^{2l-n}\ln(\|x-a\|)}{\gamma'_{l-1}}, & \text{for even } n \text{ and } l=\dfrac{n}{2},\dfrac{n}{2}+1,\ldots\end{cases}, \qquad (4.7)$$

where the constants γ_l and γ'_l are presented in [2, p.8]. Let's examine first the case when n is odd. Considering the binomial series

$$(1-x)^{-s}=\sum_{k=0}^{+\infty}\binom{s+k-1}{k}x^k,\quad |x|<1$$

we obtain the series expansion

$$\begin{aligned}u_l(x)&=\frac{1}{\gamma_{l-1}}\left((x_1-a_1)^2+\|\widehat{x}-\widehat{a}\|^2\right)^{-\frac{n-2l}{2}}\\ &=\frac{1}{\gamma_{l-1}}\|\widehat{x}-\widehat{a}\|^{-(n-l)}\left(1+\frac{(x_1-a_1)^2}{\|\widehat{x}-\widehat{a}\|^2}\right)^{-\frac{n-2l}{2}}\\ &=\frac{1}{\gamma_{l-1}}\sum_{k=0}^{+\infty}(-1)^k\binom{\frac{n}{2}-l+k-1}{k}\frac{(x_1-a_1)^{2k}}{\|\widehat{x}-\widehat{a}\|^{n-2l+2k}},\end{aligned} \qquad (4.8)$$

with $\|\widehat{x}-\widehat{a}\|^2=\sum_{i=2}^n(x_i-a_i)^2$, and $\dfrac{(x_1-a_1)^2}{\|\widehat{x}-\widehat{a}\|^2}<1$. From (4.4) putting $i+1=l$ with $l\geq 2$, $f_{l,0}(\widehat{x})=u_l(a_1,\widehat{x})=\frac{\|\widehat{x}-\widehat{a}\|^{2l-n}}{\gamma_{l-1}}$, $f_{l,1}(\widehat{x})=(\partial_{x_1}u_l)(a_1,\widehat{x})=0$, and $G_{l-1}(x)=\frac{\|x-a\|^{2l-n-2}}{\gamma_{l-2}}$ we obtain a new fundamental solution of Δ^l given by

$$\begin{aligned}G_l(x)&=\frac{1}{\gamma_{l-1}}\sum_{k=0}^{+\infty}\frac{(-1)^k(x_1-a_1)^{2k}}{\Gamma(2k+1)}\left(\widehat{\Delta}\right)^k\|\widehat{x}-\widehat{a}\|^{2l-n}\\ &\quad+\frac{1}{\gamma_{l-2}}\sum_{k=0}^{+\infty}\left(-\widehat{\Delta}\right)^k I^{2k+2}_{a_1^+}\|x-a\|^{2l-n-2}.\end{aligned} \qquad (4.9)$$

From formula (1.5) in [2] we have that

$$\begin{aligned}\left(\widehat{\Delta}\right)^k \|\widehat{x}-\widehat{a}\|^{2l-n} &= \frac{2^{2k}\,\Gamma\left(l-\frac{n}{2}+1\right)\,\Gamma\left(l-\frac{1}{2}\right)}{\Gamma\left(l-\frac{n}{2}+1-k\right)\,\Gamma\left(l-\frac{1}{2}-k\right)}\|\widehat{x}-\widehat{a}\|^{2l-n-2k}\\ &= \frac{2^{2k}\,\Gamma\left(l-\frac{n}{2}+1\right)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(l-\frac{n}{2}+1-k\right)\,\Gamma\left(\frac{1}{2}-k\right)}\frac{\Gamma\left(\frac{1}{2}-k\right)\,\Gamma\left(l-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(l-\frac{1}{2}-k\right)}\|\widehat{x}-\widehat{a}\|^{2l-n-2k}\\ &= \frac{\Gamma\left(\frac{n}{2}-l+k\right)\,\Gamma\left(2k+1\right)}{\Gamma\left(\frac{n}{2}-l\right)\,\Gamma\left(k+1\right)}\frac{\Gamma\left(\frac{1}{2}-k\right)\,\Gamma\left(l-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(l-\frac{1}{2}-k\right)}\|\widehat{x}-\widehat{a}\|^{2l-n-2k}\\ &= \binom{\frac{n}{2}-l+k-1}{k}\Gamma(2k+1)\frac{\left(\frac{1}{2}\right)_{l-1}}{\left(\frac{1}{2}-k\right)_{l-1}}\|\widehat{x}-\widehat{a}\|^{2l-n-2k}, \qquad (4.10)\end{aligned}$$

where the last identities follow from straightforward calculations involving the properties of the Gamma function and the Pochhammer symbol. Putting (4.10) in (4.9) we obtain

$$\begin{aligned}G_l(x) &= \frac{1}{\gamma_{l-1}}\sum_{k=0}^{+\infty}(-1)^k\binom{\frac{n}{2}-l+k-1}{k}\frac{\left(\frac{1}{2}\right)_{l-1}}{\left(\frac{1}{2}-k\right)_{l-1}}\frac{(x_1-a_1)^{2k}}{\|\widehat{x}-\widehat{a}\|^{n-2l+2k}}\\ &\quad+\frac{1}{\gamma_{l-2}}\sum_{k=0}^{+\infty}\left(-\widehat{\Delta}\right)^k I_{a_1^+}^{2k+2}\|x-a\|^{2l-n-2}. \qquad (4.11)\end{aligned}$$

Comparing (4.8) with the first term of (4.11) we immediately see that they differ from the factor $\frac{(\frac{1}{2})_{l-1}}{(\frac{1}{2}-k)_{l-1}}$, which is a constant under the action of the operator Δ^l. Moreover, the restriction $G_l(a_1,\widehat{x})$ is equal to $u_l(a_1,\widehat{x}) = \frac{\|\widehat{x}-\widehat{a}\|^{2l-n}}{\gamma_{l-1}}$. For the second term in (4.11) we observe that it could be omitted since the first term corresponds to the characteristic singular solution of the equation $\Delta^l u = 0$. Since $G_l(x)$ and $u_l(x)$ are fundamental solutions for the equation $\Delta^l u = 0$ they differ from a polyharmonic function of least degree l. In fact, putting $F(x) = G_l(x) - u_l(x)$ we have that $G_l(x) = u_l(x) + F(x)$.

We proceed now with the case when n is even. For $l \leq \frac{n}{2}-1$ the fundamental solution coincide with (4.11). For $l \geq \frac{n}{2}$, considering in (4.4) $i+1 = l$ with $l \geq 2$, $f_{l,0}(\widehat{x}) = u_l(a_1,\widehat{x}) = -\frac{\|\widehat{x}-\widehat{a}\|^{2l-n}\ln(\|\widehat{x}-\widehat{a}\|)}{\gamma'_{l-1}}$, $f_{l,1}(\widehat{x}) = \left(\partial_{x_1}u_l\right)(a_1,\widehat{x}) = 0$, and $G_{l-1}(x)$ given by

$$G_{l-1}(x) = \begin{cases}\dfrac{\|\widehat{x}-\widehat{a}\|^{-2}}{\gamma'_{\frac{n}{2}-2}}, & l = \dfrac{n}{2}\\ -\dfrac{\|\widehat{x}-\widehat{a}\|^{2l-n-2}\ln\left(\|\widehat{x}-\widehat{a}\|\right)}{\gamma'_{l-2}}, & l > \dfrac{n}{2}\end{cases} \qquad (4.12)$$

we obtain a new fundamental solution of Δ^l given by

$$\begin{aligned}G_l(x) &= -\frac{1}{\gamma'_{l-2}}\sum_{k=0}^{+\infty}\frac{(-1)^k(x_1-a_1)^{2k}}{\Gamma(2k+1)}\left(\widehat{\Delta}\right)^k\left(\|\widehat{x}-\widehat{a}\|^{2l-n}\ln\left(\|\widehat{x}-\widehat{a}\|\right)\right)\\ &\quad+\sum_{k=0}^{+\infty}\left(-\widehat{\Delta}\right)^k I_{a_1^+}^{2k+2}G_{l-1}(x). \qquad (4.13)\end{aligned}$$

From formula (1.7) in [2] we have that

$$\begin{aligned}
&\left(\widehat{\Delta}\right)^k \left(\|\widehat{x}-\widehat{a}\|^{2l-n}\,\ln\left(\|\widehat{x}-\widehat{a}\|\right)\right)\\
&\quad= \frac{2^{2k}\,\Gamma\left(l-\frac{n}{2}+1\right)\,\Gamma\left(l-\frac{1}{2}\right)}{\Gamma\left(l-\frac{n}{2}+1-k\right)\,\Gamma\left(l-\frac{1}{2}-k\right)}\,\|\widehat{x}-\widehat{a}\|^{2l-n-2k}\\
&\qquad\times\left[\ln\left(\|\widehat{x}-\widehat{a}\|\right)+\sum_{i=1}^{k}\left(\frac{1}{2l-n-2i+2}+\frac{1}{2l-2i}\right)\right]\\
&\quad= \binom{\frac{n}{2}-l+k-1}{k}\,\Gamma(2k+1)\,\frac{\left(\frac{1}{2}\right)_{l-1}}{\left(\frac{1}{2}-k\right)_{l-1}}\,\|\widehat{x}-\widehat{a}\|^{2l-n-2k}\\
&\qquad\times\left[\ln\left(\|\widehat{x}-\widehat{a}\|\right)+\sum_{i=1}^{k}\left(\frac{1}{2l-n-2i+2}+\frac{1}{2l-2i}\right)\right], \qquad (4.14)
\end{aligned}$$

where the last identity follows from straightforward calculations involving the properties of the Gamma function and the Pochhammer symbol. Putting (4.14) in (4.13) we obtain a new fundamental solution for the equation f $\Delta^l u = 0$, as it was done in (4.11).

5. Fundamental Solution for Natural Powers of the Fractional Dirac Operator

The line or reasoning is similar to the case of the natural powers of the fractional Laplace operator, and we assume the conditions indicated in [5] that ensure the semigroup property (2.6).

For the even powers of the fractional Dirac operator we have that $(D^\alpha_{a^+})^{2i} = (-\Delta^\alpha_{a^+})^i = (-1)^i(\Delta^\alpha_{a^+})^i$, with $i \in \mathbb{N}$, therefore the fundamental solution of $(D^\alpha_{a^+})^{2i}$ coincide with the fundamental solution of $(\Delta^\alpha_{a^+})^i$.

For the odd case, consider a function $\mathcal{G}^\alpha_1$ be a fundamental solution of $D^\alpha_{a^+}$ and a function $\mathcal{G}^\alpha_2$ such that $\Delta^\alpha_{a^+}\mathcal{G}^\alpha_2 = \mathcal{G}^\alpha_1$. Then $\mathcal{G}^\alpha_2$ is a fundamental solution of $(D^\alpha_{a^+})^3$, since $(D^\alpha_{a^+})^3\mathcal{G}^\alpha_2 = D^\alpha_{a^+}(\Delta^\alpha_{a^+}\mathcal{G}^\alpha_2) = D^\alpha_{a^+}\mathcal{G}^\alpha_1 = \delta$. In a similar way, if $\mathcal{G}^\alpha_3$ is such that $\Delta^\alpha_{a^+}\mathcal{G}^\alpha_3 = \mathcal{G}^\alpha_2$ then $\mathcal{G}^\alpha_3$ is a fundamental solution of $(D^\alpha_{a^+})^5$. Hence we can deduce by induction the following theorem.

Theorem 5.1. *Let $\mathcal{G}^\alpha_i$, with $i \in \mathbb{N}$, be a fundamental solution of $(D^\alpha_{a^+})^{2i-1}$. Then the function G^α_{i+1} such that $\Delta^\alpha_{a^+}\mathcal{G}^\alpha_{i+1} = \mathcal{G}^\alpha_i$ is a fundamental solution of $(D^\alpha_{a^+})^{2i+1}$.*

Using Theorem 5.1 we can deduce and expression for $\mathcal{G}^\alpha_i$. We start recalling the expression for the fundamental solution of $D^\alpha_{a^+}$ deduced in [7], which corresponds to the function $\mathcal{G}^\alpha_1$.

Theorem 5.2. *A family of fundamental solutions of the fractional Dirac operator $D^\alpha_{a^+}$ is given by*

$$\mathcal{G}^\alpha_1(x) = \sum_{j=1}^{n} e_j\,\left(\mathcal{G}^\alpha_1\right)_j(x),$$

where the function components are given by

$$(\mathcal{G}_1^\alpha)_1(x) = (x_1-a_1)^{\frac{\alpha_1-3}{2}} E_{1+\alpha_1,\frac{\alpha_1-1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}^\alpha_{a^+}\right)(f_{1,0})_1(\widehat{x}) + (x_1-a_1)^{\frac{\alpha_1-1}{2}} E_{1+\alpha_1,\frac{1+\alpha_1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}^\alpha_{a^+}\right)(f_{1,1})_1(\widehat{x}), \tag{5.1}$$

and for $j=2,\dots,n$

$$(\mathcal{G}_1^\alpha)_j(x) = (x_1-a_1)^{\alpha_1-1}\left(E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}^\alpha_{a^+}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)(f_{1,0})_j(\widehat{x}) + (x_1-a_1)^{\alpha_1}\left(E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}^\alpha_{a^+}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)(f_{1,1})_j(\widehat{x}), \tag{5.2}$$

where $(f_{1,0})_j(\widehat{x}) = (I_{a_1^+}^{1-\alpha_1}(\mathcal{G}_1^\alpha)_j)(a_1,\widehat{x})$, *and* $(f_{1,1})_j(\widehat{x}) = (\partial_{x_1^+}^{\alpha_1}(\mathcal{G}_1^\alpha)_j)(a_1,\widehat{x})$ *with* $j=1,\dots,n$.

Taking into account Theorem 3.1, Theorem 5.2, and the transition relation $\Delta^\alpha_{a^+}\mathcal{G}_2^\alpha = \mathcal{G}_1^\alpha$, we can deduce the following expression for $\mathcal{G}_2^\alpha$, which is the fundamental solution of $(D^\alpha_{a^+})^3$, in terms of $\mathcal{G}_1^\alpha$:

$$\mathcal{G}_2^\alpha(x) = \sum_{j=1}^n e_j\,(\mathcal{G}_2^\alpha)_j(x),$$

where

$$(\mathcal{G}_2^\alpha)_1(x) = (x_1-a_1)^{\alpha_1-1} E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}^\alpha_{a^+}\right)(f_{2,0})_1(\widehat{x}) + (x_1-a_1)^{\alpha_1} E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}^\alpha_{a^+}\right)(f_{2,1})_1(\widehat{x}) + \sum_{k=0}^{+\infty}\left(-\widehat{\Delta}^\alpha_{a^+}\right)^k I_{a_1^+}^{(1+\alpha_1)\left(k+\frac{1}{2}\right)}(\mathcal{G}_1^\alpha)_1(x), \tag{5.3}$$

and for $j=2,\dots,n$

$$(\mathcal{G}_2^\alpha)_j(x) = (x_1-a_1)^{\alpha_1-1}\left(E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}^\alpha_{a^+}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)(f_{2,0})_j(\widehat{x}) + (x_1-a_1)^{\alpha_1}\left(E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}^\alpha_{a^+}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)(f_{2,1})_j(\widehat{x}) + \sum_{k=0}^{+\infty}\left(-\widehat{\Delta}^\alpha_{a^+}\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)}\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}(\mathcal{G}_1^\alpha)_j(x), \tag{5.4}$$

where $(f_{2,0})_j(\widehat{x}) = (I_{a_1^+}^{1-\alpha}(\mathcal{G}_2^\alpha)_j)(a_1,\widehat{x})$, and $(f_{2,1})_j(\widehat{x}) = (\partial_{x_1^+}^{\alpha_1}(\mathcal{G}_2^\alpha)_j)(a_1,\widehat{x})$, with $j=1,\dots,n$. By induction and using the transition relation $\Delta^\alpha_{a^+}\mathcal{G}_{i+1}^\alpha = \mathcal{G}_i^\alpha$ we can deduce the following result:

Theorem 5.3. *For* $i\in\mathbb{N}$, *a family of fundamental solutions* $\mathcal{G}_{i+1}^\alpha$ *for the operator* $\left(D^\alpha_{a^+}\right)^{2i+1}$ *is given by*

$$\mathcal{G}_{i+1}^\alpha(x) = \sum_{j=1}^n e_j\left(\mathcal{G}_{i+1}^\alpha\right)_j(x),$$

where

$$\left(\mathcal{G}_{i+1}^{\alpha}\right)_1(x) = (x_1-a_1)^{\alpha_1-1}\, E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)(f_{i+1,0})_1(\widehat{x}) + (x_1-a_1)^{\alpha_1}\, E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)(f_{i+1,1})_1(\widehat{x}) + \sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k I_{a_1^+}^{(1+\alpha_1)\left(k+\frac{1}{2}\right)}(\mathcal{G}_i^{\alpha})_1(x),$$

and for $j=2,\dots,n$

$$\left(\mathcal{G}_{i+1}^{\alpha}\right)_j(x) = (x_1-a_1)^{\alpha_1-1}\left(E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)(f_{i+1,0})_j(\widehat{x}) + (x_1-a_1)^{\alpha_1}\left(E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)(f_{i+1,1})_j(\widehat{x}) + \sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)}\,\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}(\mathcal{G}_i^{\alpha})_j(x),$$

with, for $j=1,\dots,n$, $(f_{i+1,0})_j(\widehat{x}) = (I_{a_1^+}^{1-\alpha}(\mathcal{G}_{i+1}^{\alpha})_j)(a_1,\widehat{x})$, $(f_{i+1,1})_j(\widehat{x}) = (\partial_{x_1^+}^{\alpha_1}(\mathcal{G}_{i+1}^{\alpha})_j)(a_1,\widehat{x})$, *and* $\mathcal{G}_i^{\alpha}$ *is a fundamental solution of* $(D_{a^+}^{\alpha})^{2i-1}$.

Example 5.4. Here we present another expression for $\mathcal{G}_2^{\alpha}$. Substituting (5.1) and (5.2) into (5.3) and (5.4), respectively, they become equal to

$$\left(\mathcal{G}_2^{\alpha}\right)_1(x) = (x_1-a_1)^{\alpha_1-1}\, E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)(f_{2,0})_1(\widehat{x}) + (x_1-a_1)^{\alpha_1}\, E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)(f_{2,1})_1(\widehat{x}) + \sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k I_{a_1^+}^{(1+\alpha_1)\left(k+\frac{1}{2}\right)}\Big[(x_1-a_1)^{\frac{\alpha_1-3}{2}} \times E_{1+\alpha_1,\frac{\alpha_1-1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)(f_{1,0})_1(\widehat{x})\Big] + \sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k I_{a_1^+}^{(1+\alpha_1)\left(k+\frac{1}{2}\right)}\Big[(x_1-a_1)^{\frac{\alpha_1-1}{2}} \times E_{1+\alpha_1,\frac{1+\alpha_1}{2}}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)(f_{1,1})_1(\widehat{x})\Big],$$

and for $j=2,\dots,n$

$$\left(\mathcal{G}_2^{\alpha}\right)_j(x) = (x_1-a_1)^{\alpha_1-1}\left(E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)(f_{2,0})_j(\widehat{x}) + (x_1-a_1)^{\alpha_1}\left(E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)(f_{2,1})_j(\widehat{x}) + \sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)} \times \partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\Big[(x_1-a_1)^{\alpha_1-1}E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\,\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}(f_{1,0})_j(\widehat{x})\Big]$$

$$
\begin{aligned}
&+\sum_{k=0}^{+\infty}\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k I_{a_1^+}^{(1+\alpha_1)(k+1)} \\
&\times \partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\left[(x_1-a_1)^{\alpha_1}E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\left(f_{1,1}\right)_j(\widehat{x})\right].
\end{aligned} \tag{5.5}
$$

Making use of the integral formula (2.9) to calculate the fractional integrals that appear in the last two terms of the two previous expressions, we finally obtain

$$
\begin{aligned}
\mathcal{G}_{2,1}^{\alpha}(x) =&(x_1-a_1)^{\alpha_1-1}E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}_{a^+}^{\alpha}\right)\left(f_{2,0}\right)_1(\widehat{x}) \\
&+(x_1-a_1)^{\alpha_1}E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}_{a^+}^{\alpha}\right)\left(f_{2,1}\right)_1(\widehat{x}) \\
&+(x_1-a_1)^{\alpha_1-1}\sum_{k=0}^{+\infty}(x_1-a_1)^{(1+\alpha_1)k} \\
&\times\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k E_{1+\alpha_1,\alpha_1+(1+\alpha_1)k}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}_{a^+}^{\alpha}\right)\left(f_{1,0}\right)_1(\widehat{x}) \\
&(x_1-a_1)^{\alpha_1}\sum_{k=0}^{+\infty}(x_1-a_1)^{(1+\alpha_1)k} \\
&\times\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k E_{1+\alpha_1,(1+\alpha_1)(k+1)}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}_{a^+}^{\alpha}\right)\left(f_{1,1}\right)_1(\widehat{x}),
\end{aligned}
$$

and for $j=2,\ldots,n$

$$
\begin{aligned}
\left(\mathcal{G}_2^{\alpha}\right)_j(x) =& (x_1-a_1)^{\alpha_1-1}\left(E_{1+\alpha_1,\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)\left(f_{2,0}\right)_j(\widehat{x}) \\
&+(x_1-a_1)^{\alpha_1}\left(E_{1+\alpha_1,1+\alpha_1}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{\frac{1+\alpha_j}{2}}\right)\left(f_{2,1}\right)_j(\widehat{x}) \\
&+(x_1-a_1)^{2\alpha_1}\sum_{k=0}^{+\infty}(x_1-a_1)^{(1+\alpha_1)k} \\
&\times\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k E_{1+\alpha_1,1+2\alpha_1+(1+\alpha_1)k}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{1+\alpha_j}\left(f_{1,0}\right)_j(\widehat{x}) \\
&+(x_1-a_1)^{1+2\alpha_1}\sum_{k=0}^{+\infty}(x_1-a_1)^{(1+\alpha_1)k} \\
&\times\left(-\widehat{\Delta}_{a^+}^{\alpha}\right)^k E_{1+\alpha_1,(1+\alpha_1)(k+2)}\left(-(x_1-a_1)^{1+\alpha_1}\widehat{\Delta}_{a^+}^{\alpha}\right)\partial_{x_j^+}^{1+\alpha_j}\left(f_{1,1}\right)_j(\widehat{x}).
\end{aligned} \tag{5.6}
$$

6. Conclusions and Future Work

In this paper, we presented an expression for the family of fundamental solutions of natural powers of the n-parameter fractional Laplace operator, as well for the family of fundamental solutions of natural powers of the fractional Dirac operator. However, it is desirable to find an explicit expression for the functions f_0 and f_1 in the Cauchy's fractional conditions (3.6), in order to obtain more explicit expressions for the results obtained in Sects. 4 and 5. This will be subject to future work.

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