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RESEARCH ARTICLE

The Nine Lemma and the push forward construction for special Schreier extensions of monoids with operations

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Abstract
We show that the Nine Lemma holds for special Schreier extensions of monoids with operations. This fact is used to obtain a push forward construction for special Schreier extensions with abelian kernel. This construction permits to give a functorial description of the Baer sum of such extensions.

Keywords Monoids with operations · Special Schreier extension · Nine Lemma · Push forward · Baer sum

1 Introduction

Actions of a group B on a group X are classically defined as group homomorphisms from B to the group $\mathrm{Aut}(X)$ of automorphisms of X. There is a well known equivalence between group actions and split extensions, obtained via the semidirect product construction. Monoid actions are defined similarly: an action of a monoid B on a monoid X is a monoid homomorphism from B to the monoid $\mathrm{End}(X)$ of endomorphisms of X. It is not difficult to see that these actions do not correspond to all split epimorphisms of monoids; hence the question of what are the split extensions of monoids

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that correspond to such actions arises naturally. Such split extensions were identified in [11,17]: they are the so-called *Schreier split epimorphisms*. A split epimorphism

$$A \underset{f}{\overset{s}{\leftrightarrows}} B$$

of monoids is a Schreier split epimorphism if every element $a \in A$ can be decomposed uniquely as $a = x \cdot sf(a)$ for some x in the kernel of f.

It turns out that the class of Schreier split epimorphisms has a much better behavior than the class of all split epimorphisms of monoids, in the sense that several homological and algebraic properties of split epimorphisms of groups are still valid for Schreier split epimorphisms, but not for all split epimorphisms of monoids. A paradigmatic example is the Split Short Five Lemma [3, Theorem 4.2]. Another important one is the fact that a Schreier split epimorphism is the cokernel of its kernel [3, Proposition 2.6]. Other important properties of Schreier split epimorphisms have been studied in [2,4], and extended to other algebraic structures, like semirings and, more generally, *monoids with operations* [11]. Schreier split extensions correspond bijectively to actions in every category of monoids with operations, so these algebraic structures have a behavior which is very similar to the one of monoids. We recall in Sect. 2 how this bijection is obtained.

Other interesting properties appear when we consider Schreier relations. An internal reflexive relation (i.e. a reflexive relation which is compatible with the operations) on an object A is called a *Schreier reflexive relation* [2,3] if the split epimorphism given by the first projection and the reflexivity morphism is a Schreier one (see Sect. 2 below for more details). It happens that every Schreier reflexive relation is transitive, and it is symmetric if and only if the kernel of the first projection is a group [3, Theorem 5.5]. So, Schreier reflexive relations have a property which is typical of all reflexive relations in Mal'tsev varieties [9].

The notions of Schreier reflexive relation and of Schreier congruence allowed to introduce the one of *special Schreier homomorphism* [2]. A homomorphism $f: A \to B$ in a category of monoids with operations is special Schreier if its kernel congruence is a Schreier one. A special Schreier homomorphism induces a partial subtraction on its domain: the subtraction between two elements of A exists if they have the same image under f (again, see Sect. 2 below). In particular, the kernel of a special Schreier homomorphism is a group. Moreover, the Short Five Lemma holds for special Schreier extensions, i.e. for special Schreier surjective homomorphisms [2, Proposition 7.2.1].

A special Schreier extension $f: A \to B$ with abelian kernel X determines an action of B on X, as it is explained at the beginning of Sect. 4. Then it is a natural question whether there is an abelian group structure on the set $\mathrm{SExt}(B, X, \varphi)$ of isomorphic classes of special Schreier extensions of an object B by an abelian object X inducing the action φ, which generalizes the classical Baer sum of group extensions. The existence of the Baer sum for monoids was deduced in [2] by using categorical arguments. An explicit description of the Baer sum, in terms of factor sets, was then presented in [12]. This gives an interpretation in terms of extensions of the low dimensional cohomology theory for monoids described in [13,15,16], which was obtained by generalizing to monoids the classical bar resolution used to compute group cohomology. Different approaches to the cohomology theory of monoids have been considered in [5,8,22], where different notions of extensions were considered. Looking at a monoid as a category with one object, our approach can also be compared to the one of [6,7], where the low-dimensional cohomology of small categories was described by means of suitable extensions, that particularize to special Schreier ones in the case of monoids.

The aim of the present paper is to set the basis for a description of the cohomology of every category of monoids with operations by means of special Schreier extensions. This would give an interpretation of cohomology for several algebraic structures beyond monoids; some of the main examples are semirings and semilattices. In order to start the development of such cohomology theories, we first show that the Nine Lemma holds for special Schreier extensions in every category of monoids with operations (Sect. 3). The fact that this classical homological lemma holds for such extensions is a further evidence that the categories of monoids with operations have, relatively to these extensions, a homological behavior which is very similar to the one of group-like structures. In fact, we will use the Nine Lemma in Sect. 4 to describe a push forward construction for special Schreier extensions with abelian kernel. More specifically, given a special Schreier extension $f: A \to B$ with abelian kernel X, inducing the action φ of B on X, and a morphism $g: X \to Y$ of abelian objects which is equivariant with respect to the action φ and to a given action ψ of B on Y, we build a special Schreier extension of B by Y which induces the action ψ and which is universal with respect to all special Schreier extensions of B (in a sense that will be explained in Theorem 4.2 below). This will allow us to give, in Sect. 5, an alternative, functorial description of the Baer sum of special Schreier extensions with abelian kernel for the case of monoids. This new description is important to give a description of cohomology of monoids which is independent from the bar resolution.

Constructing the push forward for special Schreier extensions of any length would give a complete description of the cohomology of monoids with operations in terms of special Schreier extensions. This is material for a future work.

2 Schreier split epimorphisms and special Schreier extensions

The aim of this section is to recall from [2,3,11] the notions of Schreier split epimorphism, Schreier congruence and special Schreier extension that will be used in the rest of the paper.

2.1 Schreier split epimorphisms

We start by recalling the definition of monoids with operations, introduced in [11] in order to extend to a wider context the description of crossed semimodules of monoids obtained by Patchkoria [17]. The definition (as well as the main result in [11]) is inspired by Porter's definition of *groups with operations* [20], which is itself a generalization of the one of *categories of interest* in the sense of [14].

Definition 2.1 Let Ω be a set of finitary operations such that the following conditions hold: if Ω_i is the set of i-ary operations in Ω, then:

(1) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
(2) There is a binary operation $+ \in \Omega_2$ (not necessarily commutative) and a constant $0 \in \Omega_0$ satisfying the usual axioms for monoids;
(3) $\Omega_0 = \{0\}$;
(4) Let $\Omega_2' = \Omega_2 \backslash \{+\}$; if $* \in \Omega_2'$, then $*^\circ$, defined by $x *^\circ y = y * x$, is also in Ω_2';
(5) Any $* \in \Omega_2'$ is left distributive w.r.t. $+$, i.e.:

$$a * (b + c) = a * b + a * c;$$

(6) For any $* \in \Omega_2'$ we have $b * 0 = 0$;
(7) Any $\omega \in \Omega_1$ satisfies the following conditions:

- $\omega(x + y) = \omega(x) + \omega(y)$;
- for any $* \in \Omega_2'$, $\omega(a * b) = \omega(a) * b$.

Let moreover E be a set of axioms including the ones above. We will denote by $\mathbb{C}$ the category of (Ω, E)-algebras. We call the objects of $\mathbb{C}$ *monoids with operations*.

Examples of categories of monoids with operations are the categories of monoids, commutative monoids, semirings (i.e. rings where the additive structure is not necessarily a group, but just a commutative monoid), join-semilattices with a bottom element, distributive lattices with a bottom element (or a top one). The algebraic structures covered by Porter's definition, such as groups, rings, associative algebras, Lie algebras and many others, can also be seen as examples of monoids with operations (although, in order to include these examples, condition (7) above should be slightly modified, see [11] for more details).

Definition 2.2 ([11], Definition 2.6). A split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$ in a category $\mathbb{C}$ of monoids with operations is said to be a *Schreier split epimorphism* when, for any $a \in A$, there exists a unique x in the kernel $\mathrm{Ker}(f)$ of f such that $a = x + sf(a)$.

In other terms, a Schreier split epimorphism is a split epimorphism (A, B, f, s) equipped with a unique set-theoretical map $q : A \dashrightarrow \mathrm{Ker}(f)$, called the *Schreier retraction* of (A, B, f, s), with the property that, for any $a \in A$, we have:

$$a = q(a) + sf(a).$$

The following proposition was originally proved for the case of monoids, but it holds (with the same proof) for monoids with operations:

Proposition 2.3 ([3], Proposition 2.4). *A split epimorphism (A, B, f, s) is a Schreier split epimorphism if and only if there exists a set-theoretical map $q : A \dashrightarrow Ker(f)$ such that:*

$$q(a) + sf(a) = a$$
$$q(x + s(b)) = x$$

for every $a \in A$, $x \in Ker(f)$ and $b \in B$.

The definition of Schreier split epimorphism for the case of monoids was first implicitly considered in [17], in connection with the notion of Schreier internal category. Later, in [10], the definition of Schreier split epimorphism was considered in the wider context of Jónsson–Tarski varieties, i.e. varieties (in the sense of universal algebra) whose corresponding theories contain a unique constant 0 and a binary operation + satisfying the equalities $0 + x = x + 0 = x$ for any x. In the present paper, we restrict our attention only to the case of monoids with operations. The reason is that, in such a context, the Schreier split epimorphisms are equivalent to actions in the sense explained here below.

Definition 2.4 ([11], Definitions 2.4 and 2.5). Let X and B be two objects of a category $\mathbb{C}$ of monoids with operations. A *pre-action* of B on X is a set, indexed by the set Ω_2 of binary operations, of set-theoretical maps $\alpha_* \colon B \times X \to X$, $* \in \Omega_2$.

Given a pre-action $\alpha = \{\alpha_* | * \in \Omega_2\}$ of B on X, the *semidirect product* $X \rtimes_\alpha B$ of X and B with respect to α is the Ω-algebra with underlying set $X \times B$ and operations defined by:

$$(x_1, b_1) + (x_2, b_2) = (x_1 + \alpha_+(b_1, x_2), b_1 + b_2),$$
$$(x_1, b_1) * (x_2, b_2) = (x_1 * x_2 + \alpha_*(b_1, x_2) + \alpha_{*^\circ}(b_2, x_1), b_1 * b_2), \text{ for } * \in \Omega_2',$$
$$\omega(x, b) = (\omega(x), \omega(b)), \text{ for } \omega \in \Omega_1.$$

We say that the pre-action α is an *action* if the semidirect product $X \rtimes_\alpha B$ is an object of $\mathbb{C}$.

The equivalence between Schreier split extensions and actions is obtained in the following way. Given a Schreier split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$ with kernel $k \colon X \to A$, the corresponding action α of B on X is given by

$$\alpha_+(b, x) = q(s(b) + k(x)),$$
$$\alpha_*(b, x) = q(s(b) * k(x)), \text{ for } * \in \Omega_2'.$$

Conversely, given an action α of B on X, we can build a Schreier split epimorphism $X \rtimes_\alpha B \underset{\pi_B}{\overset{\langle 0,1\rangle}{\leftrightarrows}} B$ where π_B is the canonical projection on B and $\langle 0, 1\rangle$ sends b to $(0, b)$. We refer to [11] for more details on this equivalence. If we consider the particular case of monoids, the actions we are considering are just the classical monoid actions: an action of a monoid B on a monoid X is a monoid homomorphism $\varphi \colon B \to \mathrm{End}(X)$, where $\mathrm{End}(X)$ is the monoid of endomorphisms of X.

Proposition 2.5 ([3], Proposition 3.4). *Every split epimorphism* (A, B, f, s) *such that* $(B, +)$ *is a group is a Schreier split epimorphism.*

Proof It suffices to write any $a \in A$ as $a = (a - sf(a)) + sf(a)$. □

2.2 Schreier internal relations

An *internal relation* on an object A in a category of monoids with operations is a relation R which is compatible with all the operations. It can be described equivalently as a subobject of the product $A \times A$. By considering the homomorphic inclusion

$$R \rightarrowtail A \times A$$

and by composing it with the two projections of the product, we get two parallel homomorphisms

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A,$$

that are the first and the second projection of the relation. More explicitly, denoting an element of R by a pair (x, y), such that x and y belong to A and are linked by the relation R, we have that $r_1(x, y) = x$ and $r_2(x, y) = y$.

An internal relation is reflexive when r_1 and r_2 have a common section $\sigma : A \to R$. In the notation above, we have that $\sigma(a) = (a, a)$ for any $a \in A$.

Definition 2.6 ([3], Definition 5.1). An internal reflexive relation

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} A \quad (\sigma : A \to R)$$

is a *Schreier reflexive relation* if the split epimorphism (R, A, r_1, σ) is a Schreier one.

It is well known that, in a Malt'sev variety [9], every internal reflexive relation is a congruence. This is false for the varieties of monoids with operations. However, a partial version of this result can be recovered for Schreier reflexive relations:

Theorem 2.7 ([4], Corollary 6.11). *Any Schreier reflexive relation is transitive. It is a congruence if and only if* $(Ker(r_1), +)$ *is a group.*

We will call *Schreier congruence* a Schreier reflexive relation which is a congruence. Since all categories of monoids with operations are Barr-exact [1], all congruences are *kernel congruences*: given a homomorphism $f : A \to B$, the corresponding kernel congruence $\mathrm{Eq}(f)$ is defined by: $a_1 \mathrm{Eq}(f) a_2$ if and only if $f(a_1) = f(a_2)$. $\mathrm{Eq}(f)$ can be represented by the following diagram:

$$\mathrm{Eq}(f) \underset{f_2}{\overset{f_1}{\rightrightarrows}} A, \quad (\langle 1, 1\rangle : A \to \mathrm{Eq}(f))$$

where $\langle 1, 1\rangle$ is the diagonal of A: $\langle 1, 1\rangle(a) = (a, a)$. Thanks to the symmetry of the relation, the split epimorphisms $(f_1, \langle 1, 1\rangle)$ and $(f_2, \langle 1, 1\rangle)$ are isomorphic. Hence, if one of the two is a Schreier split epimorphism, the other is such, too.

2.3 Special Schreier homomorphisms

We recall from [2,4] the following notion:

Definition 2.8 A homomorphism $f : A \to B$ is a *special Schreier homomorphism* if the kernel congruence $\mathrm{Eq}(f)$ is a Schreier congruence.

In other terms, a homomorphism $f : A \to B$ is a special Schreier homomorphism if and only if the split epimorphism

$$X \xrightarrow{\langle k,0\rangle} \mathrm{Eq}(f) \underset{f_2}{\overset{\langle 1,1\rangle}{\leftrightarrows}} A,$$

where $\langle k, 0\rangle$ is the morphism sending $x \in X$ to $(k(x), 0)$, is a Schreier split epimorphism.

A weaker notion of Schreier extensions for monoids was introduced in [21], and further studed in [16,18] in connection with cohomology. The notion we are considering is stronger, whence the name "special Schreier".

As a consequence of Theorem 2.7, we have that the kernel of a special Schreier homomorphism, which is isomorphic to the kernel of each of the projections $f_1, f_2 : \mathrm{Eq}(f) \to A$, is a group w.r.t. the operation $+$. A Schreier split epimorphism is not always a special Schreier homomorphism: it happens if and only if its kernel is a group w.r.t. $+$ ([4], Proposition 6.9).

We are interested, in particular, in special Schreier surjective homomorphisms. We recall some relevant facts about them that will be used in the rest of the paper. We start with the following proposition, whose proof is the same as the one of Proposition 7.1.3 in [2]:

Proposition 2.9 *Every special Schreier surjective homomorphism $f : A \to B$ is the cokernel of its kernel. In other terms, the following sequence is an extension of B by $Ker(f)$:*

$$Ker(f) \rightarrowtail^{k} A \xrightarrow{f} B.$$

Thanks to the previous proposition, a special Schreier surjective homomorphism can be called a *special Schreier extension*.

Lemma 2.10 *Let $f : A \to B$ be a special Schreier extension. Denote by $k : X \to A$ the kernel of f. Then there exists a (unique) map $q : Eq(f) \dashrightarrow X$ which satisfies the following conditions, for every $a \in A$, $(a_1, a_2), (a'_1, a'_2) \in Eq(f)$, $x \in X$ and $* \in \Omega'_2$:*

(i) $kq(a_1, a_2) + a_2 = a_1$;
(ii) $q(k(x) + a, a) = x$;
(iii) $kq(a + k(x), a) + a = a + k(x)$;
(iv) $q(a_1 + a'_1, a_2 + a'_2) = q(a_1, a_2) + q(a_2 + kq(a'_1, a'_2), a_2)$;
(v) $q(a * k(x), 0) = a * k(x)$;
(vi) $kq(a_1 * a'_1, a_2 * a'_2) = kq(a_1, a_2) * kq(a'_1, a'_2) + a_2 * kq(a'_1, a'_2) + a'_2 *^{\circ} kq(a_1, a_2)$.

Proof The conditions of the lemma are straightforward consequences of the definition of a special Schreier extension and of the properties of the Schreier retraction q: see the proofs of Corollary 2.8 in [12] and of Proposition 6.0.11 in [2] for more details in the particular cases of monoids and semirings. □

We observe that Condition (i) in the previous lemma means that the map q endows the monoid $(A, +)$ with a *partial subtraction*: the subtraction between two elements of A exists when they have the same image by the homomorphism f. More precisely:

Corollary 2.11 *A monoid homomorphism $f: A \to B$ is a special Schreier homomorphism if and only if, for every $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ there exists a unique element x in the kernel of f such that $a_2 = x + a_1$.*

Proposition 2.12 ([2], Proposition 7.2.1). *The Short Five Lemma holds for special Schreier extensions: given a commutative diagram*

$$\begin{array}{ccccc} X & \overset{k}{\rightarrowtriangle} & A & \overset{f}{\twoheadrightarrow} & B \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ X' & \underset{k'}{\rightarrowtriangle} & A' & \underset{f'}{\twoheadrightarrow} & B' \end{array}$$

whose rows are special Schreier extensions, if u and w are isomorphisms, then also v is.

3 The Nine Lemma

We prove now, separately, the three possible versions of the Nine Lemma for special Schreier extensions (observe that they are independent from each other). We start by recalling the following well-known lemma.

Lemma 3.1 *In any category, given a commutative diagram of the form*

$$\begin{array}{ccccc} A & \overset{l}{\rightarrowtriangle} & B & \overset{g}{\longrightarrow} & C \\ {\scriptstyle t}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle m} \\ X & \underset{k}{\rightarrowtriangle} & Y & \underset{f}{\longrightarrow} & Z, \end{array}$$

where $l = Ker(g)$ and $k = Ker(f)$, if m is a monomorphism, then the left-hand side square is a pullback.

Theorem 3.2 (the Lower Nine Lemma). *Given a commutative diagram of homomorphisms in a category $\mathbb{C}$ of monoids with operations:*

$$
\begin{array}{ccccc}
N & \overset{\eta}{\rightarrowtail} & H & \overset{\lambda}{\twoheadrightarrow} & K \\
{\scriptstyle l}\downarrow & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\
X & \overset{\sigma}{\rightarrowtail} & Y & \overset{\varphi}{\twoheadrightarrow} & Z \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\
A & \underset{\alpha}{\longrightarrow} & B & \underset{\beta}{\longrightarrow} & C,
\end{array}
\tag{1}
$$

suppose that the three columns and the first two rows are special Schreier extensions. Then the lower row also is.

Proof We divide the proof in several steps:

(1) β is a surjective homomorphism, because $\beta g = p\varphi$ is.

(2) $\beta\alpha$ is the zero homomorphism, i.e. $\beta\alpha(a) = 0$ for all $a \in a$. Indeed, $\beta\alpha f = \beta g\sigma = p\varphi\sigma = 0$ and this implies that $\beta\alpha = 0$ since f is surjective.

(3) α is a monomorphism. In order to prove this, let $a_1, a_2 \in A$ be such that $\alpha(a_1) = \alpha(a_2)$. Since f is surjective, there exist $x_i \in X$ such that $f(x_i) = a_i$. Then $g\sigma(x_1) = \alpha f(x_1) = \alpha f(x_2) = g\sigma(x_2)$, which means that $(\sigma(x_1), \sigma(x_2)) \in \mathrm{Eq}(g)$. Since g is a special Schreier homomorphism, there exists a unique $h \in H$ such that $\sigma(x_2) = r(h) + \sigma(x_1)$. Being $\varphi\sigma = 0$, we get that

$$0 = \varphi\sigma(x_2) = \varphi(r(h) + \sigma(x_1)) = \varphi r(h) + \varphi\sigma(x_1) = s\lambda(h) + 0 = s\lambda(h).$$

From the fact that s is a monomorphism we obtain that $\lambda(h) = 0$ or, in other terms, that h belongs to the kernel of λ. Hence there exists a (unique) $n \in N$ such that $\eta(n) = h$. So

$$\sigma(x_2) = r\eta(n) + \sigma(x_1) = \sigma l(n) + \sigma(x_1) = \sigma(l(n) + x_1).$$

σ is a monomorphism, hence $x_2 = l(n) + x_1$. From $fl = 0$ we conclude that

$$a_2 = f(x_2) = fl(n) + f(x_1) = f(x_1) = a_1.$$

(4) A is the kernel of β. From points (2) and (3) we already know that A is contained in the kernel of β. For the other inclusion, suppose that $\beta(b) = 0$. Being g surjective, there exists $y \in Y$ such that $g(y) = b$. Since

$$p\varphi(y) = \beta g(y) = \beta(b) = 0,$$

there exists a (unique) $k \in K$ such that $s(k) = \varphi(y)$. Let $h \in H$ be such that $\lambda(h) = k$. Then

$$\varphi r(h) = s\lambda(h) = s(k) = \varphi(y).$$

But $(H, +)$ is a group, because it is the kernel of a special Schreier homomorphism. Hence we get that $\varphi(y - r(h)) = 0$. So there is $x \in X$ such that $\sigma(x) = y - r(h)$. Call $a = f(x)$. Then

$$\alpha(a) = \alpha f(x) = g\sigma(x) = g(y) - gr(h) = g(y) = b$$

and hence $\mathrm{Ker}(\beta) \subseteq A$.

(5) β is a special Schreier homomorphism. We have to prove that, for all $b_1, b_2 \in B$ such that $\beta(b_1) = \beta(b_2)$ there exists a unique $a \in A$ such that $b_2 = \alpha(a) + b_1$. Let us first prove the existence of such an a. Given b_1 and b_2 as above, let $y_i \in Y$ be such that $g(y_i) = b_i$. We have

$$p\varphi(y_1) = \beta g(y_1) = \beta(b_1) = \beta(b_2) = \beta g(y_2) = p\varphi(y_2),$$

hence $(\varphi(y_1), \varphi(y_2)) \in \mathrm{Eq}(p)$. From the fact that p is a special Schreier homomorphism we deduce that there exists a unique $k \in K$ such that $\varphi(y_2) = s(k) + \varphi(y_1)$. Choosing $h \in H$ such that $\lambda(h) = k$, we get

$$\varphi(y_2) = s\lambda(h) + \varphi(y_1) = \varphi r(h) + \varphi(y_1) = \varphi(r(h) + y_1),$$

and so $(r(h) + y_1, y_2) \in \mathrm{Eq}(\varphi)$. φ is a special Schreier homomorphism, hence there exists a unique $x \in X$ such that $y_2 = \sigma(x) + r(h) + y_1$. So we obtain that

$$\begin{aligned} b_2 = g(y_2) &= g(\sigma(x) + r(h) + y_1) = g\sigma(x) + gr(h) + g(y_1) \\ &= \alpha f(x) + 0 + g(y_1) = \alpha f(x) + b_1, \end{aligned}$$

hence $a = f(x)$ is the element of A we were looking for. To conclude the proof, we need to show that such an a is unique. Suppose that $\bar{a} \in A$ is such that $b_2 = \alpha(\bar{a}) + b_1$. Let $\bar{x} \in X$ be such that $f(\bar{x}) = \bar{a}$; moreover, let $h \in H$ be such that $\varphi(y_2) = \varphi r(h) + \varphi(y_1)$ as above. Then

$$\begin{aligned} g(\sigma(\bar{x}) + r(h) + y_1) &= g\sigma(\bar{x}) + gr(h) + g(y_1) = \alpha f(\bar{x}) + b_1 \\ &= \alpha(\bar{a}) + b_1 = b_2 = g(y_2), \end{aligned}$$

so that $(\sigma(\bar{x}) + r(h) + y_1, y_2) \in \mathrm{Eq}(g)$. Being g a special Schreier homomorphism, there exists a unique $\bar{h} \in H$ such that $y_2 = r(\bar{h}) + \sigma(\bar{x}) + r(h) + y_1$. Observe now that, since H and X are normal subgroups of the monoid $(Y, +)$, kernels of g and φ, respectively, then the element $r(\bar{h}) + \sigma(\bar{x}) - r(\bar{h}) - \sigma(\bar{x})$ belongs to $H \cap X$. Indeed, it is immediate to see that it belongs both to the kernels of g and φ. But the intersection $H \cap X$ is N, because the upper left square in Diagram (1) is a pullback (thanks to Lemma 3.1). This means that there exists $n \in N$ such that

$$\sigma l(n) = r(\bar{h}) + \sigma(\bar{x}) - r(\bar{h}) - \sigma(\bar{x})$$

or, in other terms,

$$r(\bar{h}) + \sigma(\bar{x}) = \sigma l(n) + \sigma(\bar{x}) + r(\bar{h}).$$

Hence

$$y_2 = \sigma l(n) + \sigma(\bar{x}) + r(\bar{h}) + r(h) + y_1 = \sigma(l(n) + \bar{x}) + r(\bar{h} + h) + y_1.$$

Applying φ to this last equality and using that $\varphi\sigma = 0$ we get

$$\begin{aligned} \varphi(y_2) &= \varphi\sigma(l(n) + \bar{x}) + \varphi r(\bar{h} + h) + \varphi(y_1) = \varphi r(\bar{h} + h) + \varphi(y_1) \\ &= s\lambda(\bar{h} + h) + \varphi(y_1). \end{aligned}$$

But, being p a special Schreier homomorphism, we know that there exists a unique $k \in K$ such that $\varphi(y_2) = s(k) + \varphi(y_1)$. We proved that both $\lambda(h)$ and $\lambda(\bar{h} + h)$ satisfy this equation, and hence

$$\lambda(\bar{h}) + \lambda(h) = \lambda(\bar{h} + h) = \lambda(h).$$

Since $(H, +)$ is a group, this implies that $\lambda(\bar{h}) = 0$. So there exists $\bar{n} \in N$ such that $\eta(\bar{n}) = \bar{h}$. From this we get

$$\begin{aligned} y_2 &= r(\bar{h}) + \sigma(\bar{x}) + r(h) + y_1 = r\eta(\bar{n}) + \sigma(\bar{x}) + r(h) + y_1 = \sigma(l(\bar{n}) + \bar{x}) \\ &\quad + r(h) + y_1. \end{aligned}$$

Using now the uniqueness of x as an element of X such that $y_2 = \sigma(x) + r(h) + y_1$, we obtain that $x = l(\bar{n}) + \bar{x}$. Then

$$a = f(x) = fl(\bar{n}) + f(\bar{x}) = 0 + f(\bar{x}) = f(\bar{x}) = \bar{a},$$

and this concludes the proof.

□

Theorem 3.3 (the Upper Nine Lemma). *Given a commutative diagram of homomorphisms in a category* $\mathbb{C}$ *of monoids with operations:*

$$\begin{array}{ccccc} N & \xrightarrow{\eta} & H & \xrightarrow{\lambda} & K \\ {\scriptstyle l}\downarrow & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{\sigma} & Y & \xrightarrow{\varphi} & Z \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[\alpha]{} & B & \xrightarrow[\beta]{} & C, \end{array} \tag{2}$$

suppose that the three columns and the last two rows are special Schreier extensions. Then the upper row also is.

Proof (1) η is a monomorphism, because $r\eta = \sigma l$ is.
(2) $\lambda\eta$ is the zero homomorphism. Indeed,

$$s\lambda\eta = \varphi\sigma l = 0$$

and this implies that $\lambda\eta = 0$ since s is a monomorphism.
(3) λ is a surjective homomorphism. Indeed, consider $k \in K$. Since φ is surjective, there exists $y \in Y$ such that $\varphi(y) = s(k)$. Then

$$\beta g(y) = p\varphi(y) = ps(k) = 0,$$

hence there exists $a \in A$ such that $\alpha(a) = g(y)$. Thanks to the surjectivity of f, we find $x \in X$ such that $f(x) = a$. From the equality

$$g\sigma(x) = \alpha f(x) = \alpha(a) = g(y)$$

we obtain that $(\sigma(x), y) \in \mathrm{Eq}(g)$. Being g special Schreier, there exists a unique $h \in H$ such that $y = r(h) + \sigma(x)$. $(X, +)$ is a group, so the last equality can be rewritten as $r(h) = y - \sigma(x)$. From this we get

$$s\lambda(h) = \varphi r(h) = \varphi(y) - \varphi\sigma(x) = \varphi(y) = s(k).$$

s is a monomorphism, so we conclude that $\lambda(h) = k$.
(4) N is the kernel of λ. From points (1) and (2) we already know that N is contained in the kernel of λ. Let then $h \in H$ such that $\lambda(h) = 0$. Then $\varphi r(h) = s\lambda(h) = 0$, hence there exists $x \in X$ such that $\sigma(x) = r(h)$. This means that $\sigma(x) = r(h)$ belongs to the intersection of H and X, which is N thanks to Lemma 3.1. Hence there exists $n \in N$ such that $\eta(n) = h$.
(5) Since N, H and K are groups w.r.t. $+$ (which is a consequence of the fact that the three columns are special Schreier extensions), the fact that λ is a surjective homomorphism and η is its kernel immediately implies that the upper row of Diagram (2) is a special Schreier extension. Indeed, it follows immediately from Proposition 2.5 that every extension of groups is special Schreier.

□

We would like to stress the strong asymmetry between the proofs of the Lower and the Upper Nine Lemma: the first is much more complicated than the second. This happens because, since the columns are special Schreier extensions, the upper row lies in the category of groups, in which every surjective homomorphism is a special Schreier extension.

Theorem 3.4 (the Middle Nine Lemma). *Given a commutative diagram of homomorphisms in a category $\mathbb{C}$ of monoids with operations:*

$$\begin{array}{ccccc} N & \overset{\eta}{\rightarrowtail} & H & \overset{\lambda}{\twoheadrightarrow} & K \\ {\scriptstyle l}\downarrow & & {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X & \overset{\sigma}{\longrightarrow} & Y & \overset{\varphi}{\longrightarrow} & Z \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ A & \underset{\alpha}{\rightarrowtail} & B & \underset{\beta}{\twoheadrightarrow} & C, \end{array} \tag{3}$$

suppose that the three columns, the upper and the lower row are special Schreier extensions. Suppose moreover that $\varphi\sigma = 0$. Then the middle row is a special Schreier extension, too.

Proof (1) σ is a monomorphism. In order to prove this, let $x_1, x_2 \in X$ be such that $\sigma(x_1) = \sigma(x_2)$. Then

$$\alpha f(x_1) = g\sigma(x_1) = g\sigma(x_2) = \alpha f(x_2).$$

Since α is a monomorphism, we get that $(x_1, x_2) \in \mathrm{Eq}(f)$. Being f special Schreier, there exists a unique $n \in N$ such that $x_2 = l(n) + x_1$ and hence

$$\sigma(x_1) = \sigma(x_2) = \sigma l(n) + \sigma(x_1) = r\eta(n) + \sigma(x_1).$$

Thanks to the fact that g is a special Schreier homomorphism, the last equality forces $r\eta(n) = 0$, and so $n = 0$. Then $x_1 = x_2$.

(2) φ is surjective. Indeed, consider $z \in Z$. Being β surjective, there exists $b \in B$ such that $\beta(b) = p(z)$. Choosing $y \in Y$ with $g(y) = b$ we get

$$p\varphi(y) = \beta g(y) = \beta(b) = p(z).$$

Using that p is special Schreier, we conclude that there exists a unique $k \in K$ such that $z = s(k) + \varphi(y)$. Choosing $h \in H$ such that $\lambda(h) = k$ we obtain that $s(k) = s\lambda(h) = \varphi r(h)$, hence $z = \varphi r(h) + \varphi(y) = \varphi(r(h) + y)$. So φ is surjective.

(3) X is the kernel of φ. Thanks to point (1) and the hypothesis that $\varphi\sigma = 0$, we already know that X is contained in the kernel of φ. Conversely, let $y \in Y$ be such that $\varphi(y) = 0$. Then $\beta g(y) = p\varphi(y) = 0$ and so there exists $a \in A$ such that $\alpha(a) = g(y)$. Choosing $x \in X$ such that $f(x) = a$ we get

$$g\sigma(x) = \alpha f(x) = \alpha(a) = g(y).$$

Using the fact that g is special Schreier we find a unique $h \in H$ such that $y = r(h) + \sigma(x)$. Then

$$0 = \varphi(y) = \varphi r(h) + \varphi\sigma(x) = \varphi r(h) = s\lambda(h).$$

Being s injective, we obtain that $\lambda(h) = 0$, so there is $n \in N$ such that $h = \eta(n)$. But then

$$y = r(h) + \sigma(x) = r\eta(n) + \sigma(x) = \sigma l(n) + \sigma(x) = \sigma(l(n) + x)$$

belongs to the image of X.

(4) φ is a special Schreier homomorphism. Let $y_1, y_2 \in Y$ be such that $\varphi(y_1) = \varphi(y_2)$. We have to show that there exists a unique $x \in X$ such that $y_2 = \sigma(x)+y_1$. Observe that

$$\beta g(y_1) = p\varphi(y_1) = p\varphi(y_2) = \beta g(y_2),$$

so that $(g(y_1), g(y_2)) \in \mathrm{Eq}(\beta)$. Being β special Schreier, there exists a unique $a \in A$ such that

$$g(y_2) = \alpha(a) + g(y_1).$$

Choosing $x \in X$ such that $f(x) = a$, we get that $\alpha(a) = \alpha f(x) = g\sigma(x)$, and so

$$g(y_2) = g\sigma(x) + g(y_1) = g(\sigma(x) + y_1),$$

which means that $(\sigma(x) + y_1, y_2) \in \mathrm{Eq}(g)$. Then there exists a unique $h \in H$ such that

$$y_2 = r(h) + \sigma(x) + y_1.$$

Applying φ to this equality we obtain

$$\varphi(y_2) = \varphi r(h) + \varphi\sigma(x) + \varphi(y_1) = s\lambda(h) + \varphi(y_1).$$

By assumption $\varphi(y_1) = \varphi(y_2)$, hence we have that

$$\varphi(y_1) = s\lambda(h) + \varphi(y_1).$$

But the fact that p is special Schreier forces $\lambda(h) = 0$. Then there exists $n \in N$ such that $\eta(n) = h$. Hence we have that

$$\begin{aligned} y_2 &= r(h) + \sigma(x) + y_1 = r\eta(n) + \sigma(x) + y_1 \\ &= \sigma l(n) + \sigma(x) + y_1 = \sigma(l(n) + x) + y_1, \end{aligned}$$

so $l(n) + x$ is the element of X we were looking for. It remains to show its uniqueness. For that, suppose there are $x, x' \in X$ such that

$$y_2 = \sigma(x) + y_1 = \sigma(x') + y_1.$$

Then

$$\alpha f(x) + g(y_1) = g\sigma(x) + g(y_1) = g(y_2) = g\sigma(x') + g(y_1) = \alpha f(x') + g(y_1).$$

But $(A, +)$ is a group, hence

$$g(y_1) = \alpha(-f(x) + f(x')) + g(y_1).$$

From the fact that g is special Schreier we conclude that $-f(x) + f(x') = 0$, which means that $f(x) = f(x')$. Being f special Schreier, there exists a unique $n' \in N$ such that $x' = l(n') + x$. We must show that $n' = 0$. From the equality

$$\sigma(x) + y_1 = \sigma(x') + y_1 = \sigma l(n') + \sigma(x) + y_1 = r\eta(n') + \sigma(x) + y_1,$$

and from the fact that g is special Schreier, we obtain $r\eta(n') = 0$, which means that $n' = 0$ since r and η are monomorphisms. This concludes the proof.

□

4 The push forward construction

In this section we develop a push forward construction for special Schreier extensions with abelian kernel. We start with the following

Definition 4.1 An object X in a category $\mathbb{C}$ of monoids with operations is *abelian* if $(X, +)$ is an abelian group and $x * y = 0$ for all $x, y \in X$ and all $* \in \Omega_2'$.

We observe that the abelian objects defined as above are precisely the internal abelian groups in a category $\mathbb{C}$ of monoids with operations.

We describe now how to associate an action with a special Schreier extension. Let

$$X \overset{k}{\rightarrowtail} A \overset{f}{\twoheadrightarrow} B \tag{4}$$

be a special Schreier extension with abelian kernel. This means that the split epimorphism

$$X \xrightarrow{\langle k,0\rangle} \mathrm{Eq}(f) \underset{f_2}{\overset{\langle 1,1\rangle}{\leftrightarrows}} A$$

is a Schreier split epimorphism. As we explained in Sect. 2, this split epimorphism corresponds to an action α of A on X. Putting then

$$\begin{aligned} \varphi_+(b, x) &= \alpha_+(a, x) = q(a + k(x), a) \\ \varphi_*(b, x) &= \alpha_*(a, x) = q(a * k(x), 0) \end{aligned} \tag{5}$$

for any $a \in A$ such that $f(a) = b$, we obtain an action φ of B on X: it is well defined thanks to the fact that X is an abelian object. Indeed, if $f(a) = f(a')$, then Lemma 2.10 tells us that $kq(a, a') + a' = a$. Then:

$$a + k(x) = kq(a, a') + a' + k(x) = kq(a, a') + kq(a' + k(x), a') + a';$$

this, thanks to the fact that $(X, +)$ is abelian, is equal to

$$kq(a' + k(x), a') + kq(a, a') + a' = kq(a' + k(x), a') + a.$$

On the other hand,

$$a + k(x) = kq(a + k(x), a) + a,$$

and the uniqueness in the Schreier condition says that

$$\alpha_+(a, x) = q(a + k(x), a) = q(a' + k(x), a') = \alpha_+(a', x).$$

Considering now any binary operation $* \in \Omega_2'$, we have

$$\begin{aligned} kq(a * k(x), 0) = a * k(x) &= (kq(a, a') + a') * k(x) = kq(a, a') * k(x) + a' * k(x) \\ &= 0 + a' * k(x) = a' * k(x) = kq(a' * k(x), 0), \end{aligned}$$

since $kq(a, a') * k(x) = 0$, being X abelian. This proves that

$$\alpha_*(a, x) = q(a * k(x), 0) = q(a' * k(x), 0) = \alpha_*(a', x).$$

Theorem 4.2 *Consider the following situation:*

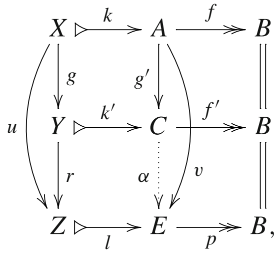
$$\begin{array}{ccccc} X & \overset{k}{\rightarrowtail} & A & \overset{f}{\twoheadrightarrow} & B, \\ {\scriptstyle g}\downarrow & & & & \\ Y & & & & \end{array} \qquad (6)$$

where:

- *f is a special Schreier extension with abelian kernel (with Schreier retraction $q\colon Eq(f) \to X$);*
- *φ is the corresponding action of B on X, defined as in (5);*
- *Y is an abelian object, equipped with an action ψ of B on it;*
- *g is a morphism which is equivariant w.r.t. the actions, which means that, for all $b \in B$ and all $x \in X$,*

$$g(\varphi_+(b, x)) = \psi_+(b, g(x)) \quad \textit{and} \quad g(\varphi_*(b, x)) = \psi_*(b, g(x)).$$

Then there exists a special Schreier extension f' with kernel Y and codomain B, which induces the action ψ and is universal among all such extensions, meaning that given any diagram of the form



(7)

where p is a special Schreier extension with abelian kernel Z, r is an equivariant morphism, (u, v) is a morphism of extensions and $u = rg$, then there exists a unique homomorphism α such that $v = \alpha g'$ and (r, α) is a morphism of extensions.

Proof The morphism f and the action ψ induce an action ζ of A on Y given by

$$\zeta_+(a, y) = \psi_+(f(a), y), \qquad \zeta_*(a, y) = \psi_*(f(a), y).$$

We can then build the semidirect product $Y \rtimes_\zeta A$ of Y and A w.r.t. ζ. Since Y is an abelian object, this gives us a special Schreier extension with abelian kernel:

$$Y \xrightarrow{\langle 1,0\rangle} Y \rtimes_\zeta A \underset{\pi_A}{\overset{\langle 0,1\rangle}{\rightleftarrows}} A.$$

Consider now the map $h\colon X \to Y \rtimes_\zeta A$ defined by

$$h(x) = (-g(x), k(x)).$$

It is clearly injective, since k is. Moreover, it is a homomorphism, indeed:

$$\begin{aligned} h(x_1) + h(x_2) &= (-g(x_1), k(x_1)) + (-g(x_2), k(x_2)) \\ &= (-g(x_1) + \zeta_+(k(x_1), -g(x_2)), k(x_1) + k(x_2)) \\ &= (-g(x_1) + \psi_+(fk(x_1), -g(x_2)), k(x_1) + k(x_2)), \end{aligned}$$

and since k is the kernel of f the last expression is equal to

$$(-g(x_1) - g(x_2), k(x_1) + k(x_2)) = (-(g(x_1 + x_2), k(x_1 + x_2)) = h(x_1 + x_2),$$

where the first equality holds because $(Y, +)$ is an abelian group. Moreover, if $* \in \Omega_2'$,

$$\begin{aligned} &h(x_1) * h(x_2) = (-g(x_1), k(x_1)) * (-g(x_2), k(x_2)) \\ &\quad = ((-g(x_1)) * (-g(x_2)) + \zeta_*(k(x_1), -g(x_2)) + \zeta_{*^\circ}(k(x_2), -g(x_1)), k(x_1) * k(x_2)) \\ &\quad = (0 + \psi_*(fk(x_1), -g(x_2)) + \psi_{*^\circ}(fk(x_2), -g(x_1)), k(x_1 * x_2)) = (0, 0) = h(x_1 * x_2), \end{aligned}$$

since $x_1, x_2 \in \mathrm{Ker}(f)$ and X is abelian.

Let $c\colon Y \rtimes_\zeta A \to C$ be the cokernel of h, i.e. the quotient w.r.t. the congruence R_h on $Y \rtimes_\zeta A$ generated by $h(X)$. We first observe that this congruence R_h has a very simple description. Indeed, consider the following relation on $Y \rtimes_\zeta A$:

$$\begin{aligned}&(y_1, a_1)R(y_2, a_2) \quad \text{if} \quad \exists\, x \in X \quad \text{such that}\\&(y_2, a_2) = (-g(x), k(x)) + (y_1, a_1) = (-g(x) + y_1, k(x) + a_1),\end{aligned} \tag{8}$$

where the last equality holds because the elements in the image of k act trivially on A. This relation R is clearly an equivalence relation (symmetry comes from the fact that $(X, +)$ is a group). Let us show that it is a congruence, i.e. that it is compatible with the operations in $Y \rtimes_\zeta A$. Then it will be necessarily the congruence R_h generated by $h(X)$. In order to do this, suppose that

$$(y_1, a_1)R(y_2, a_2) \quad \text{and} \quad (y_1', a_1')R(y_2', a_2'),$$

so that there exist $x, x' \in X$ such that

$$(y_2, a_2) = (-g(x), k(x)) + (y_1, a_1) = (-g(x) + y_1, k(x) + a_1)$$

and

$$(y_2', a_2') = (-g(x'), k(x')) + (y_1', a_1') = (-g(x') + y_1', k(x') + a_1').$$

We want to prove that there exist $\bar{x}, \tilde{x} \in X$ such that

$$(y_2, a_2) + (y_2', a_2') = (-g(\bar{x}), k(\bar{x})) + (y_1, a_1) + (y_1', a_1'), \tag{9}$$

and

$$(y_2, a_2) * (y_2', a_2') = (-g(\tilde{x}), k(\tilde{x})) + ((y_1, a_1) * (y_1', a_1')). \tag{10}$$

We have

$$\begin{aligned}(y_2, a_2) + (y_2', a_2') &= (-g(x) + y_1, k(x) + a_1) + (-g(x') + y_1', k(x') + a_1')\\&= (-g(x) + y_1 + \zeta_+((k(x) + a_1), -g(x') + y_1'), k(x) + a_1 + k(x') + a_1').\end{aligned} \tag{11}$$

Observe that $(a_1 + k(x'), a_1) \in \mathrm{Eq}(f)$; hence, by Lemma 2.10, we have

$$kq(a_1 + k(x'), a_1) + a_1 = a_1 + k(x')$$

and so

$$k(x) + a_1 + k(x') + a_1' = k(x) + kq(a_1 + k(x'), a_1) + a_1 + a_1'.$$

This gives us a candidate for the element $\bar{x}$ we were looking for, namely $\bar{x} = x + q(a_1 + k(x'), a_1)$. Replacing this expression in the right side of (9), we get

$$\begin{aligned}&(-g(x + q(a_1 + k(x'), a_1)), k(x + q(a_1 + k(x'), a_1))) + (y_1, a_1) + (y_1', a_1')\\ &\quad = (-g(x) - gq(a_1 + k(x'), a_1), k(x) + kq(a_1 + k(x'), a_1)) + (y_1\\ &\qquad + \psi_+(f(a_1), y_1'), a_1 + a_1')\end{aligned}$$

and, using the fact that the elements of $k(X)$ act trivially, this is equal to

$$\begin{aligned}&(-g(x) - gq(a_1 + k(x'), a_1) + y_1 + \psi_+(f(a_1), y_1'), k(x)\\ &\quad + kq(a_1 + k(x'), a_1) + a_1 + a_1').\end{aligned}$$

We already proved that the second component is the same as in (11). Let us check that this is the case for the first component, too. Using the fact that $q(a_1 + k(x'), a_1) = \varphi_+(f(a_1), x')$, the first component is equal to

$$-g(x) - g\varphi_+(f(a_1), x') + y_1 + \psi_+(f(a_1), y_1').$$

Using equivariance of g, this is equal to

$$-g(x) - \psi_+(f(a_1), g(x')) + y_1 + \psi_+(f(a_1), y_1').$$

The first component in (11) is

$$\begin{aligned}&-g(x) + y_1 + \zeta_+(k(x) + a_1, -g(x') + y_1')\\ &\quad = -g(x) + y_1 + \zeta_+(a_1, -g(x') + y_1')\\ &\quad = -g(x) + y_1 - \psi_+(f(a_1), g(x')) + \psi_+(f(a_1), y_1'),\end{aligned}$$

and the two expressions are the same because $(Y, +)$ is an abelian group.

Let now $* \in \Omega_2'$; in order to prove (10), we first observe that

$$\begin{aligned}&(y_2, a_2) * (y_2', a_2') = (-g(x) + y_1, k(x) + a_1) * (-g(x') + y_1', k(x') + a_1')\\ &\quad = ((-g(x) + y_1) * (-g(x') + y_1') + \psi_*(f(k(x) + a_1), -g(x') + y_1')\\ &\qquad + \psi_{*^\circ}(f(k(x') + a_1'), -g(x) + y_1), (k(x) + a_1) * (k(x') + a_1)),\end{aligned}$$

and using the distributivity and the fact that X and Y are abelian, this is equal to

$$\begin{aligned}&(\psi_*(f(a_1), -g(x')) + \psi_*(f(a_1), y_1') + \psi_{*^\circ}(f(a_1'), -g(x)) + \psi_{*^\circ}\\ &\quad (f(a_1'), y_1), k(x) * a_1' + a_1 * k(x') + a_1 * a_1').\end{aligned} \tag{12}$$

Since $k(x) * a_1' + a_1 * k(x') \in \mathrm{Ker}(f)$, we can choose

$$\tilde{x} = q(k(x) * a_1' + a_1 * k(x'), 0)$$

and we get

$$
\begin{aligned}
&(-g(\tilde{x}), k(\tilde{x})) + ((y_1, a_1) * (y_1', a_1'))\\
&\quad = (-gq(k(x) * a_1' + a_1 * k(x'), 0), kq(k(x) * a_1' + a_1 * k(x'), 0))\\
&\qquad + (y_1 * y_1' + \psi_*(f(a_1), y_1') + \psi_{*^\circ}(f(a_1'), y_1), a_1 * a_1')\\
&\quad = (-gq(k(x) * a_1' + a_1 * k(x'), 0), k(x) * a_1' + a_1 * k(x')) + (\psi_*(f(a_1), y_1')\\
&\qquad + \psi_{*^\circ}(f(a_1'), y_1), a_1 * a_1')\\
&\quad = (-gq(k(x) * a_1' + a_1 * k(x'), 0) + \psi_*(f(a_1), y_1') + \psi_{*^\circ}(f(a_1'), y_1), k(x) * a_1'\\
&\qquad + a_1 * k(x') + a_1 * a_1'), \qquad (13)
\end{aligned}
$$

where the last equality holds since the elements of $\mathrm{Ker}(f)$ act trivially. Now, since the second components in (12) and in (13) are equal, it suffices to show that the first components also are. Using the fact that $(Y, +)$ is an abelian group, it is the same to show that

$$
-gq(k(x) * a_1' + a_1 * k(x'), 0) = \psi_*(f(a_1), -g(x')) + \psi_{*^\circ}(f(a_1'), -g(x)).
$$

Thanks to the equivariance of g, we have that

$$
\begin{aligned}
&\psi_*(f(a_1), -g(x')) + \psi_{*^\circ}(f(a_1'), -g(x)) = -g\varphi_*(f(a_1), x') - g\varphi_{*^\circ}(f(a_1'), x)\\
&\quad = -g(q(a_1 * k(x'), 0) + q(a_1' *^\circ k(x), 0)) = -g(q(a_1 * k(x'), 0) + q(k(x) * a_1', 0))
\end{aligned}
$$

and, applying the monomorphism k, it is immediate to check that

$$
q(k(x) * a_1' + a_1 * k(x'), 0) = q(a_1 * k(x'), 0) + q(k(x) * a_1', 0).
$$

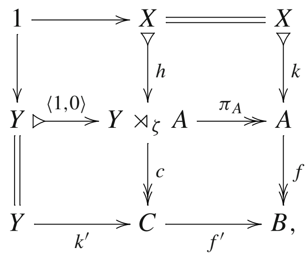
This concludes the proof that R is a congruence and that it coincides with R_h.

Knowing now that $c\colon Y \rtimes_\zeta A \to C$ is the quotient w.r.t. the congruence (8), it is immediate to see that $h(X)$ is the kernel of c, i.e. the zero-class of the relation R_h. Moreover, c is a special Schreier extension. Indeed, suppose that $c(y_1, a_1) = c(y_2, a_2)$. This means that $(y_1, a_1) R_h (y_2, a_2)$, so that there exists $x \in X$ such that

$$
(y_2, a_2) = (-g(x), k(x)) + (y_1, a_1) = (-g(x) + y_1, k(x) + a_1).
$$

In particular, this says that $a_2 = k(x) + a_1$. But then $(a_2, a_1) \in \mathrm{Eq}(f)$, and f is a special Schreier extension, so x is necessarily equal to $q(a_2, a_1)$. Hence there is a unique $x \in X$ with the required property.

Consider now the following commutative diagram:



where $k' = c\langle 1, 0\rangle$ and f' is induced by the universal property of the cokernel c. By hypothesis and by what we just proved, the three columns and the first two rows are special Schreier extensions. The Lower Nine Lemma (Theorem 3.2) gives then that the lower row also is. This is the push forward of f along g we were looking for.

We still have to prove that the action of B on Y determined by f' coincides with ψ and that our construction is universal. Let us denote by $[(y, a)]$ an element of C, i.e. an equivalence class of the relation R_h. Then

$$f'([(y, a)]) = f'c(y, a) = f\pi_A(y, a) = f(a).$$

Denoting by q' the unique map $\mathrm{Eq}(f') \to Y$ determined by the fact that f' is special Schreier, we have that the action χ of B on Y induced by f' is given by

$$\chi_+(b, y) = q'([(\bar{y}, \bar{a})] + [(y, 0)], [\bar{y}, \bar{a}]),$$
$$\chi_*(b, y) = q'([(\bar{y}, \bar{a})] * [(y, 0)], 0)$$

for all $\bar{a} \in A$ such that $f(\bar{a}) = b$. In particular, we can choose $\bar{y} = 0$. Hence:

$$\chi_+(b, y) = q'([(0, \bar{a})] + [(y, 0)], [(0, \bar{a})]) = q'([(\psi_+(f(\bar{a}), y), \bar{a})], [(0, \bar{a})])$$

and

$$\chi_*(b, y) = q'([(0, \bar{a})] * [(y, 0)], 0) = q'([(\psi_*(f(\bar{a}), y), 0)], 0).$$

But $q'([(\psi_+(f(\bar{a}), y), \bar{a})], [(0, \bar{a})]) = q'([(\psi_+(b, y), \bar{a})], [(0, \bar{a})])$ is the unique element $t \in Y$ such that

$$k'(t) + [(0, \bar{a})] = [(\psi_+(b, y), \bar{a})];$$

computing the left-hand side of the equality, we obtain

$$k'(t) + [(0, \bar{a})] = [(t, 0)] + [(0, \bar{a})] = [(t, \bar{a})],$$

and so $\chi_+(b, y) = \psi_+(b, y)$. Similarly, $q'([(\psi_*(f(\bar{a}), y), 0)], 0) = q'([(\psi_*(b, y), 0)], 0)$ is the unique element $w \in Y$ such that

$$k'(w) = [(\psi_*(b, y), 0)],$$

and so $\chi_*(b, y) = \psi_*(b, y)$.

In order to prove the universality of our construction, consider Diagram (7). Let us denote by τ the action of B on Z determined by p. We first define a map $\beta\colon Y \rtimes_\zeta A \to E$ by putting

$$\beta(y, a) = lr(y) + v(a).$$

It is a homomorphism, indeed

$$\begin{aligned}\beta((y_1, a_1) + (y_2, a_2)) &= \beta(y_1 + \psi_+(f(a_1), y_2), a_1 + a_2)\\ &= lr(y_1) + lr\psi_+(f(a_1), y_2) + v(a_1) + v(a_2)\\ &= lr(y_1) + l\tau_+(f(a_1), r(y_2)) + v(a_1) + v(a_2),\end{aligned}$$

where the last equality holds because r is equivariant. Since τ is induced by p, the last expression is equal to

$$\begin{aligned}&lr(y_1) + l\tau_+(pv(a_1), r(y_2)) + v(a_1) + v(a_2)\\ &\quad = lr(y_1) + lq_p(v(a_1) + lr(y_2), v(a_1)) + v(a_1) + v(a_2),\end{aligned}$$

where q_p is the Schreier retraction associated with the special Schreier extension p. The properties of q_p give that the last expression is

$$lr(y_1) + v(a_1) + lr(y_2) + v(a_2) = \beta(y_1, a_1) + \beta(y_2, a_2).$$

If $* \in \Omega_2'$, then

$$\begin{aligned}\beta((y_1, a_1) * (y_2, a_2)) &= \beta(y_1 * y_2 + \psi_*(f(a_1), y_2) + \psi_{*^\circ}(f(a_2), y_1), a_1 * a_2)\\ &= lr\psi_*(f(a_1), y_2) + lr\psi_{*^\circ}(f(a_2), y_1) + v(a_1 * a_2).\end{aligned}$$

By equivariance of r, this is equal to

$$\begin{aligned}&l\tau_*(pv(a_1), r(y_2)) + l\tau_{*^\circ}(pv(a_2), r(y_1)) + v(a_1) * v(a_2)\\ &\quad = lq_p(v(a_1) * lr(y_2), 0) + lq_p(lr(y_1) * v(a_2), 0) + v(a_1) * v(a_2).\end{aligned}$$

The properties of q_p give that this is equal to

$$\begin{aligned}&v(a_1) * lr(y_2) + lr(y_1) * v(a_2) + v(a_1) * v(a_2)\\ &\quad = lr(y_1) * lr(y_2) + v(a_1) * lr(y_2) + lr(y_1) * v(a_2) + v(a_1) * v(a_2)\\ &\quad = (lr(y_1) + v(a_1)) * (lr(y_2) + v(a_2)) = \beta(y_1, a_1) * \beta(y_2, a_2).\end{aligned}$$

Moreover, we have that

$$\beta h(x) = \beta(-g(x), k(x)) = -lrg(x) + vk(x) = -lu(x) + lu(x) = 0$$

for all $x \in X$. Being c the cokernel of h, we conclude that there exists a unique morphism $\alpha\colon C \to E$ such that $\alpha c = \beta$, and so

$$\alpha g' = \alpha c\langle 0, 1\rangle = \beta\langle 0, 1\rangle = v.$$

Moreover, (r, α) is a morphism of extensions, indeed:

$$\alpha k'(y) = \alpha c(y, 0) = \beta(y, 0) = lr(y)$$

and

$$p\alpha([(y, a)]) = p\beta(y, a) = plr(y) + pv(a) = 0 + pv(a) = f(a) = f'([(y, a)]).$$

□

We conclude this section by mentioning that a similar push forward construction, in the particular case of monoids, has been obtained independently in [19] for a wider class of extensions, whose kernels are commutative monoids but not necessarily abelian groups. However, in [19] a weaker universality of the construction is proved: the existence of a morphism α as in Diagram (7) was obtained only when r is an identity.

5 The Baer sum of special Schreier extensions with abelian kernel

We now show that the push forward construction described in the previous section allows, in the particular case of monoids, to define the Baer sum of special Schreier extensions with abelian kernel. A construction of the Baer sum was already given in [12], using factor sets as in the case of classical group extensions. We remark that a similar construction was announced in [13]. We will show that the two approaches give the same result. The advantage of the approach via the push forward is that it is functorial, and this can be useful to give an interpretation of cohomology of monoids in terms of special Schreier extensions. An extension of the same functorial construction of the Baer sum to the case of any category of monoids with operations, together with a description of the cohomology of such structures by means of special Schreier extensions is material for a future work.

We start by recalling briefly the construction given in [12]. For the sake of uniformity with [12], in this section we will use the multiplicative notation for the monoid operation.

Definition 5.1 ([12], Definition 3.1). Given a monoid B, an abelian group X and an action $\varphi\colon B \to \mathrm{End}(X)$ of B on X, a *factor set* is a map $g\colon B \times B \to X$ which satisfies, for all $b, b_1, b_2, b_3 \in B$, the following conditions:

(i) $g(b, 1) = g(1, b) = 1$;
(ii) $g(b_1, b_2) \cdot g(b_1 \cdot b_2, b_3) = \varphi(b_1)(g(b_2, b_3)) \cdot g(b_1, b_2 \cdot b_3)$.

Given a special Schreier extension with abelian kernel

$$X \rhd\!\xrightarrow{k} A \xrightarrow{f}\!\!\!\!\rightarrow B, \tag{14}$$

we can associate with it a factor set in the following way: let $s\colon B \to A$ be a set-theoretical section of f (it exists, since f is surjective). Let us choose s such that $s(1) = 1$. Then, for any $b_1, b_2 \in B$:

$$f(s(b_1) \cdot s(b_2)) = b_1 \cdot b_2 = f(s(b_1 \cdot b_2)).$$

Hence the pair $(s(b_1)\cdot s(b_2), s(b_1\cdot b_2))$ belongs to $\mathrm{Eq}(f)$. We define a map $g\colon B\times B \to X$ by putting:

$$g(b_1, b_2) = q(s(b_1) \cdot s(b_2), s(b_1 \cdot b_2)),$$

where q is the Schreier map associated with f. Such a map g is a factor set ([12], Proposition 3.3). Moreover, thanks to Proposition 3.4 in [12], the extension (14) is isomorphic to an extension of the form

$$X \rhd\!\xrightarrow{\langle 1,0\rangle} X \times B \xrightarrow{\pi_B}\!\!\!\!\rightarrow B,$$

where the monoid operation on $X \times B$ is defined by:

$$(x_1, b_1) \cdot (x_2, b_2) = (x_1 \cdot \varphi(b_1)(x_2) \cdot g(b_1, b_2), b_1 \cdot b_2).$$

Choosing two different sections for f, the corresponding factor sets differ by an *inner factor set*:

Definition 5.2 A factor set g is an *inner factor set* if it is of the form

$$g(b_1, b_2) = h(b_1) \cdot \varphi(b_1)(h(b_2)) \cdot h(b_1 \cdot b_2)^{-1}$$

for some map $h\colon B \to X$ such that $h(1) = 1$.

The set $\mathcal{F}(B, X, \varphi)$ of all the factor sets corresponding to a given action $\varphi\colon B \to \mathrm{End}(X)$ is a subgroup of the abelian group $X^{B\times B}$, where the group operation is the pointwise multiplication. Its subset $\mathcal{IF}(B, X, \varphi)$ of inner factor sets is a normal subgroup of $\mathcal{F}(B, X, \varphi)$. Let us denote by $\mathrm{SExt}(B, X, \varphi)$ the set of isomorphic classes of special Schreier extensions of a monoid B by an abelian group X inducing the action $\varphi\colon B \to \mathrm{End}(X)$. Since the Short Five Lemma holds for special Schreier extensions ([2], Proposition 7.2.1), two special Schreier extensions of B by X are isomorphic as soon as there exists a morphism of extensions between them. We have the following

Theorem 5.3 ([12], Theorem 3.7). *The set* $SExt(B, X, \varphi)$ *of isomorphic classes of special Schreier extensions of a monoid* B *by an abelian group* X *inducing the action* $\varphi\colon B \to End(X)$ *is in bijection with the factor abelian group*

$$\frac{\mathcal{F}(B, X, \varphi)}{\mathcal{IF}(B, X, \varphi)}.$$

By means of this bijection, we can endow $\mathrm{SExt}(B, X, \varphi)$ with an abelian group structure, which we call *the Baer sum*. The unit of this abelian group is the isomorphic class of the split extension obtained by taking the semidirect product of X and B with respect to the action φ.

We propose now an alternative description of the Baer sum. Given two special Schreier extensions

$$X \overset{k_1}{\rightarrowtail} A_1 \overset{f_1}{\twoheadrightarrow} B \quad \text{and} \quad X \overset{k_2}{\rightarrowtail} A_2 \overset{f_2}{\twoheadrightarrow} B$$

with abelian kernel X which induce the same action $\varphi\colon B \to \mathrm{End}(X)$, let us first consider their direct product:

$$X \times X \overset{k_1 \times k_2}{\rightarrowtail} A_1 \times A_2 \overset{f_1 \times f_2}{\twoheadrightarrow} B \times B$$

and pull it back along the diagonal morphism $\Delta_B\colon B \to B \times B$ defined by $\Delta_B(b) = (b, b)$:

$$\begin{array}{ccccc}
X \times X & \overset{\langle k_1, k_2 \rangle}{\rightarrowtail} & P & \overset{\bar{f}}{\twoheadrightarrow} & B \\
\| & & \downarrow & & \downarrow {\scriptstyle \Delta_B} \\
X \times X & \underset{k_1 \times k_2}{\rightarrowtail} & A_1 \times A_2 & \underset{f_1 \times f_2}{\twoheadrightarrow} & B \times B.
\end{array}$$

Special Schreier extensions are stable under pullback along any morphism ([2], Proposition 7.1.4), hence $\bar{f}$ is a special Schreier extension. Moreover, it is easy to check that the corresponding action $\bar{\varphi}\colon B \to \mathrm{End}(X \times X)$ is given by

$$\bar{\varphi}(b)(x_1, x_2) = (\varphi(b)(x_1), \varphi(b)(x_2)).$$

Since X is an abelian group, its multiplication $m\colon X \times X \to X$ is a homomorphism, and it is equivariant w.r.t. the actions $\bar{\varphi}$ and φ, since

$$\varphi(b)(x_1 \cdot x_2) = \varphi(b)(x_1) \cdot \varphi(b)(x_2) = m(\bar{\varphi}(b)(x_1, x_2)).$$

We can then take the push forward of $\bar{f}$ along m:

$$\begin{array}{ccccc} X \times X & \xrightarrow{\langle k_1, k_2 \rangle} & P & \xrightarrow{\bar{f}} & B \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle c} & & \| \\ X & \xrightarrow[k']{} & C & \xrightarrow[f']{} & B, \end{array}$$

thus obtaining a special Schreier extension f' which induces the same action φ. We now show that such an extension is the same that we would obtain by taking the Baer sum of f_1 and f_2 by means of factor sets.

Let us choose two sections s_1 and s_2 of f_1 and f_2, respectively, with the property that $s_i(1) = 1$. The corresponding factor sets are then given by

$$g_1(b, b') = q_1(s_1(b) \cdot s_1(b'), s_1(b \cdot b')),$$
$$g_2(b, b') = q_2(s_2(b) \cdot s_2(b'), s_2(b \cdot b')),$$

where q_1 and q_2 are the Schreier maps associated with f_1 and f_2. We observe that the pullback P is the set

$$P = \{(a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2)\}.$$

The monoid C is then a quotient of the semidirect product $X \rtimes P$. We can then consider the section s' of f' defined by

$$s'(b) = [(1, s_1(b), s_2(b))].$$

The corresponding factor set is

$$g'(b, b') = q'(s'(b) \cdot s'(b'), s'(b \cdot b')),$$

where q' is the Schreier retraction associated with f'. We want to prove that

$$g'(b, b') = g_1(b, b') \cdot g_2(b, b').$$

Thanks to the properties of q', it suffices to prove that the element $g_1(b, b') \cdot g_2(b, b')$ of X is such that

$$k'(g_1(b, b') \cdot g_2(b, b')) \cdot [(1, s_1(b \cdot b'), s_2(b \cdot b'))] = [(1, s_1(b) \cdot s_1(b'), s_2(b) \cdot s_2(b'))].$$

But

$$k'(g_1(b, b') \cdot g_2(b, b')) = [(g_1(b, b') \cdot g_2(b, b'), 1, 1)],$$

so we have to show that

$$[(g_1(b,b')\cdot g_2(b,b'),1,1)]\cdot[(1,s_1(b\cdot b'),s_2(b\cdot b'))]$$
$$=[(1,s_1(b)\cdot s_1(b'),s_2(b)\cdot s_2(b'))]$$

or, in other terms,

$$[(g_1(b,b')\cdot g_2(b,b'),s_1(b\cdot b'),s_2(b\cdot b'))]=[(1,s_1(b)\cdot s_1(b'),s_2(b)\cdot s_2(b'))].$$

The two equivalence classes coincide if and only if there is a pair $(x_1,x_2)\in X\times X$ such that

$$(1,s_1(b)\cdot s_1(b'),s_2(b)\cdot s_2(b'))=h(x_1,x_2)\cdot(g_1(b,b')\cdot g_2(b,b'),s_1(b\cdot b'),s_2(b\cdot b')),$$

where $h\colon X\times X\to X\rtimes P$ is the monomorphism given by

$$h(x_1,x_2)=((x_1\cdot x_2)^{-1},k_1(x_1),k_2(x_2)).$$

If we choose $x_i=g_i(b,b')$, we get:

$$h(x_1,x_2)\cdot(g_1(b,b')\cdot g_2(b,b'),s_1(b\cdot b'),s_2(b\cdot b'))$$
$$=((g_1(b,b')\cdot g_2(b,b'))^{-1},k_1g_1(b,b'),k_2g_2(b,b'))$$
$$\cdot(g_1(b,b')\cdot g_2(b,b'),s_1(b\cdot b'),s_2(b\cdot b')).$$

Since the elements of P of the form $(k_1(x_1),k_2(x_2))$ act trivially on X (because $\bar{f}(k_1(x_1),k_2(x_2))=1$), the last expression is equal to

$$((g_1(b,b')\cdot g_2(b,b'))^{-1}\cdot g_1(b,b')\cdot g_2(b,b'),k_1g_1(b,b')\cdot s_1(b,b'),$$
$$k_2g_2(b,b')\cdot s_2(b,b'))=(1,s_1(b)\cdot s_1(b'),s_2(b)\cdot s_2(b'))$$

and the proof is completed.

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