Theory and Applications of Categories, Vol. 27, No. 5, 2012, pp. 65–79.

WEAKLY MAL'TSEV CATEGORIES AND STRONG RELATIONS

ZURAB JANELIDZE AND NELSON MARTINS-FERREIRA

Abstract. We define a *strong relation* in a category $\mathbb{C}$ to be a span which is "orthogonal" to the class of jointly epimorphic pairs of morphisms. Under the presence of finite limits, a strong relation is simply a strong monomorphism $R \to X \times Y$. We show that a category $\mathbb{C}$ with pullbacks and equalizers is a weakly Mal'tsev category if and only if every reflexive strong relation in $\mathbb{C}$ is an equivalence relation. In fact, we obtain a more general result which includes, as its another particular instance, a similar well-known characterization of Mal'tsev categories.

1. Introduction

In [15], the notion of a *weakly Mal'tsev category* was introduced as a generalization of the notion of a Mal'tsev category [4, 5], where one retains the description of internal categories as multiplicative reflexive graphs (originally obtained for Mal'tsev varieties in [8] and extended to Mal'tsev categories in [5]).

The category of distributive lattices is an example of a weakly Mal'tsev category which is not a Mal'tsev category (see [16] where it is shown that a variety of lattices is a weakly Mal'tsev variety if and only if it is a variety of distributive lattices).

The definition of a weakly Mal'tsev category is based on a particular reformulation of the classical Mal'tsev property (that every reflexive relation is an equivalence relation), due to D. Bourn [2]. In the present paper we show that among categories with pullbacks and equalizers, the weakly Mal'tsev categories are those where certain reflexive relations, called *strong relations*, are equivalence relations. In particular, when finite limits exist, strong relations are those which are given by strong monomorphisms $R \to X \times Y$. We also show that, as in the classical Mal'tsev case, here all such reflexive relations are equivalence relations if and only if all such relations are difunctional.

We obtain these new characterizations of weakly Mal'tsev categories as an application of a more general result (see Theorem 3.10 below) which unifies these characterizations with the corresponding known characterizations of Mal'tsev categories (see [5] and the references there).

The first author acknowledges the support of South African National Research Foundation and Georgian National Science Foundation (GNSF/ST09_730_3-105). The second author's research was partially supported by FCT grant SFRH/BPD/43216/2008; also by ESTG and CDRSP from the Polytechnical Institute of Leiria.

Received by the editors 2011-08-29 and, in revised form, 2012-07-18.

Published on 2012-07-27 in the volume of articles from CT2011.

2000 Mathematics Subject Classification: 18C99, 18A20.

Key words and phrases: weakly Mal'tsev category, Mal'tsev category, difunctional relation, factorization system.

2. The definition of a weakly Mal'tsev category

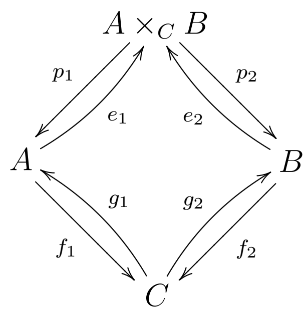
Recall from [2] that a finitely complete category $\mathbb{C}$ is a Mal'tsev category [4, 5] if and only if for every object C in $\mathbb{C}$, the pointed category

$$\mathrm{Pt}(C) = ((C, 1_C) \downarrow (\mathbb{C} \downarrow C))$$

is unital, i.e. product injections

$$A \xrightarrow{e_1=(1,0)} A \times B \xleftarrow{e_2=(0,1)} B$$

are jointly strongly epimorphic in $\mathrm{Pt}(C)$. Product injections in $\mathrm{Pt}(C)$ arise as sections of pullback projections



in $\mathbb{C}$, where f_1 and f_2 are split epimorphisms with right inverses g_1 and g_2, respectively, while e_1 and e_2 are the canonical morphisms into the pullback defined by the equalities $p_1e_1 = 1_A$, $p_2e_1 = g_2f_1$, and, $p_2e_2 = 1_B$, $p_1e_2 = g_1f_2$, respectively. A pullback diagram of this form will be called a *local product* in $\mathbb{C}$, and the pair of morphisms e_1, e_2 will be called a *local product injection pair* in $\mathbb{C}$. We say that $\mathbb{C}$ *has local products* if all such pullbacks exists (i.e. pullbacks of split epimorphisms along split epimorphisms exist).

Under the presence of pullbacks, a pair of morphisms in $\mathrm{Pt}(C)$ is jointly strongly epimorphic in $\mathrm{Pt}(C)$ if and only if it is jointly strongly epimorphic in $\mathbb{C}$ (this can be proved straightforwardly, using the fact that the presence of pullbacks guarantees that a morphism in $\mathrm{Pt}(C)$ is a monomorphism in $\mathrm{Pt}(C)$ if and only if it is a monomorphism in $\mathbb{C}$, which itself follows from the fact that a morphism is a monomorphism if and only if its kernel pair has equal components). Thus, the above mentioned result of [2] can be reformulated as follows:

2.1. THEOREM. *In a finitely complete category $\mathbb{C}$ all reflexive relations are equivalence relations (i.e. $\mathbb{C}$ is a Mal'tsev category) if and only if every local product injection pair in $\mathbb{C}$ is jointly strongly epimorphic.*

Now, in this new terminology, the definition of a weakly Mal'tsev category given in [15] states:

2.2. DEFINITION. *A category $\mathbb{C}$ is said to be a weakly Mal'tsev category when $\mathbb{C}$ has local products and every local product injection pair in $\mathbb{C}$ is jointly epimorphic.*

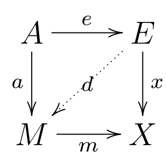
One of our aims in the present paper is to present a unified proof of Theorem 2.1 above and the following new result:

2.3. Theorem. *In a finitely complete category $\mathbb{C}$ all reflexive relations given by strong monomorphisms $R \to X \times X$ are equivalence relations if and only if every local product injection pair in $\mathbb{C}$ is jointly epimorphic (i.e. $\mathbb{C}$ is a weakly Mal'tsev category).*

2.4. Remark. Our usage of the term "local product" is the same as in [10], where local product projections were studied. In particular, in [10] a syntactical characterization of algebraic categories where local product projections are "normal" (i.e. product projections in $\mathrm{Pt}(C)$ are normal epimorphisms, for every object C) was obtained (see also [9]). The condition that local product injection pairs are jointly epimorphic implies normality of local product projections (in other words, every weakly Mal'tsev category has normal local projections). This follows from an observation that if a product (in a pointed category) has jointly epimorphic pair of product injections, then the product projections are normal epimorphisms, which is due to G. M. Kelly (who noted this during a talk by the first author at the Australian Category Seminar in 2003, when he was illustrating the proof of the fact that every unital category has normal product projections). Pointed categories where product injections are jointly epimorphic were called "weakly unital" in [14].

3. Formulating the main result

A *prefactorization system* in the sense of P. Freyd and G. M. Kelly [6] consists of a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms in a category $\mathbb{C}$, such that $\mathcal{E}$ is the class of all morphisms which are "orthogonal" to $\mathcal{M}$, and vice versa; here by a morphism e being *orthogonal* to a morphism m, written as in [6] as $e \downarrow m$, is meant that for any commutative diagram



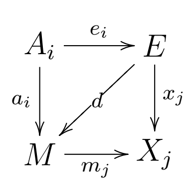
of solid arrows, there exists a unique dotted arrow such that the new diagram of the dotted and the solid arrows is still commutative. The relation of orthogonality can be naturally extended to the case when e is a *cospan*, i.e. a family of morphisms having common codomain, and m is a *span*, i.e. a family of morphisms having common domain:

$$e = (e_i : A_i \to E)_{i \in I}, \quad m = (m_j : M \to X_j)_{j \in J}.$$

Then $e \downarrow m$ means that for any two similar families

$$a = (a_i : A_i \to M)_{i \in I}, \quad x = (x_j : E \to X_j)_{j \in J}$$

such that $x_j e_i = m_j a_i$ for all $i \in I$ and $j \in J$, there exists a unique morphism $d : E \to M$ such that for all $i \in I$ and $j \in J$ the diagram



commutes (for examples of the use of such generalized orthogonality see e.g. [7] and also [1]). In the case when the coproduct $\coprod_{i\in I} A_i$ and the product $\prod_{j\in J} X_j$ exist, $e \downarrow m$ if and only if we have orthogonality of the canonical morphisms $[e_i]_{i\in I} : \coprod_{i\in I} A_i \to E$ and $\langle m_j\rangle_{j\in J} : M \to \prod_{j\in J} X_j$.

The relation of orthogonality gives rise to a Galois connection between classes of cospans and classes of spans: for a class $\mathcal{E}$ of cospans, by $\mathcal{E}^{\downarrow}$ we denote the corresponding Galois closed class of all spans m such that $e \downarrow m$ for all $e \in \mathcal{E}$; for a class $\mathcal{M}$ of spans, by ${}^{\downarrow}\mathcal{M}$ we denote the corresponding Galois closed class of all cospans e such that $e \downarrow m$ for all $m \in \mathcal{M}$. Now, for given classes κ and σ of families of morphisms, this Galois connection restricts to a Galois connection between cospan subclasses of κ and span subclasses of σ,

$$\mathcal{E}^{\downarrow\sigma} = \mathcal{E}^{\downarrow} \cap \sigma, \quad {}^{\kappa\downarrow}\mathcal{M} = \kappa \cap {}^{\downarrow}\mathcal{M}.$$

In particular, when $\kappa = \sigma = \mathbf{1}$ is the class of families over a fixed singleton index set, we can identify corresponding classes of cospans and spans with classes of morphisms of the category, and then the Galois connection becomes the one considered in [6]. A prefactorization system in the sense of [6] is thus the *same* as a prefactorization system of type $\mathbf{1}/\mathbf{1}$ in the following sense:

3.1. Definition. *A pair of Galois closed classes*

$$\mathcal{E} = {}^{\kappa\downarrow}\mathcal{M}, \quad \mathcal{M} = \mathcal{E}^{\downarrow\sigma},$$

is called a prefactorization system of type κ/σ.

By Δ we denote the class of all families of identical isomorphisms (i.e. those families of isomorphisms where any two members are equal). Then

$$\Delta_{\mathbf{1}} = \Delta \cap \mathbf{1}$$

is the class of isomorphisms, and we have:

- $(\Delta_{\mathbf{1}})^{\downarrow} = (\varnothing)^{\downarrow}$ is the class of all spans;
- ${}^{\downarrow}(\Delta_{\mathbf{1}}) = {}^{\downarrow}(\varnothing)$ is the class of all cospans.

Now let $\mathbf{2}$ denote the class of all pairs of morphisms of the category (i.e. families over a fixed index set $\{1, 2\}$), and define

$$\Delta_{\mathbf{2}} = \Delta \cap \mathbf{2}.$$

Then:

3.2. Lemma. *In any category $\mathbb{C}$, we have:*

(a) $(\Delta_{\mathbf{2}})^{\downarrow}$ is the class of all jointly monomorphic spans. Further, $(\Delta_{\mathbf{2}})^{\downarrow} = \Delta^{\downarrow}$.

(b) ${}^{\downarrow}(\Delta_{\mathbf{2}})$ is the class of all jointly epimorphic cospans. Further, ${}^{\downarrow}(\Delta_{\mathbf{2}}) = {}^{\downarrow}\Delta$.

Recall that a span $(m_j : M \to X_j)_{j\in J}$ is said to be *jointly monomorphic* when for any two morphisms $u, v : L \to M$ we have: if $m_j u = m_j v$ for all $j \in J$, then $u = v$. A jointly epimorphic cospan is defined dually.

PROOF OF LEMMA 3.2 It suffices to prove (a), since (b) is dual to (a). Let $(m_j : M \to X_j)_{j\in J}$ be a span from $(\Delta_\mathbf{2})^\downarrow$. Take any two morphisms $a_1, a_2 : A \to M$ such that $m_j a_1 = m_j a_2$ for all $j \in J$. The fact that $(m_j : M \to X_j)_{j\in J}$ is orthogonal to the cospan $(1_A, 1_A)$ implies $a_1 = a_2$. This proves that the span $(m_j : M \to X_j)_{j\in J}$ is jointly monomorphic. Since $\Delta^\downarrow \subseteq (\Delta_\mathbf{2})^\downarrow$, it remains to observe that any cospan from Δ is orthogonal to any jointly monomorphic span. ■

3.3. DEFINITION. *A prefactorization system* $(\mathcal{E}, \mathcal{M})$ *(of any given type* κ/σ*) is said to be proper, when any cospan in* $\mathcal{E}$ *is jointly epimorphic and any span in* $\mathcal{M}$ *is jointly monomorphic.*

In particular, for the type $\mathbf{1}/\mathbf{1}$ this becomes the usual notion of a proper prefactorization system [6]. For this type of prefactorization systems we always have

$$\Delta_\mathbf{1} = \mathcal{E} \cap \mathcal{M}.$$

For the purposes of the present paper, we are interested in prefactorization systems of type $\mathbf{2}/\mathbf{2}$. For such prefactorization systems, the inclusion

$$\Delta_\mathbf{2} \subseteq \mathcal{E} \cap \mathcal{M} \tag{1}$$

alone characterizes the proper prefactorization systems (see also Example 3.7). More generally, we have:

3.4. THEOREM. *In any category* $\mathbb{C}$*, the following conditions on a type* $\mathbf{2}/\mathbf{2}$ *prefactorization system* $(\mathcal{E}, \mathcal{M})$ *are equivalent:*

(a) The inclusion (1) holds.

(b) $(\mathcal{E}, \mathcal{M})$ *is a proper prefactorization system.*

(c) The inclusions $({}^{\mathbf{2}\downarrow}(\Delta_\mathbf{2}))^{\downarrow\mathbf{2}} \subseteq \mathcal{M}$ *and* ${}^{\mathbf{2}\downarrow}((\Delta_\mathbf{2})^{\downarrow\mathbf{2}}) \subseteq \mathcal{E}$ *hold.*

(d) The inclusions $({}^{\mathbf{2}\downarrow}(\Delta_\mathbf{2}))^{\downarrow\mathbf{2}} \subseteq \mathcal{M} \subseteq (\Delta_\mathbf{2})^{\downarrow\mathbf{2}}$ *hold.*

(e) The inclusions $\Delta_\mathbf{2} \subseteq \mathcal{M} \subseteq (\Delta_\mathbf{2})^{\downarrow\mathbf{2}}$ *hold.*

PROOF. The inclusion (1) implies, via the Galois connection, the inclusions

$$\mathcal{E} \subseteq {}^{\mathbf{2}\downarrow}(\Delta_\mathbf{2}) \quad \text{and} \quad \mathcal{M} \subseteq (\Delta_\mathbf{2})^{\downarrow\mathbf{2}}.$$

By Lemma 3.2, these inclusions state precisely that the prefactorization system $(\mathcal{E}, \mathcal{M})$ is proper. Thus, (a)⇒(b). Moreover, another application of Galois connection allows to deduce the inclusions in (c) from the ones above, and so (b)⇒(c). The remaining implications (c)⇒(d)⇒(e)⇒(a) follow in a similar manner. ■

3.5. Proposition. *Let $(\mathcal{E},\mathcal{M})$ be a proper type $\mathbf{2}/\mathbf{2}$ prefactorization system. Consider a cospan*

$$A_1 \xrightarrow{e_1} E \xleftarrow{e_2} A_1$$

The conditions below are related by the implications (a)⇒(b)⇒(c).

(a) $(e_1, e_2) \in \mathcal{E}$.

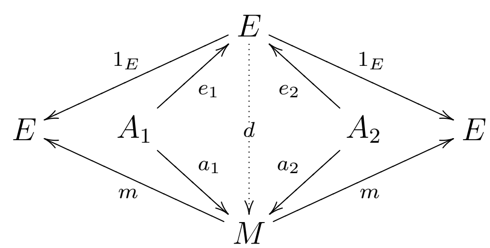
(b) (e_1, e_2) *is orthogonal to all* m *(regarded as a singleton family) such that* $(m, m) \in \mathcal{M}$.

(c) $(e_1, e_2) = (ma_1, ma_2)$ *and* $(m, m) \in \mathcal{M}$ *implies* m *is an isomorphism.*

Moreover, under the presence of pushouts and coequalizers we have (b)⇒(a), while under the presence of pullbacks and equalizers we have (c)⇒(a).

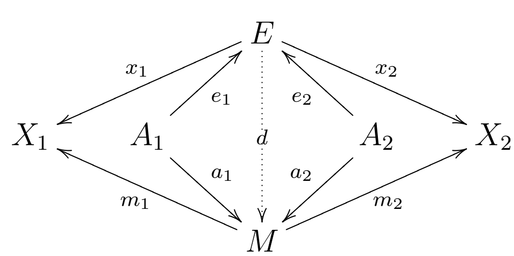
Proof. (a)⇒(b) is straightforward.

(b)⇒(c): Suppose $(e_1, e_2) = (ma_1, ma_2)$, as exhibited by commutativity of the following diagram of solid arrows:



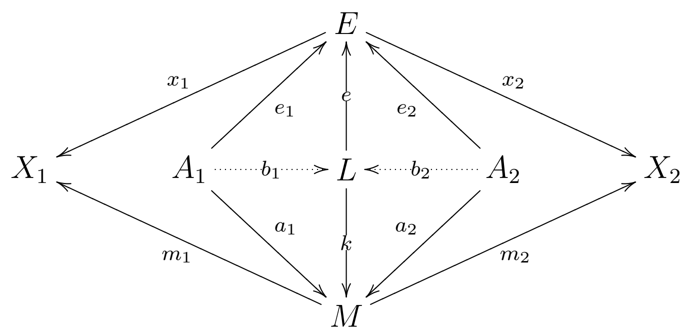
If $(m, m) \in \mathcal{M}$ then, by (b), there exists a morphism d indicated by the dotted arrow above, such that $md = 1_E$. Since $(\mathcal{E},\mathcal{M})$ is proper, $(m, m) \in \mathcal{M}$ implies that (m, m) is jointly monomorphic, and hence m is a monomorphism. This and $md = 1_E$ together imply that m is an isomorphism.

(b)⇒(a) under the presence of pushouts and coequalizers: We prove the dual formulation of this fact, which states that under the presence of pullbacks and equalizers, if $e \downarrow (m_1, m_2)$ for all e such that $(e, e) \in \mathcal{E}$, then $(m_1, m_2) \in \mathcal{M}$. Assume that we have $e \downarrow (m_1, m_2)$ for all e such that $(e, e) \in \mathcal{E}$. To show $(m_1, m_2) \in \mathcal{M}$ we must show that for any commutative diagram of solid arrows



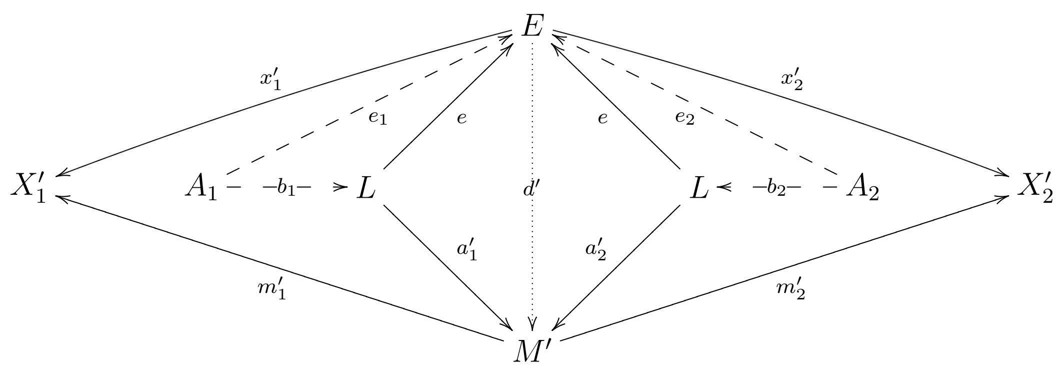
(2)

with $(e_1, e_2) \in \mathcal{E}$, there exists unique dotted arrow d such that the entire diagram commutes. Since $(\mathcal{E},\mathcal{M})$ is proper, the uniqueness of d follows from the existence. To show the existence of d, we complete the above diagram with a limiting cone over the outer diamond (which can be obtained using a pullback and an equalizer), and the induced morphisms as indicated by the dotted arrows in the following diagram:



(3)

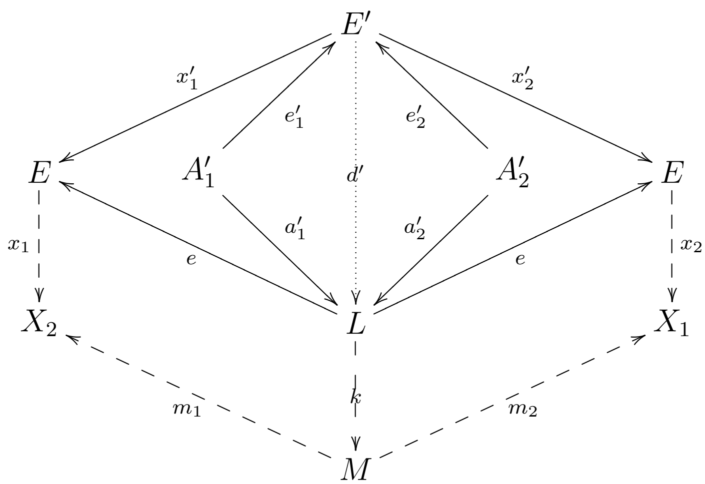
If we can show $(e, e) \in \mathcal{E}$, then by the assumption on (m_1, m_2) we will get a morphism $d : E \to M$ such that $de = k$ and the two outer triangles in (3), which are the same as those in (2), commute. Commutativity of the two inner triangles in (2) will then follow from commutativity of the four inner triangles in (3) and the equality $de = k$, and so this d will be the desired one. We now show that indeed $(e, e) \in \mathcal{E}$, by showing that $(e, e) \downarrow (m'_1, m'_2)$ for any span $(m'_1, m'_2) \in \mathcal{M}$. For this, we must show that for any commutative diagram of solid arrows



(4)

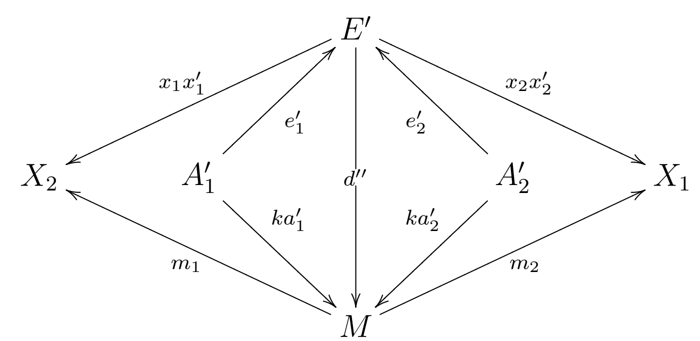
(with $(m'_1, m'_2) \in \mathcal{M}$), there exists unique dotted arrow d' such that the diagram of dotted and solid arrows commutes (ignore the dashed arrows for now). Again, since $(\mathcal{E}, \mathcal{M})$ is proper, it suffices to show only the existence of d'. For this, we extend the diagram (4) with the dashed arrows. Since $(e_1, e_2) \in \mathcal{E}$ and $(m'_1, m'_2) \in \mathcal{M}$, we get the dotted arrow d' making the two outer triangles in (4) commute (as well as satisfying the equalities $d'e_1 = a'_1 b_1$ and $d'e_2 = a'_2 b_2$, but we will not use these equalities). To show that this d' is the desired one, it remains to show $a'_1 = d'e = a'_2$. This follows from the fact that (m'_1, m'_2) is jointly monomorphic, and that the diagram of solid arrows in (4) commutes.

(c)⇒(a) under the presence of pullbacks and equalizers: To show $(e_1, e_2) \in \mathcal{E}$, we consider a diagram of solid arrows (2) with $(m_1, m_2) \in \mathcal{M}$ and show that there exists unique dotted arrow d making the entire diagram (2) commute. Again, uniqueness will follow from the fact that (m_1, m_2) is jointly monomorphic (as $(\mathcal{E}, \mathcal{M})$ is proper). To show the existence of d, we consider the same construction as above, where we complete the diagram (2) with the limiting cone of the outer diamond and the induced dotted morphisms, as displayed in (3). To get the d in (2) it suffices to show that e is an isomorphism, since then we can set $d = ke^{-1}$. We have $(e_1, e_2) = (eb_1, eb_2)$ and so, by (c), to show that e is an isomorphism it suffices to show $(e, e) \in \mathcal{M}$. For this, we show that for any commutative diagram of solid arrows



(5)

where $(e_1', e_2') \in \mathcal{E}$, there exists a unique dotted arrow d' such that the diagram of dotted and solid arrows commutes. Extending the diagram with the dashed arrows (taken from diagram (3)), we note that since $(m_1, m_2) \in \mathcal{M}$, there exists a unique morphism $d'' : E' \to M$ making the following diagram commutative:



To show that d'' lifts to d' in (5), it suffices to show $x_1' = x_2'$. Since $(e_1', e_2') \in \mathcal{E}$ and $(\mathcal{E}, \mathcal{M})$ is proper, the pair (e_1', e_2') is jointly epimorphic. This, together with commutativity of the diagram of solid arrows in (5) gives $x_1' = x_2'$. ■

Prefactorization systems $(\mathcal{E}, \mathcal{M})$ can be ordered as follows: $(\mathcal{E}, \mathcal{M}) \leqslant (\mathcal{E}', \mathcal{M}')$ when $\mathcal{E}' \subseteq \mathcal{E}$ and $\mathcal{M} \subseteq \mathcal{M}'$. Below we consider some prefactorization systems which can be characterized via this order:

3.6. Examples. Theorem 3.4 shows that the pair

$$({}^{\mathbf{2}\downarrow}((\Delta_{\mathbf{2}})^{\downarrow\mathbf{2}}), (\Delta_{\mathbf{2}})^{\downarrow\mathbf{2}})$$

is the largest proper prefactorization system of type $\mathbf{2}/\mathbf{2}$, while the pair

$$({}^{\mathbf{2}\downarrow}(\Delta_{\mathbf{2}}), ({}^{\mathbf{2}\downarrow}(\Delta_{\mathbf{2}}))^{\downarrow\mathbf{2}})$$

is the smallest such. We make few observations and introduce special terminology:

(a) By Lemma 3.2(a), $(\Delta_{\mathbf{2}})^{\downarrow\mathbf{2}}$ is the class of jointly monomorphic spans (m_1, m_2) — henceforth we call them *(binary) relations*. It follows from Proposition 3.5 that in the case when the category has pullbacks and equalizers, the corresponding class ${}^{\mathbf{2}\downarrow}((\Delta_{\mathbf{2}})^{\downarrow\mathbf{2}})$ coincides with the class of jointly strongly epimorphic pairs, which are also the same as jointly extremal epimorphic pairs. We remind the reader that a cospan (e_1, e_2) is said to be a *jointly strongly epimorphic pair* when it is orthogonal to any singleton span consisting of a monomorphism, and it is said to be a *jointly extremal epimorphic pair* when any monomorphism m which arises in a decomposition $(e_1, e_2) = (me'_1, me'_2)$ is necessarily an isomorphism.

(b) By Lemma 3.2(b), ${}^{\mathbf{2}\downarrow}(\Delta_{\mathbf{2}})$ is the class of jointly epimorphic cospans (e_1, e_2). Spans in the corresponding class $({}^{\mathbf{2}\downarrow}(\Delta_{\mathbf{2}}))^{\downarrow\mathbf{2}}$ will be called *(binary) strong relations*. In particular, it is easy to verify that all pullback projection pairs are strong relations. It follows from Lemma 3.2(a) that every strong relation is a relation. By the dual of Proposition 3.5, when the category has pullbacks and equalizers, strong relations are precisely the jointly strongly monomorphic pairs.

3.7. Example. In a category with pullbacks and equalizers, the smallest prefactorization system of type $\mathbf{2}/\mathbf{2}$ is the pair $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ consists of all cospans (e_1, e_2), and $\mathcal{M}$ is the class of precisely those spans

$$X_1 \xleftarrow{m_1} M \xrightarrow{m_2} X_2$$

which are product diagrams. This shows that there are prefactorization systems of type $\mathbf{2}/\mathbf{2}$ where we have neither $\Delta_{\mathbf{2}} \subseteq \mathcal{E} \cap \mathcal{M}$ nor $\mathcal{E} \cap \mathcal{M} \subseteq \Delta_{\mathbf{2}}$.

3.8. Remark. All observations on type $\mathbf{2}/\mathbf{2}$ prefactorization systems above can be also extended to prefactorization systems of other types. In particular, Proposition 3.5 can be seen as a direct adaptation of well-known observations for prefactorization systems of type $\mathbf{1}/\mathbf{1}$, and also extends to prefactorization systems of higher types. At the same time, while Theorem 3.4 is somewhat unexpected from the point of view of prefactorization systems of type $\mathbf{1}/\mathbf{1}$, it can also be extended to prefactorization systems of higher types.

3.9. Definition. *A class $\mathcal{M}$ of binary relations is said to be proper if it is part of a proper prefactorization system $(\mathcal{E}, \mathcal{M})$ of type $\mathbf{2}/\mathbf{2}$.*

We are now ready to formulate the main result of the paper:

3.10. THEOREM. *Let $\mathbb{C}$ be a category with pullbacks and equalizers. For any proper class $\mathcal{M}$ of binary relations, the following conditions are equivalent:*

(a) The corresponding class $\mathcal{E}$ of cospans contains local product injection pairs.

(b) Any reflexive relation in $\mathcal{M}$ is symmetric.

(c) Any reflexive relation in $\mathcal{M}$ is transitive.

(d) Any reflexive relation in $\mathcal{M}$ is an equivalence relation.

(e) Any relation in $\mathcal{M}$ is difunctional.

Before proving the theorem, which is carried out in the subsequent section, we remind the reader that a binary relation R is said to be difunctional if it satisfies the deduction rule

$$\frac{\begin{array}{ccc} a & R & d \\ c & R & d \\ c & R & b \end{array}}{\begin{array}{ccc} a & R & b \end{array}}$$

Via the Yoneda embedding, the same condition can be repeated for an internal relation

$$X_1 \xleftarrow{r_1} R \xrightarrow{r_2} X_2$$

in a category. Then, a, c above become *generalized elements* of X_1, and b, d those of X_2, all being morphisms having the same domain. For a span

$$X_1 \xleftarrow{x} C \xrightarrow{y} X_2$$

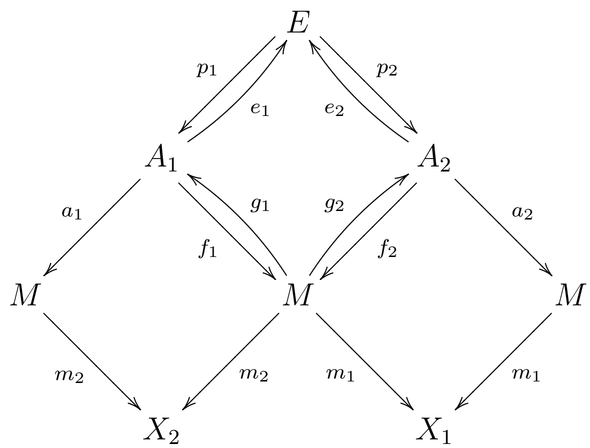
consisting of such generalized elements, the meaning of xRy is understood as the existence of a morphism $f : C \to R$ such that $(r_1 f, r_2 f) = (x, y)$. Reflexivity, symmetry and transitivity of an internal relation is defined in a similar way using the language of generalized elements.

4. Proof of Theorem 3.10

(a)$\Rightarrow$(e): From a relation

$$X_1 \xleftarrow{m_1} M \xrightarrow{m_2} X_2$$

construct a diagram

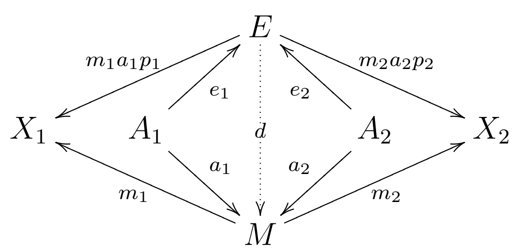


(6)

by taking the downward directed diamonds to be pullbacks, the top diamond to be a local product, and g_1, g_2 to be defined by the equalities

$$a_1 g_1 = f_1 g_1 = 1_M = f_2 g_2 = a_2 g_2.$$

The relation (m_1, m_2) is difunctional if and only if there exists a morphism $d : E \to M$ such that the diagram

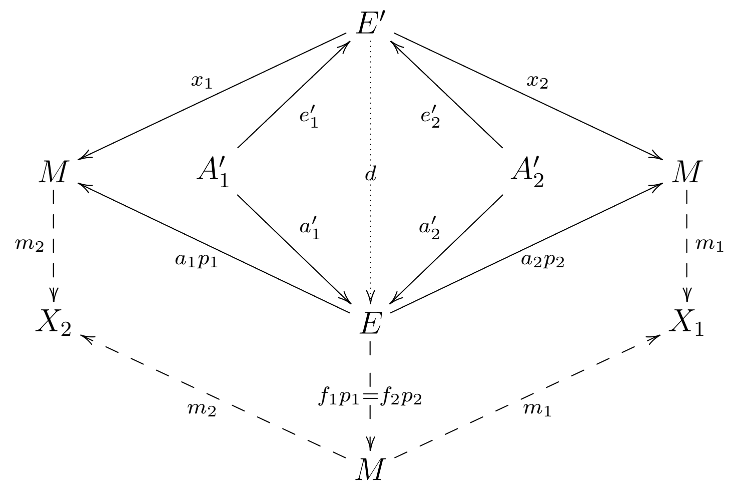


commutes (see [5]). This is evidently the case when $(e_1, e_2) \downarrow (m_1, m_2)$.

(e)⇒(d) since a reflexive relation is an equivalence relation if and only if it is difunctional.

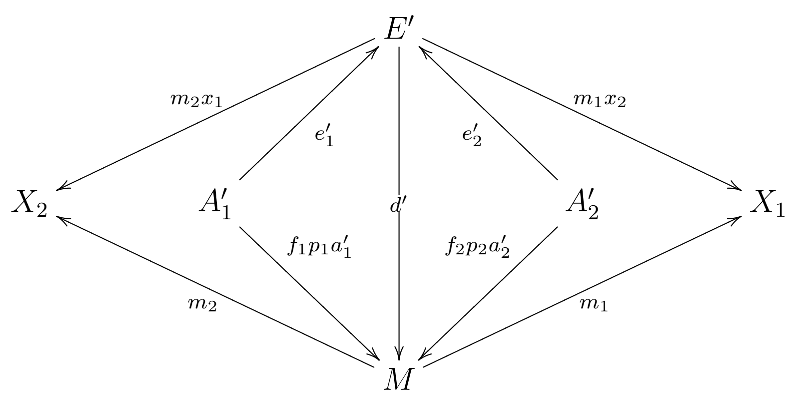
(d)⇒(c) and (d)⇒(b) are trivial.

For (b)⇒(e) and (c)⇒(e) we repeat the classical argument from [5]: In the construction (6) the span $m' = (a_1 p_1, a_2 p_2)$ is a reflexive relation which is symmetric/transitive if and only if the relation $m = (m_1, m_2)$ is difunctional. It remains to verify that $m' = (a_1 p_1, a_2 p_2) \in \mathcal{M}$. Given any commutative diagram of solid arrows (ignore the dotted/dashed arrows for now)



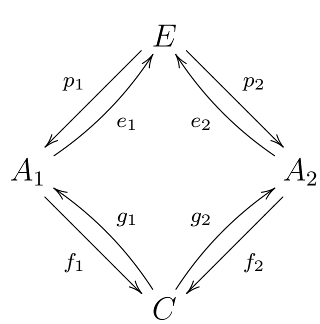
(7)

with $(e'_1, e'_2) \in \mathcal{E} = {}^{\mathbf{2}\downarrow}\mathcal{M}$, we have to show the existence of a unique dotted arrow d such that the entire diagram commutes. For this, we extend the diagram with the dashed arrows. Since $(m_1, m_2) \in \mathcal{M}$, we get $d' : E' \to M$ making the diagram

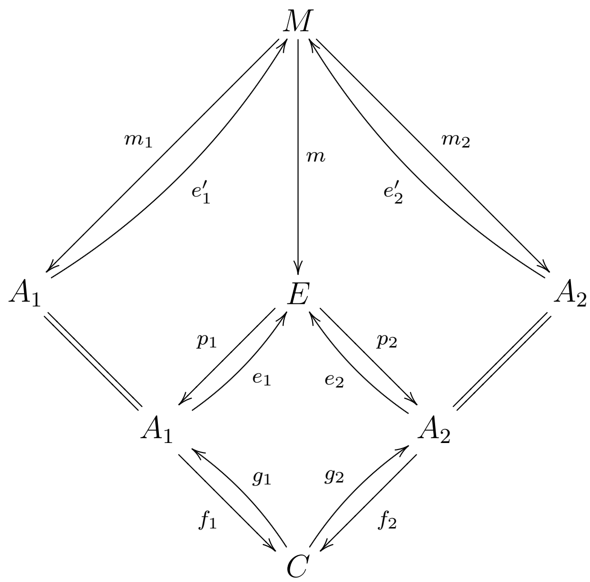


commute. The morphism d' induces the morphism d in (7) since the cone coming out of the object E in (7) is a limiting cone over the diagram of dashed arrows. Uniqueness of d follows from the fact that (e'_1, e'_2) is jointly epimorphic.

(e)⇒(a): Consider a local product



By Proposition 3.5, to show $(e_1, e_2) \in \mathcal{E}$ it suffices to show that whenever $(e_1, e_2) = (me_1', me_2')$ and $(m, m) \in \mathcal{M}$, the morphism m is an isomorphism. For this, consider the composites $m_1 = p_1 m$ and $m_2 = p_2 m$:



Since (p_1, p_2) is a strong relation, it belongs to $\mathcal{M}$ (see Theorem 3.4(d) and Example 3.6(b)). It can be straightforwardly shown that if further $(m, m) \in \mathcal{M}$ then also $(m_1, m_2) = (p_1 m, p_2 m) \in \mathcal{M}$. By (e), the relation (m_1, m_2) is difunctional. This implies that m is an isomorphism. Indeed, m is a monomorphism and for any generalized element $(a, b) \in E = A_1 \times_C A_2$ we have:

$$\begin{array}{rcl} a & M & g_2 f_1 a \\ g_1 f_2 b & M & g_2 f_1 a \\ g_1 f_2 b & M & b \\ \hline a & M & b \end{array}$$

The proof of Theorem 3.10 is now complete.

5. Conclusion

The equivalence of the conditions (b-e) in Theorem 3.10, with $\mathcal{M}$ being the class of all relations, was proved for finitely complete categories in [5], where they are used to define Mal'tsev categories (whereas in [4] only regular Mal'tsev categories were considered). As already explained in Section 2, characterization of Mal'tsev categories via the condition (a) of Theorem 3.10 is a simple reformulation of a result due to D. Bourn [2]. The fact that the equivalence of the conditions (a-e) in Theorem 3.10 can be established by using only pullbacks and equalizers (but avoiding products) allows to slightly expand the class of categories where these conditions have been looked at, including in it, for instance, the category of fields.

Taking $\mathcal{M}$ in Theorem 3.10 to be the class of strong relations (see Example 3.6(b)), we obtain the following new result:

5.1. Corollary. *Let $\mathbb{C}$ be a category having pullbacks and equalizers. Then the following conditions are equivalent:*

(a) $\mathbb{C}$ is a weakly Mal'tsev category.

(b) Any reflexive strong relation is symmetric.

(c) Any reflexive strong relation is transitive.

(d) Any reflexive strong relation is an equivalence relation.

(e) Any strong relation is difunctional.

5.2. Remark. As we can see from Example 3.7, the conditions (a-e) in Theorem 3.10 are still equivalent to each other when $\mathcal{M}$ is the class of product projection pairs (indeed, in this case all conditions are always satisfied), even though such $\mathcal{M}$ is not a proper class of relations. This suggests that perhaps it is possible to further generalize Theorem 3.10.

5.3. Remark. The connection between difunctionality, as well as symmetry and transitivity of reflexive relations, and the classical universal-algebraic condition asserting the existence of a Mal'tsev term (i.e. a term p satisfying $p(x, y, y) = x = p(y, y, x)$, see [13, 17]) has a long history, briefly exhibited in the Introduction of [3]. See also [11, 12] which gives a general method for producing such relational properties. In [3], Mal'tsev categories were characterized by suitable "categorical Mal'tsev terms". These results can be directly extended to the more general context proposed in the present paper, and in particular, as suggested by Corollary 5.1 above, they can be adopted for weakly Mal'tsev categories, by simply replacing relations with strong relations.

References

[1] J. Adamek, H. Herrlich, and G. Strecker, *Abstract and concrete categories: The joy of cats*, Reprints in Theory and Applications of Categories, 2006.

[2] D. Bourn, Mal'cev categories and fibration of pointed objects, *Applied Categorical Structures* 4, 1996, 307–327.

[3] D. Bourn and Z. Janelidze, Approximate Mal'tsev operations, *Theory and Applications of Categories* 21, 2008, 152-171.

[4] A. Carboni, J. Lambek, and M. C. Pedicchio, Diagram chasing in Mal'cev categories, *Journal of Pure and Applied Algebra* 69, 1990, 271-284.

[5] A. Carboni, M. C. Pedicchio, and N. Pirovano, Internal graphs and internal groupoids in Mal'tsev categories, *Canadian Mathematical Society Conference Proceedings* 13, 1992, 97-109.

[6] P. J. Freyd and G. M. Kelly, Categories of continuous functors I, *Journal of Pure and Applied Algebra* 2, 1972, 169-191.

[7] H. Herrlich, Topological functors, *General Topology and its Applications* 4, 1974, 125-142.

[8] G. Janelidze, *Internal categories in Maltsev varieties*, York University, 1990 (preprint).

[9] Z. Janelidze, Characterization of pointed varieties of universal algebras with normal projections, *Theory and Applications of Categories* 11, 2003, 212-214.

[10] Z. Janelidze, Varieties of universal algebras with normal local projections, *Georgian Mathematical Journal* 11, 2004, 93-98.

[11] Z. Janelidze, Closedness properties of internal relations I: A unified approach to Mal'tsev, unital and subtractive categories, *Theory and Applications of Categories* 16, 2006, 236-261.

[12] Z. Janelidze, Closedness properties of internal relations VI: Approximate operations, *Cahiers de Topologie et Géométrie Différentielle Catégoriques* 50, 2009, 298-319.

[13] A. I. Mal'tsev, On the general theory of algebraic systems, *Mat. Sbornik*, N.S. 35 (77), 1954, 3-20 (in Russian); English translation: *American Mathematical Society Translations* (2) 27, 1963, 125-341.

[14] N. Martins-Ferreira, *Low-dimensional internal categorial structures in weakly Mal'cev sesquicategories*, PhD Thesis, University of Cape Town, 2008.

[15] N. Martins-Ferreira, Weakly Mal'cev categories, *Theory and Applications of Categories* 21, 2008, 91-117.

[16] N. Martins-Ferreira, On distributive lattices and weakly Mal'tsev categories, *Journal of Pure and Applied Algebra* 216, 2012, 1961-1963.

[17] J. D. H. Smith, *Mal'cev varieties*, Lecture Notes in Mathematics 554, Springer, 1976.

Department of Mathematical Sciences, Stellenbosch University, South Africa

Departamento de Matemática, Escola Superior de Tecnologia e Gestão,
Centro para o Desenvolvimento Rápido e Sustentado do Produto,
Instituto Politécnico de Leiria, Portugal
Email: zurab@sun.ac.za
martins.ferreira@ipleiria.pt

This article may be accessed at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/27/5/27-05.{dvi,ps,pdf}

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at http://www.tac.mta.ca/tac/ and by ftp. It is archived electronically and in printed paper format.

Subscription information Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

Information for authors The typesetting language of the journal is TEX, and LATEX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at http://www.tac.mta.ca/tac/.