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Research paper

Factorization à la Dirac applied to the time-fractional telegraph equation

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ABSTRACT

This paper examines a coupled system of two-term time-fractional diffusion Dirac-type equations. The system is derived by factorizing the multi-dimensional time-fractional telegraph equation with Hilfer fractional derivatives, using the Dirac method and a triplet of Pauli matrices. Solutions are obtained using operational methods provided by the combination of the Fourier transform in the space variable and the Laplace transform in the time variable. Key results include the discovery of novel Fourier transform pairs. These pairs relate specific Fourier kernels of bivariate Mittag-Leffler functions to Fox H-functions of two variables. This allows to obtain explicit solutions of the system in both Fourier-time and space-time domains. The asymptotic behaviour of these solutions is rigorously analysed, and graphical representations are generated. Further, we show that the factorization allows for the use of alternative triplets of Pauli matrices yielding related solutions. The results obtained can be generalised to the case of ψ-Hilfer derivatives.

1. Introduction

The telegraph equation can be understood as the link between the wave equation and the diffusion equation and is employed in various fields of physics such as electricity, elasticity, and viscoelasticity, etc. (see [8] and references therein). It was established by Heaviside for the transmission of electrical signals over lines (telegraph wires), and it has the following form

$$\frac{\partial^2 f}{\partial x^2}(x,t) - LC\,\frac{\partial^2 f}{\partial t^2}(x,t) = (RC+GL)\,\frac{\partial f}{\partial t}(x,t) + RG\,f(x,t), \tag{1}$$

where R, L, C, and G represent, respectively, the resistance, the inductance, the capacity, and the conductance. The left-hand side of (1) contains the propagation terms, while the right-hand side accounts for wave distortion and attenuation. Therefore, it is a wave equation that takes into account the parameters of the medium that determine the propagation speed and the transport parameters. The telegraph equation can also be considered to be a simple generalization of the diffusion equation where we consider the Maxwell-Cattaneo's law (which takes the rate of the flow into account)

$$\frac{\partial^2 f}{\partial t^2}(x,t) + \frac{1}{\tau}\,\frac{\partial f}{\partial t}(x,t) = \frac{\kappa}{\tau}\,\nabla^2 f(x,t). \tag{2}$$

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Another way of looking at the telegraph equation is to consider it as a nondissipative wave equation in which viscous and thermal damping have been added by hand:

$$\frac{\partial^2 f}{\partial t^2}(x,t) + a\,\frac{\partial f}{\partial t}(x,t) - c^2\,\frac{\partial^2 f}{\partial x^2}(x,t) = 0, \tag{3}$$

where c is the speed of the wave and a is the attenuation coefficient. As pointed out in [36], even though Eqs. (2) and (3) have the same mathematical form, they are different in nature. Indeed, (2) is built according to the prescriptions that are given above, whereas (3) is not. In many applications, the introduction of wave damping is done by hand, i.e., without a proven scientific justification and it may obscure the understanding of the energy transfer mechanisms between the wave and the propagation medium. The only justification for the term $a\,\frac{\partial f}{\partial t}(x,t)$ is that the dispersion equation leads to complex-valued wave numbers that are associated with damped waves. In this work we consider the time-fractional version of (3):

$$^{H}\partial_t^{\alpha_2,\mu_2} u(x,t) + c_1^2\,{}^{H}\partial_t^{\alpha_1,\mu_1} u(x,t) - c_0^2\,\Delta_x u(x,t) = 0, \tag{4}$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and the time-fractional derivatives are in the Hilfer sense. In [33], we generalized the previous equation to the case of fractional derivatives with distributed order. Using operational techniques the solutions were expressed in terms of Fox H-functions, highlighting parallels in transform-based methods for multi-order systems. Other recent approach [24] include the use of the Elzaki transform for tackling linear and nonlinear time-fractional telegraph equations in electromagnetism.

Inspired by Dirac's work on creation and annihilation operators and by Pauli and Weyl's work on spherical harmonics with spin, Schrödinger was one of the first researchers to apply the factorization method to the case of hypergeometric equations. Subsequently, numerous authors have developed different factorization methods applicable to specific contexts/equations. For example, we can mention the work of Infeld and Hull (see [21]), where they applied this method to a broad class of differential equations to determine the eigenvalues and establish a procedure for generating the normalised eigenfunctions; of Mielnik (see [25]), where the factorization method is applied to the construction of a one-parameter family of potentials in one dimension; of Gendenshtein (see [15]), where supersymmetries for the Schrödinger operator are investigated using this method; of Smirnov (see [29]), where this method is applied in the context of orthogonal polynomials; of Rosu (see [26]), where supersymmetric quantum mechanics is investigated using the factorization method; and by Dattoli and his collaborators (see [2,3] and the references cited therein), who demonstrated that the Dirac factorization method can be successfully employed to address problems involving fractional order operators. In [4] the authors introduced a new class of time-fractional Dirac-type operators with time-variable coefficients, building on a Witt basis. This work provides explicit solutions to the associated fractional Cauchy problems, demonstrating their applicability for both analytical and computational purposes, including the resolution of an inverse problem. In [11], Faustino presented a wide class of space- and time-fractional semidiscrete Dirac operators of Lévy-Leblond type. His work relies on operational methods, including algebraic manipulations and discrete Fourier analysis, to study Cauchy problems on the semidiscrete space-time lattice.

A detailed analysis of the work of Dattoli and his collaborators reveals that they initially started by replacing a second-order differential equation with a pair of first-order differential equations. They generalised the classical factorization method at several levels. Firstly, it applies very naturally to second-order partial differential equations, making it convenient and efficient for determining solutions to wave equations, telegraph equations, and generalised diffusion equations. Another advantage of the method is that it yields various forms of solutions. By its nature, it results in a system of lower-order equations, which can be either independent or coupled. Furthermore, it can be generalised to equations of order higher than two, and to initial equations with odd-order derivatives, leading to differential equations containing fractional derivatives. Consequently, this method incorporates the memory effects that arise during the evolution of the studied processes. This is a significant aspect, as memory effects often exert a strong influence on transport phenomena, leading, for example, to non-stationarity. All these advantages render the Dirac factorization an effective method for the interpretation of physical processes.

Dirac's factorization method, as outlined in Fellah et al. [8], states that an operator can be defined as the square root of the sum of the squares of two operators, i.e., $\mathrm{O} = \sqrt{A^2 + B^2}$. This operator O can be expressed in the form $\mathrm{O} = \gamma_1 A + \gamma_2 B$, where γ_1 and γ_2 satisfy $\gamma_1^2 = \gamma_2^2 = 1$ and $\gamma_1\gamma_2 + \gamma_2\gamma_1 = 0$. The Pauli matrices (see Section 4) satisfy these conditions.

The Dirac method offers a structured approach to study the telegraph equation. This framework allows for the reformulation of the Euler equation and the constitutive relation as fractional partial differential equations (see [8] for a detailed discussion). By employing fractional calculus, the dynamic density and dynamic compressibility of the fluid can be introduced into our system. These quantities are treated as time-fractional differential operators acting on the wave that excites the fluid, each comprising an immediate response component and a memory-dependent term.

This paper applies the Dirac factorization method, employing a triplet of Pauli matrices, to factorize the telegraph equation in higher dimensions. The resulting coupled system of two-term time-fractional diffusion Dirac-type equations is solved using integral transforms. This process required the derivation of novel Fourier pairs. These pairs relate Fourier kernels, composed of bivariate Mittag-Leffler functions and the Fourier symbol of Laplace and/or Laplace and Dirac operators, to two-variable Fox H-functions in the space-time domain. These relations, which were not previously documented in the literature, are fundamental to this work. The solution of the coupled system involved technical procedures, resulting in explicit expressions for the solution in the Fourier-time domain, using bivariate Mittag-Leffler functions, and in the space-time domain, using two-variable Fox H-functions or double hypergeometric series.

The paper is structured as follows: Section 2 provides a review of fundamental concepts related to integro-differential fractional operators employed in this work, together with a review of the special functions and their main operational properties necessary for the analysis. Particular attention is given to both single and bivariate forms of Mittag-Leffler functions and Fox H-functions. Section 3 establishes a set of new Fourier transform pairs. These pairs link bivariate Mittag-Leffler functions in the Fourier-time domain to Fox H-functions of two variables in the space-time domain. In Section 4, the Dirac factorization method, using Pauli matrices, is applied to the time-fractional telegraph equation with Hilfer derivatives. The resulting coupled system of two-term time-fractional diffusion Dirac-type equations is solved using the Fourier transform with respect to the space variable and the Laplace transform with respect to the time variable. The Fourier pairs established in Section 3 are then used to derive explicit expressions for the solutions in the space-time domain, represented by Fox H-functions of two variables. For odd spatial dimensions, double series representations for the space-time solutions are derived. The asymptotic behaviour of the solutions is analysed in both Fourier-time and space-time domains, considering both large and small values of the respective variables. Conditions ensuring the L_p-integrability of the solutions in the Fourier-time domain are also established. Graphical representations and analysis of the solutions in both domains are included in Sections 4 and 5 examines particular cases of the solutions obtained in the preceding section. Specifically, the integer case ($\alpha_2 = 2$ and $\alpha_1 = 1$ in (4)) is considered, which, following the Dirac factorization, results in fractional derivatives of orders 1 and $\frac{1}{2}$. Sections 4 and 5 also analyse the case where $c_1 = 0$ in (4), which corresponds to the wave equation. The concluding section presents remarks concerning the role of the Pauli matrices in Section 4 and discusses the potential generalisation of the results to the case where ψ-Hilfer derivatives are considered.

2. Preliminaries

2.1. Definitions and properties

We begin by reviewing basic concepts related to Hilfer fractional integrals and derivatives. Let $a, b \in \mathbb{R}$ with $a < b$ and $\gamma > 0$. The left Riemann-Liouville fractional integral $I^{\gamma}_{a^+}$ of order $\gamma > 0$ of a function f is given by (see [22])

$$\left(I^{\gamma}_{a^+} f\right)(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{f(w)}{(t-w)^{1-\gamma}}\, dw, \qquad x > a. \tag{5}$$

The left Hilfer (or composite) fractional derivative ${}^H D^{\gamma,\nu}_{a^+}$ of order $\gamma > 0$ and type $0 \le \mu \le 1$ of a function f is given by (see [18–20,32])

$$\left({}^H D^{\gamma,\mu}_{a^+} f\right)(t) = \left(I^{\mu(m-\gamma)}_{a^+} \frac{d^m}{dt^m} \left(I^{(1-\mu)(m-\gamma)}_{a^+} f\right)(t),\right. \tag{6}$$

where $m = \lfloor \gamma \rfloor + 1$ and $\lfloor \gamma \rfloor$ is the floor function of γ. Note that when $\mu = 0$ we recover the left Riemann-Liouville fractional derivative of order γ and when $\mu = 1$ we recover the left Caputo fractional derivative of order γ. The latter is given by

$$\left({}^C D^{\gamma}_{a^+} f\right)(t) = \frac{1}{\Gamma(m-\gamma)} \int_a^t \frac{f^{(m)}(w)}{(t-w)^{1-m+\gamma}}\, dw. \tag{7}$$

The previous definitions of fractional integrals and derivatives can be naturally extended to functions of n variables by considering partial fractional integrals and derivatives (see [27, Ch. 5]).

Concerning the composition of the operators introduced previously, we have that ${}^H D^{\gamma,\mu}_{a^+} I^{\gamma}_{a^+} f(t) = f(t)$. For the composition $I^{\gamma}_{a^+}\, {}^H D^{\gamma,\mu}_{a^+}$ we have (see [18–20,32]):

$$I^{\gamma}_{a^+}\, {}^H D^{\gamma,\mu}_{a^+} f(t) = f(t) - \sum_{k=1}^{m} \frac{(t-a)^{\gamma+\mu(k-\gamma)-k}}{\Gamma(\gamma+\mu(k-\gamma)-k+1)} I^{(1-\mu)(m-\gamma)-(m-k)}_{a^+} f(a). \tag{8}$$

For the power function, we have the following differentiation rule:

$${}^H D^{\gamma,\mu}_{a^+}\left[(t-a)^{\lambda-1}\right] = \begin{cases} \frac{\Gamma(\lambda)}{\Gamma(\lambda-\gamma)}(t-a)^{\lambda-1-\gamma}, & \lambda - 1 + (1-\mu)(m-\gamma) > \lfloor \gamma \rfloor \\ 0, & \lambda - 1 + (1-\mu)(m-\gamma) = p,\ \ p = 0, \dots, m-1 \end{cases} \tag{9}$$

as can be seen by (6).

In this work, we make use of the n-dimensional Fourier transform and the Laplace transform. For a real-valued Lebesgue integrable function f on $\mathbb{R}^n$, the n-dimensional Fourier transform is defined by (see [22])

$$\mathcal{F}\{f(x)\}(\kappa) = \hat{f}(\kappa) = \int_{\mathbb{R}^n} e^{i\kappa\cdot x} f(x)\, dx, \qquad x, \kappa \in \mathbb{R}^n,$$

where $\kappa \cdot x$ represents the dot product of vectors κ and x, while the corresponding inverse Fourier transform is given by

$$f(x) = \mathcal{F}^{-1}\left\{\hat{f}(\kappa)\right\}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\kappa\cdot x} \hat{f}(\kappa)\, d\kappa. \tag{10}$$

We will also use the following well-known Convolution Theorem:

$$\mathcal{F}\left\{(f *_x g)(x)\right\}(\kappa) = \mathcal{F}\{f\}(\kappa)\, \mathcal{F}\{g\}(\kappa), \tag{11}$$

where the convolution $*_x$ is given by

$$(f *_x g)(x) = \int_{\mathbb{R}^n} f(x-z)\, g(z)\, dz. \tag{12}$$

For $g : [0,\infty) \to \mathbb{R}$ exponentially bounded, the Laplace transform is defined by (see [22])

$$\mathcal{L}\{g(t)\}(s) = \widetilde{g}(s) = \int_0^{\infty} e^{-st}\, g(t)\, dt, \tag{13}$$

which is an analytic function of s for $\mathrm{Re}(s) > 0$. The inverse Laplace transform is defined by

$$\mathcal{L}^{-1}\{\widetilde{g}(s)\}(t) = \lim_{R\to\infty} \frac{1}{2\pi\, i} \int_{c-iR}^{c+iR} e^{st}\, \widetilde{g}(s)\, ds. \tag{14}$$

For functions $f, g : [0,\infty) \to \mathbb{R}$ the standard convolution reduces to the Laplace convolution with finite limits of integration, which is defined by

$$(f *_t g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau. \tag{15}$$

The Laplace convolution theorem is then given by

$$\mathcal{L}\{(f *_t g)(t)\} = \widetilde{f}(s)\, \widetilde{g}(s) \tag{16}$$

or, equivalently,

$$\mathcal{L}^{-1}\{\widetilde{f}(s)\, \widetilde{g}(s)\} = (f *_t g)(t). \tag{17}$$

The Laplace transform of the Hilfer derivative (6) is (see [17]):

$$\mathcal{L}\{{}^{H}D_{0^+}^{\gamma,\mu} f(t)\}(s) = s^{\gamma}\, \widetilde{f}(s) - \sum_{j=0}^{m-1} s^{m-j-\mu(m-\gamma)-1} \left[\frac{d^j}{dt^j} \left({}_t I_{0^+}^{(1-\mu)(m-\gamma)} f \right) \right](0^+), \tag{18}$$

where the initial-value terms are evaluated as $t \to 0^+$.

2.2. Special functions

2.2.1. Fox H-functions

This section reviews some special functions used in this work and their main properties. The Fox H-function $H_{p,q}^{m,n}(z)$ is defined by means of a Mellin-Barnes type integral, a complex contour integral (see [23,31]):

$$H_{p,q}^{m,n}\left[z \;\middle|\; \begin{matrix} (a_1,\alpha_1),\dots,(a_p,\alpha_p) \\ (b_1,\beta_1),\dots,(b_q,\beta_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma\big(b_j + \beta_j s\big) \prod_{i=1}^{n} \Gamma\big(1 - a_i - \alpha_i s\big)}{\prod_{i=n+1}^{p} \Gamma\big(a_i + \alpha_i s\big) \prod_{j=m+1}^{q} \Gamma\big(1 - b_j - \beta_j s\big)}\, z^{-s}\, ds, \tag{19}$$

in which $z^{-s} = \exp\big[-s(\ln|z| + i \arg z)\big]$ where $z \neq 0$, $\log|z|$ represents the natural logarithm of $|z|$, and $\arg z$ is not necessarily the principal value. An empty product is interpreted as unity. Also, m, n, p, and q are non-negative integers satisfying $0 \le n \le p$, $1 \le m \le q$, the parameters $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}^+$, for $i = 1,\dots,p$ and $j = 1,\dots,q$. Also, a_j and b_j are such that none of the poles of $\Gamma\big(b_j + \beta_j s\big)$, $j = 1,\dots,m$, and $\Gamma\big(1 - a_i - \alpha_i s\big)$, $i = 1,\dots,n$, coincide with one another. The contour $\mathcal{L}$ is an infinite contour, indented if necessary, such that the poles of $\Gamma\big(b_j + \beta_j s\big)$, $j = 1,\dots,m$, lie to the right of $\mathcal{L}$ and the poles of $\Gamma\big(1 - a_i - \alpha_i s\big)$, $i = 1,\dots,n$ lie to the left of $\mathcal{L}$, and has one of the following forms:

- $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;
- $\mathcal{L} = \mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty + i\varphi_1$ and terminating at the point $+\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$;
- $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour parallel to the imaginary axis, starting at the point $\gamma - i\infty$ and terminating at the point $\gamma + i\infty$, where $\gamma \in \mathbb{R}$.

The conditions on the parameters for the analyticity and convergence of the Fox H-function, and the orientation of the contour $\mathcal{L}$ are presented in Theorems 1.1 and 1.2 in [23].

The H-function (19) has been extended to several complex variables, which allows for the analysis of more complex systems involving multiple parameters. For the case of two complex variables, it is defined via a double Mellin-Barnes type integral of the form (see [6,31])

$$\begin{aligned} &H_{p_1,q_1;\,p_2,q_2;\,p_3,q_3}^{0,n_1;\,m_2,n_2;\,m_3,n_3}\left[\begin{matrix} z_1 \\ z_2 \end{matrix} \;\middle|\; \begin{matrix} (a_j;\alpha_j,A_j)_{1,p_1};\,(c_j,\gamma_j)_{1,p_2};\,(e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1};\,(d_j,\delta_j)_{1,q_2};\,(f_j,F_j)_{1,q_3} \end{matrix} \right] \\ &\quad = \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_2} \int_{\mathcal{L}_1} \phi(s,w)\, \phi_1(s)\, \phi_2(w)\, z_1^{s}\, z_2^{w}\, ds\, dw, \end{aligned} \tag{20}$$

with

$$\phi(s,w) = \frac{\prod_{i=1}^{n_1} \Gamma\big(1 - a_i + \alpha_i s + A_i w\big)}{\prod_{i=n_1+1}^{p_1} \Gamma\big(a_i - \alpha_i s - A_i w\big) \prod_{j=1}^{q_1} \Gamma\big(1 - b_j + \beta_j s + B_j w\big)},$$

$$\phi_1(s) = \frac{\prod_{j=1}^{m_2} \Gamma\left(d_j - \delta_j s\right) \prod_{i=1}^{n_2} \Gamma\left(1 - c_i + \gamma_i s\right)}{\prod_{j=m_2+1}^{q_2} \Gamma\left(1 - d_j + \delta_j s\right) \prod_{i=n_2+1}^{p_2} \Gamma\left(c_i - \gamma_i s\right)},$$

$$\phi_2(w) = \frac{\prod_{j=1}^{m_3} \Gamma\left(f_j - F_j w\right) \prod_{i=1}^{n_3} \Gamma\left(1 - e_i + E_i w\right)}{\prod_{j=m_3+1}^{q_3} \Gamma\left(1 - f_j + F_j w\right) \prod_{i=n_3+1}^{p_3} \Gamma\left(e_i - E_i w\right)},$$

where $z_1, z_2 \in \mathbb{C}$, any empty product is interpreted as unity, and $m_i, n_i, p_i, q_i \in \mathbb{Z}$ such that $0 \leq n_i \leq p_i$, $0 \leq q_i$, and $0 \leq m_j \leq q_j$ $(i = 1, 2, 3; j = 2, 3)$. Also, $a_i, b_j, c_i, d_j, e_i, f_j \in \mathbb{C}$ and $\alpha_i, A_i, \beta_j, B_j, \gamma_i, \delta_j, E_i, F_j \in \mathbb{R}^+$. The contour $\mathcal{L}_1$ is in the s-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to separate the poles, ensuring that the poles of $\Gamma\left(d_j - \delta_j s\right)$ $(j = 1, \dots, m_2)$ lie to the right and the poles of $\Gamma\left(1 - c_i + \gamma_i s\right)$ $(i = 1, \dots, n_2)$ and $\Gamma\left(1 - a_i + \alpha_i s + A_i w\right)$ $(i = 1, \dots, n_1)$ lie to the left of the contour. The contour $\mathcal{L}_2$ lies in the w-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma\left(f_j - F_j w\right)$ $(j = 1, \dots, m_3)$ lie to the right and the poles of $\Gamma\left(1 - e_i + E_i w\right)$ $(i = 1, \dots, n_3)$ and $\Gamma\left(1 - a_i + \alpha_i s + A_i w\right)$ $(i = 1, \dots, n_1)$ lie to the left of the contour. The conditions for the analyticity and convergence of this special function, its general properties, and the orientation of the contours $\mathcal{L}_1$ and $\mathcal{L}_2$ are studied in [6,9,31]. The difference of $\pm s$ in the exponent of z in (19) and in z and w in (20) arises from the different definitions found in the literature. However, a change of variables allows for the same exponent s or $-s$ in the integration variable(s) of both definitions.

Let $H[z_1, z_2]$ denote the Fox H-function of two variables defined in (20). We have (see [31, Sec. 6.2]):

$$H[z_1, z_2] = \mathcal{O}(|z_1|^{\alpha} |z_2|^{\beta}), \quad \max\{|z_1|, |z_2|\} \to 0, \tag{21}$$

where

$$\alpha = \min_{1 \leq j \leq m_2} \left(\mathrm{Re}\left(\frac{d_j}{\delta_j}\right)\right) \qquad \text{and} \qquad \beta = \min_{1 \leq j \leq m_3} \left(\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right).$$

Moreover, also from [31, Sec. 6.2], we have the following asymptotic behavior at infinity:

$$H[z_1, z_2] = \mathcal{O}(|z_1|^{\alpha'} |z_2|^{\beta'}), \quad \min\{|z_1|, |z_2|\} \to \infty, \tag{22}$$

where

$$\alpha' = \min_{1 \leq i \leq n_2} \left(\mathrm{Re}\left(\frac{c_i - 1}{\gamma_i}\right)\right) \qquad \text{and} \qquad \beta' = \min_{1 \leq i \leq n_3} \left(\mathrm{Re}\left(\frac{e_i}{E_i}\right)\right).$$

Based on the series representation of the Fox H-function of two variables (see Formula (6.2.1) in [31]) we derive the following lemma:

Lemma 2.1. *(c.f. Formula (6.4.17) in [31]) Under the conditions on the sequences of poles for the Fox H-function of two variables (20), we have:*

$$\begin{aligned}
&\lim_{z_2 \to 0} H^{0,n_1;\,m_2,n_2;\,m_3,n_3}_{p_1,q_1;\,p_2,q_2;\,p_3,q_3}\left[\begin{array}{c|c} z_1 & (a_j;\alpha_j,A_j)_{1,p_1};\,(c_j,\gamma_j)_{1,p_2};\,(e_j,E_j)_{1,p_3} \\ z_2 & (b_j;\beta_j,B_j)_{1,q_1};\,(d_j,\delta_j)_{1,q_2};\,(f_j,F_j)_{1,q_3} \end{array}\right] \\
&\quad = H^{0,p_1+p_2}_{p_1+p_2,q_1+q_2}\left[\begin{array}{c|c} z_1 & (a_j,\alpha_j)_{1,p1},(c_j,\gamma_j)_{1,p_2} \\ & (b_j,\beta_j)_{1,q_1},\,(d_j,\delta_j)_{1,q_2} \end{array}\right].
\end{aligned} \tag{23}$$

2.2.2. Mittag-Leffler functions

The Mittag-Leffler function (henceforth referred to as the ML function) with two parameters, denoted by $E_{\beta_1,\beta_2}(z)$, is defined by the following power series (see [16])

$$E_{\beta_1,\beta_2}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma\left(\beta_1 k + \beta_2\right)}, \quad z \in \mathbb{C}, \ \ \mathrm{Re}(\beta_1) > 0, \, \mathrm{Re}(\beta_2) > 0. \tag{24}$$

For specific values of β_1 and β_2, (24) reduces to elementary functions, which simplifies the analysis of certain problems. For example, when $z \in \mathbb{R}^+$, we have:

$$E_{2,1}(-z) = \cos(\sqrt{z}) \qquad \text{and} \qquad E_{2,2}(-z) = \frac{\sin(\sqrt{z})}{\sqrt{z}}. \tag{25}$$

The three-parameter ML function $E^{\beta_3}_{\beta_1,\beta_2}(z)$ (see [16]), is defined in terms of power series by

$$E^{\beta_3}_{\beta_1,\beta_2}(z) = \sum_{k=0}^{\infty} \frac{(\beta_3)_k \, z^k}{k! \, \Gamma\left(\beta_1 k + \beta_2\right)}, \quad z \in \mathbb{C}, \ \ \mathrm{Re}(\beta_1) > 0, \, \mathrm{Re}(\beta_2) > 0, \, \beta_3 > 0, \tag{26}$$

where $(\beta_3)_k = \Gamma\left(\beta_3 + k\right)/\Gamma\left(\beta_3\right)$ is the Pochhammer symbol. When $\beta_3 = 1$, we recover the two-parameter ML function. Moreover, this special function admits the following asymptotic expansion along the negative semi-axes (see [16, Thm. 5.4])

$$E^{\beta_3}_{\beta_1,\beta_2}(-x) \sim \frac{x^{-\beta_3}}{\Gamma\left(\beta_3\right)} \sum_{k=0}^{+\infty} \frac{(-1)^k \, \Gamma\left(k + \beta_3\right)}{k! \, \Gamma\left(\beta_2 - \beta_1(k + \beta_3)\right)} x^{-k}, \quad x \to +\infty, \tag{27}$$

where $0 < \beta_1 < 2$. Moreover, we have the following result:

Lemma 2.2 (cf. [34, Lem. 3.2]). *Let $\beta_1, \beta_2 \in \mathbb{C}$ such that $\mathrm{Re}(\beta_1), \mathrm{Re}(\beta_2) > 0$, $\beta_3 \in \mathbb{R}$ such that $\beta_3 > 0$, $\tau \in \mathbb{R}^+$, and $\kappa \in \mathbb{R}^n$. The following Fourier-type relation is valid*

$$\mathcal{F}^{-1}\left\{ E^{\beta_3}_{\beta_1,\beta_2}\left(-\tau\|\kappa\|^2\right)\right\}(x) = \frac{1}{\pi^{\frac{n}{2}}\,\|x\|^n\,\Gamma\left(\beta_3\right)}\; H^{0,2}_{2,1}\left[\ \frac{4\tau}{\|x\|^2}\ \middle|\ \begin{array}{c} \left(1-\frac{n}{2},1\right),\,(1-\beta_3,1) \\ (1-\beta_2,\beta_1) \end{array}\ \right], \tag{28}$$

where H is the Fox H-function defined in (19).

The multivariate ML function $E_{(a_1,\dots,a_n),b}(z_1,\dots,z_n)$ of n complex variables $z_1,\dots,z_n \in \mathbb{C}$ with complex parameters $a_1,\dots,a_n,\lambda \in \mathbb{C}$ (with positive real parts) is defined by (see [28]):

$$E_{(a_1,\dots,a_n),\lambda}(z_1,\dots,z_n) = \sum_{k=0}^{+\infty}\ \sum_{\substack{l_1+\dots+l_n=k\\ l_1,\dots,l_n\geq 0}} \binom{k}{l_1,\dots,l_n} \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma\left(\lambda+\sum_{i=1}^n a_i l_i\right)}, \tag{29}$$

where the multinomial coefficients are given by

$$\binom{k}{l_1,\dots,l_n} := \frac{k!}{l_1!\times\dots\times l_n!}.$$

When $n = 2$ we obtain the bivariate ML function which can be written as

$$E_{(a_1,a_2),\lambda}(z_1,z_2) = \sum_{l_1=0}^{+\infty}\sum_{l_2=0}^{+\infty} \frac{(l_1+l_2)!}{l_1!\,l_2!}\, \frac{z_1^{l_1}\,z_2^{l_2}}{\Gamma\left(\lambda+a_1l_1+a_2l_2\right)}. \tag{30}$$

The following relation is immediate between the bivariate ML function and the two parameter ML function:

$$E_{(a_1,a_2),\lambda}(0,z_2) = E_{a_2,\lambda}(z_2). \tag{31}$$

The following lemmas will be useful in the sequel.

Lemma 2.3 (cf.[1]). *Let $a_1, a_2, \lambda \in \mathbb{C}$, with $\mathrm{Re}(a_1), \mathrm{Re}(a_2) > 0$, and $\omega_1, \omega_2 \in \mathbb{R}$. The Laplace transform of the bivariate ML function is given by*

$$\mathcal{L}\{t^{\lambda-1}\, E_{(a_1,a_2),\lambda}\left(\omega_1\, t^{a_1}, \omega_2\, t^{a_2}\right)\}(s) = s^{-\lambda}(1-\omega_1 s^{-a_1}-\omega_2 s^{-a_2})^{-1}.$$

Lemma 2.4 (see [1]). *Let $a_1, a_2, \lambda, \omega_1, \omega_2 \in \mathbb{R}$, with $\mathrm{Re}(a_1), \mathrm{Re}(a_2) > 0$. For $a_1 > a_2$, $a_1 > \lambda$, and $\mathrm{Re}(s) > 0$, the following results holds:*

$$\mathcal{L}^{-1}\left\{ \frac{s^\lambda}{s^{a_1}-\omega_2 s^{a_2}-\omega_1}\right\}(t) = t^{a_1-\lambda-1}\, E_{(a_1,a_1-a_2),a_1-\lambda}\left(\omega_1\, t^{a_1}, \omega_2\, t^{a_1-a_2}\right). \tag{32}$$

Lemma 2.5 (see [14]). *Let $a_1, a_2, \lambda, z_1, z_2 \in \mathbb{C}$, with $\mathrm{Re}(a_1), \mathrm{Re}(a_2) > 0$. The following addition formula for the bivariate ML-function holds:*

$$E_{(a_1,a_2),\lambda}(z_1,z_2) = \frac{1}{\Gamma(\lambda)} + z_1\, E_{(a_1,a_2),\lambda+a_1}(z_1,z_2) + z_2\, E_{(a_1,a_2),\lambda+a_2}(z_1,z_2). \tag{33}$$

We now provide integration and fractional derivation rules involving the univariate version of the bivariate ML-function. For $\lambda > 0$, the following fractional integration formula is valid:

$$\begin{aligned} &I^{\gamma}_{a^+}\left[(t-a)^{\lambda-1} E_{(a_1,a_2),\lambda}\left(\omega_1\,(t-a)^{a_1}, \omega_2\,(t-a)^{a_2}\right)\right] \\ &\qquad = (t-a)^{\lambda+\gamma-1} E_{(a_1,a_2),\lambda+\gamma}\left(\omega_1\,(t-a)^{a_1}, \omega_2\,(t-a)^{a_2}\right). \end{aligned} \tag{34}$$

By applying the definition of the Hilfer derivative (6) and the power function differentiation rule (9), we have the following derivative rule:

$$\begin{aligned} &{}^H D^{\gamma,\mu}_{a^+}\left[(t-a)^{\lambda-1} E_{(a_1,a_2),\lambda}\left(\omega_1\,(t-a)^{a_1}, \omega_2\,(t-a)^{a_2}\right)\right] \\ &\qquad = (t-a)^{\lambda-1-\gamma} E_{(a_1,a_2),\lambda-\gamma}\left(\omega_1\,(t-a)^{a_1}, \omega_2\,(t-a)^{a_2}\right), \end{aligned} \tag{35}$$

which is valid for $\lambda - 1 + (1-\mu)(m-\gamma) > \lfloor\gamma\rfloor$, or equivalently, $\lambda + (1-\mu)(m-\gamma) > m$. Since $0 \leq (1-\mu)(m-\gamma) < 1$, because $0 \leq \mu \leq 1$ and $0 \leq m-\gamma \leq 1$) the derivative rule (35) can be applied for $\lambda > m$. For $\lambda - 1 + (1-\mu)(m-\gamma) = p$, with $p = 0,\dots,m-1$, we have the following lemma:

Lemma 2.6. *Let $a_1, a_2, z_1, z_2 \in \mathbb{C}$, with $\mathrm{Re}(a_1), \mathrm{Re}(a_2) > 0$, $\gamma > 0$, and $\mu \in [0,1]$. Then the following derivative rule holds:*

$$\begin{aligned} &{}^H D^{\gamma,\mu}_{a^+}\left[(t-a)^{(1-\mu)(\gamma-m)-1+p} E_{(a_1,a_2),(1-\mu)(\gamma-m)+p}\left(\omega_1\,(t-a)^{a_1}, \omega_2\,(t-a)^{a_2}\right)\right] \\ &\quad = \omega_1\,(t-a)^{(1-\mu)(\gamma-m)+p+a_1-\gamma} E_{(a_1,a_2),(1-\mu)(\gamma-m)+p+a_1-\gamma+1}\left(\omega_1\,(t-a)^{a_1}, \omega_2\,(t-a)^{a_2}\right) \\ &\quad + \omega_2\,(t-a)^{(1-\mu)(\gamma-m)+p+a_2-\gamma} E_{(a_1,a_2),(1-\mu)(\gamma-m)+p+a_2-\gamma+1}\left(\omega_1\,(t-a)^{a_1}, \omega_2\,(t-a)^{a_2}\right). \end{aligned} \tag{36}$$

Proof. By (9), we have ${}^{H}D^{\gamma,\mu}_{a^+}\left[(t-a)^{\lambda-1}\right]=0$ for $\lambda-1-(1-\mu)(m-\gamma)=p$ with $p=0,\ldots,m-1$. Consequently, the first term of the series expansion of $(t-a)^{\lambda-1}E_{(a_1,a_2),\lambda}\left(\omega_1\,(t-a)^{a_1},\omega_2\,(t-a)^{a_2}\right)$ vanishes, requiring a readjustment of the series indices. Since we are dealing with double series, where both indices range from zero to infinity, this adjustment is not straightforward due to the need to account for the interplay between the two indices. Therefore, we adopt a different approach. Using the addition formula (33), we have:

$$\begin{aligned}&(t-a)^{\lambda-1}E_{(a_1,a_2),\lambda}\left(\omega_1\,(t-a)^{a_1},\omega_2\,(t-a)^{a_2}\right)\\&=\frac{(t-a)^{\lambda-1}}{\Gamma(\lambda)}+\omega_1(t-a)^{\lambda-1+a_1}\,E_{(a_1,a_2),\lambda+a_1}(\omega_1(t-a)^{a_1},\omega_2(t-a)^{a_2})\\&+\omega_2(t-a)^{\lambda-1+a_2}\,E_{(a_1,a_2),\lambda+a_2}(\omega_1(t-a)^{a_1},\omega_2(t-a)^{a_2}).\end{aligned}\tag{37}$$

Applying the Hilfer fractional derivative ${}^{H}D^{\gamma,\mu}_{a^+}$ to the right-hand side of (37), the first term vanishes because it is a constant term. For the remaining terms, we apply the derivative rule (35), which directly yields the desired result. □

Let us now pay attention to the asymptotic behaviour of some functions involving the bivariate ML function. The first terms in the power series (30) give the following asymptotic expansion with $\omega_1,\omega_2\in\mathbb{R}^+$ (cf. Expression (2.4) in [10] for multivariate ML functions):

$$(t-a)^{\lambda-1}\,E_{(a1,a2),\lambda}\left(-\omega_1(t-a)^{a_1},-\omega_2(t-a)^{a2}\right)\sim\frac{(t-a)^{\lambda-1}}{\Gamma(\lambda)},\qquad t\to a^+.\tag{38}$$

From (27), the leading term as $t\to+\infty$ is obtained as follows (cf. expression (3.4) in [10] for multivariate ML functions)

$$(t-a)^{\lambda-1}\,E_{(a1,a2),\lambda}\left(-\omega_1(t-a)^{a_1},-\omega_2(t-a)^{a2}\right)\sim\begin{cases}\omega_1^{-1}\,\dfrac{(t-a)^{\lambda-a_1-1}}{\Gamma\left(\lambda-a_1\right)}, & a_1\neq\lambda\\ -\omega_1\,\omega_2\,\dfrac{(t-a)^{-a_1+a_2-1}}{\Gamma\left(-a_1+a_2\right)}, & a_1=\lambda\end{cases},\qquad t\to+\infty.\tag{39}$$

3. Novel Fourier pairs

In this section, we prove some inverse Fourier relations involving bivariate ML functions (30) in the Fourier-time domain (frequency domain) and Fox H-functions of two variables (20) in the space-time domain. These results allow us to obtain the solution of the time-fractional diffusion Dirac-type equations in the space-time domain using the Fourier method. The new Fourier pairs that we prove are also important for the study of problems involving the Dirac or Laplace operators combined with two time-fractional derivatives, which arise in various fields such as quantum mechanics and viscoelasticity. The first lemma is a generalization of the one-dimensional Fourier-type relation in Lemma 2.2 with $\beta_3=1$.

Lemma 3.1. *Let $a_1,a_2,\lambda\in\mathbb{C}$ such that $0<\mathrm{Re}(a_1)<\mathrm{Re}(a_2)$, $c_1,c_2\in\mathbb{R}^+$, $a,t\in\mathbb{R}$ with $t>a$, $\kappa\in\mathbb{R}^n$. The following inverse Fourier relation is valid*

$$\begin{aligned}&\mathcal{F}^{-1}\left\{E_{(a_1,a_2),\lambda}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2,-c_2\,(t-a)^{a_2}\right)\right\}(x,t)\\&=\frac{1}{\pi^{\frac{n}{2}}\,\|x\|^n}\,H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c|c}\frac{4\,c_1}{\|x\|^2}\,(t-a)^{a_1} & (0;1,1);\ \left(1-\frac{n}{2},1\right);\ ---\\ c_2\,(t-a)^{a_2} & (1-\lambda;a_1,a_2);\ ---;\ (0,1)\end{array}\right],\end{aligned}\tag{40}$$

where --- denotes empty parameters.

Proof. Taking into account the definitions of the bivariate and the three-parameter ML functions given in (30) and (26), respectively, we have the following relation between these two functions

$$\begin{aligned}&E_{(a_1,a_2),\lambda}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2,-c_2\,(t-a)^{a_2}\right)\\&=\sum_{p=0}^{+\infty}\sum_{q=0}^{\infty}\frac{(p+q)!}{p!\,q!\,\Gamma\left(\lambda+a_1q+a_2p\right)}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2\right)^q\left(-c_2\,(t-a)^{a_2}\right)^p\\&=\sum_{p=0}^{+\infty}\left(-c_2\,(t-a)^{a_2}\right)^p\sum_{q=0}^{\infty}\frac{(p+1)_q}{q!\,\Gamma\left(\lambda+a_1q+a_2p\right)}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2\right)^q\\&=\sum_{p=0}^{+\infty}\left(-c_2\,(t-a)^{a_2}\right)^p\,E^{p+1}_{a_1,\lambda+a_2p}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2\right).\end{aligned}\tag{41}$$

From the asymptotic expansion presented in (27), we can clearly observe that the three-parameter ML function that appears in (41) belongs to $L_1\left(\mathbb{R}^n\right)$. Hence, applying the inverse Fourier transform and taking into account (28), we obtain

$$\begin{aligned}&\mathcal{F}^{-1}\left\{E_{(a_1,a_2),\lambda}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2,-c_2\,(t-a)^{a_2}\right)\right\}(x,t)\\&=\sum_{p=0}^{+\infty}\left(-c_2\,(t-a)^{a_2}\right)^p\,\mathcal{F}^{-1}\left\{E^{p+1}_{a_1,\lambda+a_2p}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2\right)\right\}(x,t)\end{aligned}$$

$$= \sum_{p=0}^{+\infty} \frac{\left(-c_2 (t-a)^{a_2} \right)^p}{\pi^{\frac{n}{2}} \|x\|^n \, p!} H_{2,1}^{0,2}\left[\frac{4\,c_1}{\|x\|^2} (t-a)^{a_1} \,\middle|\, \begin{matrix} \left(1-\frac{n}{2},1\right), (-p,1) \\ (1-\lambda-a_2 p, a_1) \end{matrix} \right]. \tag{42}$$

The expression (42) can be rewritten in terms of a Fox H-function of two variables, thus avoiding the appearance of the series of Fox H-functions of one variable. In fact, from the definition (19) and the residues of the Gamma function $\operatorname{Res}(\Gamma,-p) = (-1)^p/p!$ for $p = 0, 1, \ldots$, we can write

$$\mathcal{F}^{-1}\left\{ E_{(a_1,a_2),\lambda}\left(-c_1 (t-a)^{a_1} \|\kappa\|^2, -c_2 (t-a)^{a_2} \right) \right\}(x,t)$$
$$= \frac{1}{\pi^{\frac{n}{2}} \|x\|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_2} \int_{\mathcal{L}_1} \frac{\Gamma(1-s-w)\Gamma\left(\frac{n}{2}-s\right)\Gamma(w)}{\Gamma(\lambda - a_1 s - a_2 w)} \left(\frac{4\,c_1}{\|x\|^2} (t-a)^{a_1} \right)^{-s} \left(c_2 (t-a)^{a_2} \right)^{-w} ds\, dw. \tag{43}$$

Finally, considering the change of variables $-s \mapsto s$ and $-w \mapsto w$ and definition (20), we obtain

$$\mathcal{F}^{-1}\left\{ E_{(a_1,a_2),\lambda}\left(-c_1 (t-a)^{a_1} \|\kappa\|^2, -c_2 (t-a)^{a_2} \right) \right\}(x,t)$$
$$= \frac{1}{\pi^{\frac{n}{2}} \|x\|^n} H_{1,1;1,0;0,1}^{0,1;0,1;1,0}\left[\begin{matrix} \frac{4c_1}{\|x\|^2} (t-a)^{a_1} \\ c_2 (t-a)^{a_2} \end{matrix} \,\middle|\, \begin{matrix} (0;1,1);\ \left(1-\frac{n}{2},1\right);\ --- \\ (1-\lambda; a_1, a_2);\ ---;\ (0,1) \end{matrix} \right].$$

□

Next, we give the Fourier pair of the bivariate ML function of Lemma 3.1 multiplied with $-\|\kappa\|^2$, the Fourier symbol of the Laplace operator $\Delta_x = \sum_{i=1}^n \partial_{x_i}^2$.

Lemma 3.2. *Let $a_1, a_2, \lambda \in \mathbb{C}$ such that $0 < \operatorname{Re}(a_1) < \operatorname{Re}(a_2)$, $c_1, c_2 \in \mathbb{R}^+$, $a, t \in \mathbb{R}$ with $t > a$, $\kappa \in \mathbb{R}^n$. The following inverse Fourier relation is valid*

$$\mathcal{F}^{-1}\left\{ -\|\kappa\|^2\, E_{(a_1,a_2),\lambda}\left(-c_1 (t-a)^{a_1} \|\kappa\|^2, -c_2 (t-a)^{a_2} \right) \right\}(x,t)$$
$$= \frac{4}{\pi^{\frac{n}{2}} \|x\|^{n+2}} H_{1,1;2,1;0,1}^{0,1;0,2;1,0}\left[\begin{matrix} \frac{4c_1}{\|x\|^2} (t-a)^{a_1} \\ c_2 (t-a)^{a_2} \end{matrix} \,\middle|\, \begin{matrix} (0;1,1);\ \left(-\frac{n}{2},1\right), (-1,1);\ --- \\ (1-\lambda; a_1, a_2);\ (0,1);\ (0,1) \end{matrix} \right]. \tag{44}$$

Proof. Using (43) with the substitutions $-w \mapsto w$ and $-s \mapsto s$, we have

$$\mathcal{F}^{-1}\left\{ -\|\kappa\|^2\, E_{(a_1,a_2),\lambda}\left(-c_1 (t-a)^{a_1} \|\kappa\|^2, -c_2 (t-a)^{a_2} \right) \right\}(x,t)$$
$$= \Delta_x \left(\mathcal{F}^{-1}\left\{ E_{(a_1,a_2),\lambda}\left(-c_1 (t-a)^{a_1} \|\kappa\|^2, -c_2 (t-a)^{a_2} \right) \right\}(x,t) \right)$$
$$= \frac{1}{\pi^{\frac{n}{2}}} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_2} \int_{\mathcal{L}_1} \frac{\Gamma(1+s+w)\Gamma\left(\frac{n}{2}+s\right)\Gamma(-w)}{\Gamma\left(\lambda + a_1 s + a_2 w\right)} \left(4\,c_1 (t-a)^{a_1} \right)^s \left(c_2 (t-a)^{a_2} \right)^w \Delta_x\left(\|x\|^{-2s-n} \right) ds\, dw.$$

Since

$$\Delta_x\left(\|x\|^{-2s-n} \right) = (2s+n)(2s+2)\|x\|^{-2s-n-2}$$
$$= \frac{4}{\|x\|^{n+2}} \left(\frac{n}{2} + s \right) \frac{\Gamma(2+s)}{\Gamma(1+s)} \frac{1}{\|x\|^{2s}} \tag{45}$$

where we used the fact that $\Delta_x\left(\|x\|^p \right) = p(p+n-2)\|x\|^{p-2}$, then we obtain

$$\mathcal{F}^{-1}\left\{ -\|\kappa\|^2\, E_{(a_1,a_2),\lambda}\left(-c_1 (t-a)^{a_1} \|\kappa\|^2, -c_2 (t-a)^{a_2} \right) \right\}(x,t)$$
$$= \frac{4}{\pi^{\frac{n}{2}} \|x\|^{n+2}} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_2} \int_{\mathcal{L}_1} \frac{\Gamma(1+s+w)\Gamma\left(1+\frac{n}{2}+s\right)\Gamma(2+s)\Gamma(-w)}{\Gamma\left(\lambda + a_1 s + a_2 w\right)\Gamma(1+s)} \left(\frac{4\,c_1}{\|x\|^2} (t-a)^{a_1} \right)^s \left(c_2 (t-a)^{a_2} \right)^w ds\, dw$$
$$= \frac{4}{\pi^{\frac{n}{2}} \|x\|^{n+2}} H_{1,1;2,1;0,1}^{0,1;0,2;1,0}\left[\begin{matrix} \frac{4c_1}{\|x\|^2} (t-a)^{a_1} \\ c_2 (t-a)^{a_2} \end{matrix} \,\middle|\, \begin{matrix} (0;1,1);\ \left(-\frac{n}{2},1\right), (-1,1);\ --- \\ (1-\lambda; a_1, a_2);\ (0,1);\ (0,1) \end{matrix} \right].$$

□

The next lemma provides the Fourier pair of the bivariate ML function of Lemma 3.1 multiplied with the vector-valued symbol $i\kappa$, i.e., the Fourier symbol of the Dirac operator. It is defined by $\partial_x = \sum_{i=1}^n \partial_{x_i} e_i$ where $e_1, \cdots, e_n$ is an orthonormal basis in $\mathbb{R}^n$ satisfying the Clifford algebra multiplication relations:

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

where δ_{ij} represents the Kronecker's delta. In particular, $e_i^2 = -1$ for all $i = 1, \ldots, n$ and $e_i e_j = -e_j e_i$, $i \neq j$. Hence, $\partial_x^2 = -\Delta_x$, which can be verified by direct computation using the multiplication relations, meaning that the Dirac operator factorizes the Laplace operator. The Fourier symbol of the Dirac operator is given by (see [5])

$$\mathcal{F}\left\{ \partial_x f(x) \right\}(\kappa) = i\,\kappa\, \mathcal{F}\{ f(x) \}(\kappa). \tag{46}$$

Lemma 3.3. *Let $a_1, a_2, \lambda \in \mathbb{C}$ such that $0 < \mathrm{Re}(a_1) < \mathrm{Re}(a_2)$, $c_1, c_2 \in \mathbb{R}^+$, $a, t \in \mathbb{R}$ with $t > a$, $\kappa \in \mathbb{R}^n$. The following inverse Fourier relation is valid*

$$\mathcal{F}^{-1}\left\{ i\,\kappa\, E_{(a_1,a_2),\lambda}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2, -c_2\,(t-a)^{a_2} \right)\right\}(x,t)$$
$$= \frac{-2x}{\pi^{\frac{n}{2}}\,\|x\|^{n+2}}\, H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c} \frac{4\,c_1}{\|x\|^2}\,(t-a)^{a_1} \\ c_2\,(t-a)^{a_2} \end{array} \middle| \begin{array}{c} (0;1,1);\ \left(-\frac{n}{2},1\right);\ ---\\ (1-\lambda;a_1,a_2);\ ---;\ (0,1) \end{array} \right]. \tag{47}$$

Proof. Using (46) and (43) with the substitutions $-w \mapsto w$ and $-s \mapsto s$, we have

$$\mathcal{F}^{-1}\left\{ i\,\kappa\, E_{(a_1,a_2),\lambda}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2, -c_2\,(t-a)^{a_2} \right)\right\}(x,t)$$
$$= \partial_x\left(\mathcal{F}^{-1}\left\{ E_{(a_1,a_2),\lambda}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2, -c_2\,(t-a)^{a_2} \right)\right\}(x,t)\right)$$
$$= \frac{1}{\pi^{\frac{n}{2}}}\,\frac{1}{(2\pi i)^2} \int_{\mathcal{L}_2}\int_{\mathcal{L}_1} \frac{\Gamma(1+s+w)\,\Gamma\left(\frac{n}{2}+s\right)\Gamma(-w)}{\Gamma\left(\lambda+a_1 s+a_2 w\right)} \left(4\,c_1\,(t-a)^{a_1}\right)^s \left(c_2\,(t-a)^{a_2}\right)^w \partial_x\left(\|x\|^{-2s-n}\right) ds\,dw.$$

Since

$$\partial_x\left(\|x\|^{-2s-n}\right) = \frac{-2x}{\|x\|^{n+2}}\left(\frac{n}{2}+s\right)\frac{1}{\|x\|^{2s}},$$

where we used the fact that $\partial_x\left(\|x\|^p\right) = px\,\|x\|^{p-2}$, we obtain

$$\mathcal{F}^{-1}\left\{ i\,\kappa\, E_{(a_1,a_2),\lambda}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2, -c_2\,(t-a)^{a_2} \right)\right\}(x,t)$$
$$= \frac{-2x}{\pi^{\frac{n}{2}}\,\|x\|^{n+2}}\,\frac{1}{(2\pi i)^2} \int_{\mathcal{L}_2}\int_{\mathcal{L}_1} \frac{\Gamma(1+s+w)\,\Gamma\left(1+\frac{n}{2}+s\right)\Gamma(-w)}{\Gamma\left(\lambda+a_1 s+a_2 w\right)} \left(\frac{4\,c_1}{\|x\|^2}\,(t-a)^{a_1}\right)^s \left(c_2\,(t-a)^{a_2}\right)^w ds\,dw$$
$$= \frac{-2x}{\pi^{\frac{n}{2}}\,\|x\|^{n+2}}\, H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c} \frac{4\,c_1}{\|x\|^2}\,(t-a)^{a_1} \\ c_2\,(t-a)^{a_2} \end{array} \middle| \begin{array}{c} (0;1,1);\ \left(-\frac{n}{2},1\right);\ ---\\ (1-\lambda;a_1,a_2);\ ---;\ (0,1) \end{array} \right]. \tag{48}$$

□

Finally, multiplying the Fourier symbols of the Dirac and the Laplace operators with the bivariate ML function of Lemma 3.1 we obtain the following result.

Lemma 3.4. *Let $a_1, a_2, \lambda \in \mathbb{C}$ such that $0 < \mathrm{Re}(a_1) < \mathrm{Re}(a_2)$, $c_1, c_2 \in \mathbb{R}^+$, $a, t \in \mathbb{R}$ with $t > a$, $\kappa \in \mathbb{R}^n$. The following inverse Fourier relation is valid*

$$\mathcal{F}^{-1}\left\{ -i\,\kappa\|\kappa\|^2\, E_{(a_1,a_2),\lambda}\left(-c_1\,(t-a)^{a_1}\,\|\kappa\|^2, -c_2\,(t-a)^{a_2} \right)\right\}(x,t)$$
$$= \frac{-8x}{\pi^{\frac{n}{2}}\,\|x\|^{n+4}}\, H^{0,1;0,2;1,0}_{1,1;2,1;0,1}\left[\begin{array}{c} \frac{4\,c_1}{\|x\|^2}\,(t-a)^{a_1} \\ c_2\,(t-a)^{a_2} \end{array} \middle| \begin{array}{c} (0;1,1);\ \left(-1-\frac{n}{2},1\right),(-1,1);\ ---\\ (1-\lambda;a_1,a_2);\ (0,1);\ (0,1) \end{array} \right]. \tag{49}$$

Remark 3.5. *All the Fox H-functions of two variables that appear in the Fourier pairs are convergent for all $x \in \mathbb{R}^n$ and $t > a$ due to the convergence conditions in [9, Thm. 3.1].*

4. Time-fractional diffusion Dirac-type equations

In [34] we investigated the time-fractional telegraph equation with ψ-Hilfer derivatives, which generalize the classical Hilfer derivative by introducing a function $\psi(t)$ in the definition. We obtained a closed representation of its fundamental solution in terms of bivariate ML functions in the Fourier domain and convolution integrals involving Fox H-functions of two-variables in the space-time domain. This previous work provides a general framework for studying time-fractional telegraph equations, and in this paper, we concentrate on the particular case $\psi(t) = t$, which corresponds to the standard Hilfer derivative, namely

$${}^{H}\partial_t^{\alpha_2,\mu_2} u(x,t) + c_1^2\,{}^{H}\partial_t^{\alpha_1,\mu_1} u(x,t) - c_0^2\,\Delta_x u(x,t) = 0, \tag{50}$$

where $c_1 \geq 0$ represents a damping coefficient, $c_0 > 0$ represents a diffusion coefficient, $(x,t) \in \mathbb{R}^n \times I$ with $I = [0,b] \subset \mathbb{R}_0^+$, Δ_x is the Laplace operator in $\mathbb{R}^n$, and the partial time-fractional derivatives are Hilfer derivatives of orders $\alpha_1 \in]0,1]$ and $\alpha_2 \in]1,2]$, and types $\mu_1, \mu_2 \in [0,1]$, respectively. The orders α_1 and α_2 control the degree of non-locality in time, while the types μ_1 and μ_2 interpolate between the Riemann-Liouville and Caputo fractional derivatives.

When $c_1 \neq 0$, the telegraph Eq. (50) relates the wave equation and the diffusion equations due to the presence of the fractional derivatives of orders $\alpha_2 \in]1,2]$ and $\alpha_1 \in]0,1]$. Specifically, the second-order fractional derivative (α_2) contributes to the wave-like behavior, while the first-order fractional derivative (α_1) contributes to the diffusion-like behavior.

This fractional model interpolates between different stages of the classical telegraph equation, meaning it can model a range of behaviors from pure wave propagation to pure diffusion, depending on the values of the fractional orders and types. The classical telegraph equation can be interpreted as a non-dissipative wave equation with viscous and thermal damping/diffusion terms. For the particular case of $c_1 = 0$, the damping/diffusion term is eliminated, recovering the time-fractional wave equation. This simplification allows us to study pure wave propagation phenomena in a fractional time domain.

In [13] the Dirac factorization method, which uses Pauli's matrices to decompose a second-order differential operator into a product of two first-order operators was applied to the time-fractional diffusion equation. This method resulted in an uncoupled system of two time-fractional diffusion equations of Dirac type. In this spirit, we aim to extend the obtained results for the case of the time-fractional telegraph Eq. (50). As we will see, the application of the Dirac factorization method to (50) leads to a coupled system of two-term time-fractional diffusion equations of Dirac type. In contrast to uncoupled systems, where each equation can be solved independently, coupled systems require simultaneous solution due to the interdependence of the equations.

Pauli's matrices are a set of three 2×2 unitary, Hermitian, and involutory matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{51}$$

satisfying the relations

$$\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_3\sigma_1 = -i\sigma_2, \quad \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij} I_2, \tag{52}$$

where I_2 is the identity matrix of order 2 and δ_{ij} the Kronecker delta. The last relation indicates that the Pauli matrices are anti-commuting for $i \neq j$ and $\sigma_j^2 = I_2$, for $j = 1, 2, 3$. These properties are crucial for the factorization process. To factorize (50) we consider the following matrix fractional differential equation defined by three different Pauli's matrices:

$$\left[\sigma_j\, {}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + c_1\, \sigma_k\, {}^H\partial_t^{\frac{\alpha_1}{2},\mu_1} + c_0\, \sigma_l\, \partial_x\right] \Phi(x,t) = 0, \; j \neq k \neq l, \tag{53}$$

where $\Phi(x,t) = \left[\phi_1(x,t),\, \phi_2(x,t)\right]^T$ is a two component vector function, depending on x and t. By the multiplication rules (52) and assuming that the semigroup property of the fractional derivatives is fulfilled, we have the following Dirac factorization

$$\begin{aligned}
&\left[\sigma_j\, {}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + c_1\, \sigma_k\, {}^H\partial_t^{\frac{\alpha_1}{2},\mu_1} + c_0\, \sigma_l\, \partial_x\right]^2 \\
&= \sigma_j^2\, {}^H\partial_t^{\alpha_2,\mu_2} + c_1 \underbrace{\left(\sigma_j\sigma_k + \sigma_k\sigma_j\right)}_{=0} {}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\, {}^H\partial_t^{\frac{\alpha_1}{2},\mu_1} + c_0 \underbrace{\left(\sigma_j\sigma_l + \sigma_l\sigma_j\right)}_{=0} {}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\, \partial_x \\
&\quad + c_1^2\, \sigma_k^2\, {}^H\partial_t^{\alpha_1,\mu_1} + c_0\, c_1 \underbrace{\left(\sigma_k\sigma_l + \sigma_l\sigma_k\right)}_{=0} {}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\, \partial_x - c_0^2\, \sigma_l^2\, \Delta_x \\
&= \left({}^H\partial_t^{\alpha_2,\mu_2} + c_1^2\, {}^H\partial_t^{\alpha_1,\mu_1} - c_0^2\, \Delta_x\right) I_2.
\end{aligned}$$

Different choices of the triplets (j,k,l) lead to different systems, because each triplet determines a unique combination of Pauli matrices in the matrix fractional differential equation, resulting in different coupling systems. For the triplet $(j,k,l) = (3,1,2)$ we have the following coupled system of two-term time-fractional diffusion Dirac-type equations

$$\begin{cases}
{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\, \phi_1(x,t) + c_1\, {}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\, \phi_2(x,t) - i\, c_0\, \partial_x\, \phi_2(x,t) = 0 & (54) \\
-{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\, \phi_2(x,t) + c_1\, {}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\, \phi_1(x,t) + i\, c_0\, \partial_x\, \phi_1(x,t) = 0. & (55)
\end{cases}$$

This specific triplet is chosen because it simplifies to a previously studied system when $c_1 = 0$, allowing for comparisons and extensions. In fact, when $c_1 = 0$, the system (54)–(55) simplifies to a system of coupled time-fractional diffusion Dirac equations which was studied in [13].

Differently from [13] it is not possible to transform the coupled system (54)–(55) into a decoupled system by considering similarity matrices. In fact, in this case, it is not possible to find invertible matrices A and B such that $A\sigma_1 B = \sigma_3$, $A\sigma_2 B = \sigma_2$, and $A\sigma_3 B = \sigma_1$.

4.1. Classical solutions

We aim to find classical solutions of the system

$$\begin{cases}
{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\, \phi_1(x,t) + c_1\, {}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\, \phi_2(x,t) - i\, c_0\, \partial_x\, \phi_2(x,t) = f(x,t) & (56) \\
-{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\, \phi_2(x,t) + c_1\, {}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\, \phi_1(x,t) + i\, c_0\, \partial_x\, \phi_1(x,t) = g(x,t) & (57)
\end{cases}$$

subject to the following initial conditions

$$(i)\;\; I_t^{(1-\mu_2)\left(1-\frac{\alpha_2}{2}\right)}\, \phi_1(x,0^+) = h_{2,1}(x) \qquad (ii)\;\; I_t^{(1-\mu_2)\left(1-\frac{\alpha_2}{2}\right)}\, \phi_2(x,0^+) = h_{2,2}(x) \tag{58}$$

$$(iii)\;\; I_t^{(1-\mu_1)\left(1-\frac{\alpha_1}{2}\right)}\, \phi_1(x,0^+) = h_{1,1}(x) \qquad (iv)\;\; I_t^{(1-\mu_1)\left(1-\frac{\alpha_1}{2}\right)}\, \phi_2(x,0^+) = h_{1,2}(x). \tag{59}$$

Here we assume $\mu_1 \neq 1$ or $\mu_2 \neq 1$, otherwise the initial conditions (i) and (ii) and also (iii) and (iv) degenerate into just one initial condition. This corresponds to the case $\mu_1 = 1$ and $\mu_2 = 1$, that is, the system above is composed of Caputo fractional derivatives of orders α_2 and α_1. This special case will be analysed separately.

To find classical solutions for this system, we will assume that all functions involved are Fourier and Laplace transformable. By first applying the n-dimensional Fourier transform to the spatial variable, we obtain the system

$$\begin{cases} {}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\,\hat{\phi}_1(\kappa,t) + c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\hat{\phi}_2(\kappa,t) + c_0\,\kappa\,\hat{\phi}_2(\kappa,t) = \hat{f}(\kappa,t) \\ -{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\,\hat{\phi}_2(\kappa,t) + c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\hat{\phi}_1(\kappa,t) - c_0\,\kappa\,\hat{\phi}_1(\kappa,t) = \hat{g}(\kappa,t) \end{cases}$$

subject to

$$(i)\quad I_t^{(1-\mu_2)\left(1-\frac{\alpha_2}{2}\right)}\,\hat{\phi}_1(\kappa,0^+) = \hat{h}_{2,1}(\kappa) \qquad (ii)\quad I_t^{(1-\mu_2)\left(1-\frac{\alpha_2}{2}\right)}\,\hat{\phi}_2(\kappa,0^+) = \hat{h}_{2,2}(\kappa) \tag{60}$$

$$(iii)\quad I_t^{(1-\mu_1)\left(1-\frac{\alpha_1}{2}\right)}\,\hat{\phi}_1(\kappa,0^+) = \hat{h}_{1,1}(\kappa) \qquad (iv)\quad I_t^{(1-\mu_1)\left(1-\frac{\alpha_1}{2}\right)}\,\hat{\phi}_2(\kappa,0^+) = \hat{h}_{1,2}(\kappa). \tag{61}$$

Now, applying the Laplace transform with respect to the time-variable and using (18) with $m = 1$, we obtain the system:

$$\begin{cases} s^{\frac{\alpha_2}{2}}\,\tilde{\hat{\phi}}_1(\kappa,s) - s^{-\mu_2(1-\frac{\alpha_2}{2})}\,\hat{h}_{2,1}(\kappa) + c_1\left(s^{\frac{\alpha_1}{2}}\,\tilde{\hat{\phi}}_2(\kappa,s) - s^{-\mu_1(1-\frac{\alpha_1}{2})}\,\hat{h}_{1,2}(\kappa)\right) + c_0\,\kappa\,\tilde{\hat{\phi}}_2(\kappa,s) = \tilde{\hat{f}}(\kappa,s) & (62)\\ -\left(s^{\frac{\alpha_2}{2}}\,\tilde{\hat{\phi}}_2(\kappa,s) - s^{-\mu_2(1-\frac{\alpha_2}{2})}\,\hat{h}_{2,2}(\kappa)\right) + c_1\left(s^{\frac{\alpha_1}{2}}\,\tilde{\hat{\phi}}_1(\kappa,s) - s^{-\mu_1(1-\frac{\alpha_1}{2})}\,\hat{h}_{1,1}(\kappa)\right) - c_0\,\kappa\,\tilde{\hat{\phi}}_1(\kappa,s) = \tilde{\hat{g}}(\kappa,s). & (63) \end{cases}$$

Solving the equation (62) for $\tilde{\hat{\phi}}_1$ we get

$$\tilde{\hat{\phi}}_1(\kappa,s) = -(c_1\,s^{\frac{\alpha_1-\alpha_2}{2}} + c_0\,\kappa\,s^{-\frac{\alpha_2}{2}})\tilde{\hat{\phi}}_2(\kappa,s) + s^{-\mu_2(1-\frac{\alpha_2}{2})-\frac{\alpha_2}{2}}\,\hat{h}_{2,1}(\kappa) + s^{-\mu_1(1-\frac{\alpha_1}{2})-\frac{\alpha_2}{2}}\,\hat{h}_{1,2}(\kappa)s^{-\frac{\alpha_2}{2}}\,\tilde{\hat{f}}(\kappa,s). \tag{64}$$

Substituting Eq. (64) in (63), and after performing the necessary simplifications, we obtain the following expression for $\tilde{\hat{\phi}}_2$:

$$\begin{aligned} \tilde{\hat{\phi}}_2(\kappa,s) = \frac{1}{S}\Big[&\left(c_0\,\kappa\,s^{-\mu_2(1-\frac{\alpha_2}{2})-\frac{\alpha_2}{2}} - c_1\,s^{-\mu_2(1-\frac{\alpha_2}{2})-\frac{\alpha_2-\alpha_1}{2}}\right)\hat{h}_{2,1}(\kappa) - s^{-\mu_2(1-\frac{\alpha_2}{2})}\,\hat{h}_{2,2}(\kappa) \\ &+ c_1\,s^{-\mu_1(1-\frac{\alpha_1}{2})}\,\hat{h}_{1,1}(\kappa) + \left(c_0 c_1\,\kappa\,s^{-\mu_1(1-\frac{\alpha_1}{2})-\frac{\alpha_2}{2}} - c_1^2\,s^{-\mu_1(1-\frac{\alpha_1}{2})-\frac{\alpha_2-\alpha_1}{2}}\right)\hat{h}_{1,2}(\kappa) \\ &- \left(c_1\,s^{-\frac{\alpha_2-\alpha_1}{2}} - c_0\,\kappa\,s^{-\frac{\alpha_2}{2}}\right)\tilde{\hat{f}}(\kappa,s) + \tilde{\hat{g}}(\kappa,s)\Big], \end{aligned} \tag{65}$$

with

$$S := -\left(s^{\frac{\alpha_2}{2}} + c_1^2\,s^{\alpha_1-\frac{\alpha_2}{2}} + c_0^2\,\|\kappa\|^2\,s^{-\frac{\alpha_2}{2}}\right). \tag{66}$$

Substituting (65) in (64), and performing direct algebraic manipulations, we arrive at the following expression for $\tilde{\hat{\phi}}_1$:

$$\begin{aligned} \tilde{\hat{\phi}}_1(\kappa,s) = \frac{1}{S}\Big[&-s^{-\mu_2(1-\frac{\alpha_2}{2})}\,\hat{h}_{2,1}(\kappa) + \left(c_0\,\kappa\,s^{-\mu_2(1-\frac{\alpha_2}{2})-\frac{\alpha_2}{2}} + c_1\,s^{-\mu_2(1-\frac{\alpha_2}{2})-\frac{\alpha_2-\alpha_1}{2}}\right)\hat{h}_{2,2}(\kappa) \\ &+ \left(-c_0 c_1\,\kappa\,s^{-\mu_1(1-\frac{\alpha_1}{2})-\frac{\alpha_2}{2}} - c_1^2\,s^{-\mu_1(1-\frac{\alpha_1}{2})-\frac{\alpha_2-\alpha_1}{2}}\right)\hat{h}_{1,1}(\kappa) - c_1\,s^{-\mu_1(1-\frac{\alpha_1}{2})}\,\hat{h}_{1,2}(\kappa) \\ &- \tilde{\hat{f}}(\kappa,s) + \left(-c_1\,s^{-\frac{\alpha_2-\alpha_1}{2}} - c_0\,\kappa\,s^{-\frac{\alpha_2}{2}}\right)\tilde{\hat{g}}(\kappa,s)\Big]. \end{aligned} \tag{67}$$

Comparing the solutions (67) and (65) we immediately observe a structural symmetry between them, specifically a sign change and permutation of the functions, as follows:

$$c_1 \longleftrightarrow -c_1, \qquad \hat{h}_{2,1} \longleftrightarrow \hat{h}_{2,2}, \qquad \hat{h}_{1,2} \longleftrightarrow \hat{h}_{1,1} \qquad \tilde{\hat{g}} \longleftrightarrow -\tilde{\hat{f}}. \tag{68}$$

This symmetry can also be observed in the structure of equations (62) and (63), after multiplying the second equation by −1, revealing a sign and variable interchange between the two equations. To obtain the solution in the space-time domain we need to apply the inverse Laplace and the inverse Fourier transforms. Since

$$\frac{1}{S} = -\frac{s^{\frac{\alpha_2}{2}}}{s^{\alpha_2} + c_1^2\,s^{\alpha_1} + c_0^2\,\|\kappa\|^2}$$

then by applying Lemma 2.4 and the Laplace convolution formula (17), we immediately get the following solution for $\hat{\phi}_2$:

$$\begin{aligned} \hat{\phi}_2(\kappa,t) = &\left(\sum_{j=0}^{1}(-c_0)^{1-j}\,c_1^j\,\kappa^{1-j}\,t^{\mu_2(1-\frac{\alpha_2}{2})+\alpha_2-j\frac{\alpha_1}{2}-1}\,E_{(\alpha_2,\alpha_2-\alpha_1),\mu_2(1-\frac{\alpha_2}{2})+\alpha_2-j\frac{\alpha_1}{2}}(z_1,z_2)\right)\hat{h}_{2,1}(\kappa) \\ &+ t^{\mu_2(1-\frac{\alpha_2}{2})+\frac{\alpha_2}{2}-1}\,E_{(\alpha_2,\alpha_2-\alpha_1),\mu_2(1-\frac{\alpha_2}{2})+\frac{\alpha_2}{2}}(z_1,z_2)\,\hat{h}_{2,2}(\kappa) \\ &- c_1\,t^{\mu_1(1-\frac{\alpha_1}{2})+\frac{\alpha_2}{2}-1}\,E_{(\alpha_2,\alpha_2-\alpha_1),\mu_1(1-\frac{\alpha_1}{2})+\frac{\alpha_2}{2}}(z_1,z_2)\,\hat{h}_{1,1}(\kappa) \end{aligned}$$

$$+\left(\sum_{j=0}^{1}(-c_0)^{1-j}\,c_1^{1+j}\,\kappa^{1-j}\,t^{\mu_1(1-\frac{\alpha_1}{2})+\alpha_2-j\frac{\alpha_1}{2}-1}\,E_{(\alpha_2,\alpha_2-\alpha_1),\mu_1(1-\frac{\alpha_1}{2})+\alpha_2-j\frac{\alpha_1}{2}}(z_1,z_2)\right)\widehat{h}_{1,2}(\kappa), \tag{69}$$

$$+\left(\sum_{j=0}^{1}(-c_0)^{1-j}\,c_1^{j}\,\kappa^{1-j}\,t^{\alpha_2-j\frac{\alpha_1}{2}-1}\,E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2-j\frac{\alpha_1}{2}}(z_1,z_2)\right)*_t\,\widehat{f}(\kappa,t)$$

$$-\,t^{\frac{\alpha_2}{2}-1}\,E_{(\alpha_2,\alpha_2-\alpha_1),\frac{\alpha_2}{2}}(z_1,z_2)*_t\,\widehat{g}(\kappa,t) \tag{70}$$

with $z_1=-c_0^2\,\|\kappa\|^2\,t^{\alpha_2}$ and $z_2=-c_1^2\,t^{\alpha_2-\alpha_1}$, and the convolution $*_t$ is defined by (15). Similarly, we can obtain the expression for $\widehat{\phi}_1$ considering the symmetries described in (68).

To obtain the solutions in the space-time domain we need to apply the results presented in Section 3 containing the specific Fourier pairs deduced. We will do this later, in Section 4.5.

4.2. The Caputo case

As we have already pointed out, when $\mu_1=1$ and $\mu_2=1$ the initial conditions (i) and (ii), and also (iii) and (iv) will degenerate into just a single initial condition. Therefore, the Caputo system

$$\begin{cases} {}^{C}\partial_t^{\frac{\alpha_2}{2}}\,\mu_2\,\phi_1(x,t)+c_1\,{}^{C}\partial_t^{\frac{\alpha_1}{2}}\,\mu_1\,\phi_2(x,t)-i\,c_0\,\partial_x\,\phi_2(x,t)=f(x,t)\\ -{}^{C}\partial_t^{\frac{\alpha_2}{2}}\,\mu_2\,\phi_2(x,t)+c_1\,{}^{C}\partial_t^{\frac{\alpha_1}{2}}\,\mu_1\,\phi_1(x,t)+i\,c_0\,\partial_x\,\phi_1(x,t)=g(x,t)\end{cases} \tag{71}$$

is subject to the initial conditions

$$(v)\;\;\phi_1(x,0^+)=h_1(x)\qquad\text{and}\qquad(vi)\;\;\phi_2(x,0^+)=h_2(x). \tag{72}$$

In (58)–(59) this means that

$$\widehat{h}_{2,1}(\kappa)=\widehat{h}_{1,1}(\kappa)=\widehat{h}_1(\kappa)\qquad\text{and}\qquad\widehat{h}_{2,2}(\kappa)=\widehat{h}_{1,2}(\kappa)=\widehat{h}_2(\kappa). \tag{73}$$

Therefore, from (70), the solution for $\widehat{\phi}_2$ is given by

$$\widehat{\phi}_2(\kappa,t)=\left(E_{(\alpha_2,\alpha_2-\alpha_1),1}(z_1,z_2)+\sum_{j=0}^{1}(-c_0)^{1-j}\,c_1^{1+j}\,\kappa^{1-j}\,t^{\alpha_2-(1+j)\frac{\alpha_1}{2}}\,E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2-(1+j)\frac{\alpha_1}{2}+1}(z_1,z_2)\right)\widehat{h}_2(\kappa)$$

$$-\,c_0\,\kappa\,t^{\frac{\alpha_2}{2}}\,E_{(\alpha_2,\alpha_2-\alpha_1),\frac{\alpha_2}{2}+1}(z_1,z_2)\,\widehat{h}_1(\kappa)$$

$$+\left(\sum_{j=0}^{1}(-c_0)^{1-j}\,c_1^{j}\,\kappa^{1-j}\,t^{\alpha_2-j\frac{\alpha_1}{2}-1}\,E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2-j\frac{\alpha_1}{2}}(z_1,z_2)\right)*_t\,\widehat{f}(\kappa,t)$$

$$-\,t^{\frac{\alpha_2}{2}-1}\,E_{(\alpha_2,\alpha_2-\alpha_1),\frac{\alpha_2}{2}}(z_1,z_2)*_t\,\widehat{g}(\kappa,t) \tag{74}$$

with $z_1=-c_0^2\,\|\kappa\|^2\,t^{\alpha_2}$ and $z_2=-c_1^2\,t^{\alpha_2-\alpha_1}$, and the convolution $*_t$ is defined by (15). We can obtain the expression for $\widehat{\phi}_1$ by considering the following symmetries:

$$c_1\longleftrightarrow -c_1,\qquad\qquad \widehat{h}_1\longleftrightarrow\widehat{h}_2,\qquad\qquad \widetilde{\widehat{g}}\longleftrightarrow-\widetilde{\widehat{f}}. \tag{75}$$

Remark 4.1. *If we consider $c_1=0$ in (71) we obtain a system of time-fractional diffusion equations of Dirac type, which was already studied in [13]. Considering the homogeneous case $f=g\equiv 0$ we obtain the solutions*

$$\widehat{\phi}_1(k,t)=-c_0\,\kappa\,t^{\frac{\alpha_2}{2}}\,E_{\alpha_2,\frac{\alpha_2}{2}+1}(z_1)\,\widehat{h}_2(k)\,+\,E_{\alpha_2}(z_1)\,\widehat{h}_1(k) \tag{76}$$

and

$$\widehat{\phi}_2(k,t)=-c_0\,\kappa\,t^{\frac{\alpha_2}{2}}\,E_{\alpha_2,\frac{\alpha_2}{2}+1}(z_1)\,\widehat{h}_1(k)\,+\,E_{\alpha_2}(z_1)\,\widehat{h}_2(k) \tag{77}$$

with $z_1=-c_0^2\|\kappa\|^2t^{\alpha_2}$. Finally, considering $\widehat{h}_1(k)=\widehat{h}_2(k)=1$, we obtain the solutions deduced in [13]. This shows consistency of our results with previous results obtained.

4.3. Asymptotic behaviours of the solutions in the Fourier-time domain

In this section, we analyse the asymptotic behaviour of $\widehat{\phi}_2$ and $\widehat{\phi}_1$ in both variables. Analyzing the asymptotic behavior of the solutions is important to understand their long-term behavior and to verify the stability of the system. In what follows, the notation $\sim$ means that $f(x)\sim g(x)$ as $x\to x_0$ if $f(x)=g(x)+o(g(x))$. Here, $o(g(x))$ represents a function that goes to zero faster than $g(x)$ as x approaches x_0. If the ration f/g is defined this implies that $\lim_{x\to x_0} f(x)/g(x)=1$.

4.3.1. Asymptotic behaviour for large and small values of t

First, we study the asymptotic behaviour of $\hat{\phi}_2$ as t tends to infinity, assuming $c_1 \neq 0$, and the homogeneous case $\hat{f}(\kappa,t) = \hat{g}(\kappa,t) = 0$. Assuming that all the initial conditions (60) and (61) are nonzero, we apply (39) to each term of (70) to obtain:

$$\begin{aligned}\hat{\phi}_2(\kappa,t) \sim & \left(\frac{1}{c_0}\frac{\kappa}{\|\kappa\|^2}\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)\right)} - \frac{c_1}{c_0^2}\frac{1}{\|\kappa\|^2}\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)-\frac{\alpha_1}{2}-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)-\frac{\alpha_1}{2}\right)}\right)\hat{h}_{2,1}(\kappa) \\ & - \frac{1}{c_0^2}\frac{1}{\|\kappa\|^2}\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)-\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)-\frac{\alpha_2}{2}\right)}\hat{h}_{2,2}(\kappa) + \frac{c_1}{c_0^2}\frac{1}{\|\kappa\|^2}\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)-\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)-\frac{\alpha_2}{2}\right)}\hat{h}_{1,1}(\kappa) \\ & + \left(\frac{c_1}{c_0}\frac{\kappa}{\|\kappa\|^2}\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)\right)} - \frac{c_1^2}{c_0^2}\frac{1}{\|\kappa\|^2}\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)-\frac{\alpha_1}{2}-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)-\frac{\alpha_1}{2}\right)}\right)\hat{h}_{1,2}(\kappa)\end{aligned} \tag{78}$$

with $\|\kappa\| \neq 0$. By comparing terms, we obtain:

$$\hat{\phi}_2(\kappa,t) \sim \begin{cases} \frac{1}{c_0}\frac{\kappa}{\|\kappa\|^2}\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)\right)}\hat{h}_{2,1}(\kappa) & \text{if } 0 \le \mu_1 < \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} \\ \frac{c_1}{c_0}\frac{\kappa}{\|\kappa\|^2}\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)\right)}\hat{h}_{1,2}(\kappa) & \text{if } \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} < \mu_1 \le 1 \end{cases}. \tag{79}$$

When $\mu_1 = \frac{\mu_2(2-\alpha_2)}{2-\alpha_1}$, i.e. $\mu_1\left(1-\frac{\alpha_1}{2}\right) = \mu_2\left(1-\frac{\alpha_2}{2}\right)$, we get:

$$\hat{\phi}_2(\kappa,t) \sim \frac{1}{c_0}\frac{\kappa}{\|\kappa\|^2}\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)\right)}\left(c_1\,\hat{h}_{1,2}(\kappa) + \hat{h}_{2,1}(\kappa)\right). \tag{80}$$

Since the exponents of t in (79) and (80) are negative, we conclude that $\hat{\phi}_2$ decays to zero as $t \to +\infty$, in the three cases.

The preceding conclusions hold provided that $\hat{h}_{1,2}(\kappa)$ and $\hat{h}_{2,1}(\kappa)$ are nonzero. If any of these vanish the asymptotic behaviour of $\hat{\phi}_2$ differs. For instance, if $\hat{h}_{1,1}(\kappa) = \hat{h}_{1,2}(\kappa) = 0$, the dependence on μ_1 disappears, and we obtain:

$$\hat{\phi}_2(\kappa,t) \sim \frac{1}{c_0}\frac{\kappa}{\|\kappa\|^2}\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)\right)}\hat{h}_{2,1}(\kappa) \;\to\; 0, \qquad \text{as } t \to +\infty. \tag{81}$$

Similarly, if $\hat{h}_{2,1}(\kappa) = \hat{h}_{2,2}(\kappa) = 0$, the dependence on μ_2 disappears, and we obtain:

$$\hat{\phi}_2(\kappa,t) \sim \frac{c_1}{c_0}\frac{\kappa}{\|\kappa\|^2}\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)\right)}\hat{h}_{1,2}(\kappa) \;\to\; 0, \qquad \text{as } t \to +\infty. \tag{82}$$

Next, we consider the asymptotic behaviour of $\hat{\phi}_2$ as $t \to 0^+$. Assuming all initial conditions in (60) and (61) to be nonzero, we apply (38) to each term of (70) to obtain:

$$\begin{aligned}\hat{\phi}_2(\kappa,t) \sim & \left(-c_0\,\kappa\,\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2\right)} + c_1\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2-\frac{\alpha_1}{2}-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2-\frac{\alpha_1}{2}\right)}\right)\hat{h}_{2,1}(\kappa) \\ & + \frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}\right)}\hat{h}_{2,2}(\kappa) - c_1\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}\right)}\hat{h}_{1,1}(\kappa) \\ & + \left(-c_0\,c_1\kappa\,\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2\right)} + c_1^2\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2-\frac{\alpha_1}{2}-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2-\frac{\alpha_1}{2}\right)}\right)\hat{h}_{1,2}(\kappa).\end{aligned} \tag{83}$$

Comparing the exponents of t in each term, we find the leading terms, which are the terms with the smallest exponents:

$$\hat{\phi}_2(\kappa,t) \sim \begin{cases} -c_1\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}\right)}\hat{h}_{1,1}(\kappa) & \text{if } 0 \le \mu_1 < \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} \\ \frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}\right)}\hat{h}_{2,2}(\kappa) & \text{if } \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} < \mu_1 \le 1 \end{cases}. \tag{84}$$

When $\mu_1 = \frac{\mu_2(2-\alpha_2)}{2-\alpha_1}$, i.e. $\mu_1\left(1-\frac{\alpha_1}{2}\right) = \mu_2\left(1-\frac{\alpha_2}{2}\right)$, we obtain:

$$\hat{\phi}_2(\kappa,t) \sim \frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}\right)}\left(\hat{h}_{2,2}(\kappa) - c_1\,\hat{h}_{1,1}(\kappa)\right). \tag{85}$$

Hence, we conclude that $\hat{\phi}_2$ tends to infinity for small values of t, since the exponents of t in (84) and in (85) are negative. If any of the initial conditions $\hat{h}_{2,2}(\kappa)$ or $\hat{h}_{1,1}(\kappa)$ vanish, the previous conclusions are different. For instance, if $\hat{h}_{1,1}(\kappa) = \hat{h}_{1,2}(\kappa) = 0$, then the dependence on μ_1 disappears and we obtain:

$$\hat{\phi}_2(\kappa,t) \sim \frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}\right)}\hat{h}_{2,2}(\kappa) \to 0, \qquad \text{as } t \to 0^+. \tag{86}$$

Similarly, if $\hat{h}_{2,1}(\kappa) = \hat{h}_{2,2}(\kappa) = 0$, the dependence on μ_2 disappears, and we obtain:

$$\hat{\phi}_2(\kappa,t) \sim -c_1\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}\right)}\hat{h}_{1,1}(\kappa), \qquad \text{as } t \to 0^+. \tag{87}$$

In this case,

$$\hat{\phi}_2(\kappa,t) \to \begin{cases} \infty & \text{if } 0 \le \mu_1 < \frac{2-\alpha_2}{2-\alpha_1} \\ -\frac{c_1}{\Gamma\left(\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}\right)}\hat{h}_{1,1}(\kappa) & \text{if } \mu_1 = \frac{2-\alpha_2}{2-\alpha_1} \\ 0 & \text{if } \frac{2-\alpha_2}{2-\alpha_1} < \mu_1 \le 1 \end{cases}.$$

Based on the asymptotic behaviours of $\hat{\phi}_2$ as $t \to +\infty$ and $t \to 0^+$, we can deduce sufficient conditions for the L_p-integrability of $\hat{\phi}_2$ with respect to the time variable.

Theorem 4.2. *Let $\mu_1 \neq 1$ or $\mu_2 \neq 1$, $c_1 \neq 0$, and $\hat{f}(\kappa,t) = \hat{g}(\kappa,t) = 0$. Then:*

Case 1: $\hat{h}_{2,1}(\kappa)$, $\hat{h}_{2,2}(\kappa)$, $\hat{h}_{1,1}(\kappa)$, and $\hat{h}_{1,2}(\kappa)$ are all non-zero:

$$\text{if } \begin{cases} \frac{1}{1-\mu_2\left(1-\frac{\alpha_2}{2}\right)} < p < \frac{1}{1-\frac{\alpha_2}{2}-\mu_1\left(1-\frac{\alpha_1}{2}\right)} & \text{with } 0 \le \mu_1 < \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} \\ & \text{or} \\ \frac{1}{1-\mu_1\left(1-\frac{\alpha_1}{2}\right)} < p < \frac{1}{1-\frac{\alpha_2}{2}-\mu_2\left(1-\frac{\alpha_2}{2}\right)} & \text{with } \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} \le \mu_1 \le 1 \end{cases} \text{ then } \hat{\phi}_2(\cdot,t) \in L_p(\mathbb{R}^+).$$

Case 2: $\hat{h}_{1,1}(\kappa) = \hat{h}_{1,2}(\kappa) = 0$:

$$\text{if } \frac{1}{1-\mu_2\left(1-\frac{\alpha_2}{2}\right)} < p < \frac{1}{1-\frac{\alpha_2}{2}-\mu_2\left(1-\frac{\alpha_2}{2}\right)} \text{ then } \hat{\phi}_2(\cdot,t) \in L_p(\mathbb{R}^+).$$

Case 3: $\hat{h}_{2,1}(\kappa) = \hat{h}_{2,2}(\kappa) = 0$:

$$\text{if } \begin{cases} \frac{1}{1-\mu_1\left(1-\frac{\alpha_1}{2}\right)} < p < \frac{1}{1-\frac{\alpha_2}{2}-\mu_1\left(1-\frac{\alpha_1}{2}\right)} & \text{with } 0 \le \mu_1 < \frac{2-\alpha_2}{2-\alpha_1} \\ & \text{or} \\ p > \frac{1}{1-\mu_1\left(1-\frac{\alpha_1}{2}\right)} & \text{with } \frac{2-\alpha_2}{2-\alpha_1} \le \mu_1 \le 1 \end{cases} \text{ then } \hat{\phi}_2(\cdot,t) \in L_p(\mathbb{R}^+).$$

For the Caputo case, i.e., $\mu_1 = \mu_2 = 1$, applying (39) and (38) to the expression (74), and assuming $c_1 \neq 0$ and $\hat{f}(\kappa,t) = \hat{g}(\kappa,t) = 0$, we obtain the following asymptotic behaviours:

$$\hat{\phi}_2(\kappa,t) \sim \begin{cases} \frac{c_1}{c_0}\frac{\kappa}{\|\kappa\|^2}\frac{t^{-\frac{\alpha_1}{2}}}{\Gamma\left(1-\frac{\alpha_1}{2}\right)}\hat{h}_2(\kappa) & \text{if } \hat{h}_2(\kappa) \neq 0 \\ -\frac{1}{c_0}\frac{\kappa}{\|\kappa\|^2}\frac{t^{-\frac{\alpha_2}{2}}}{\Gamma\left(1-\frac{\alpha_2}{2}\right)}\hat{h}_1(\kappa) & \text{if } \hat{h}_2(\kappa) = 0 \end{cases}, \qquad t \to +\infty \tag{88}$$

and

$$\hat{\phi}_2(\kappa,t) \sim \begin{cases} \hat{h}_2(\kappa) & \text{if } \hat{h}_2(\kappa) \neq 0 \\ -c_0\,\kappa\frac{t^{\frac{\alpha_2}{2}}}{\Gamma\left(1+\frac{\alpha_2}{2}\right)}\hat{h}_1(\kappa) & \text{if } \hat{h}_2(\kappa) = 0 \end{cases}, \qquad t \to 0^+. \tag{89}$$

From (88) and (89), we have the following sufficient condition for the L_p-integrability of $\hat{\phi}_2$:

Theorem 4.3. *Let $\mu_1 = \mu_2 = 1$, $c_1 \neq 0$, and $\hat{f}(\kappa, t) = \hat{g}(\kappa, t) = 0$. Then:*

Case 1. $\hat{h}_2(\kappa) \neq 0$: *if* $p > \frac{2}{\alpha_1}$ *then* $\hat{\phi}_2(\cdot, t) \in L_p(\mathbb{R}^+)$;
Case 2. $\hat{h}_2(\kappa) = 0$: *if* $p > \frac{2}{\alpha_2}$ *then* $\hat{\phi}_2(\cdot, t) \in L_p(\mathbb{R}^+)$.

The asymptotic behaviour and the sufficient conditions for the L_p-integrability of $\hat{\phi}_1$ can be deduced analogously to those of $\hat{\phi}_2$, due to the symmetry observed between the solutions.

4.3.2. Asymptotic behaviour for large and small values of $\|\kappa\|$

In this section, we analyse the behaviour of $\hat{\phi}_2$ for large and small values of the frequency variable κ. To begin, we establish an auxiliary result regarding the asymptotic behaviour of the bivariate ML function.

Theorem 4.4. *Let $a_1, a_2, \lambda \in \mathbb{C}$ such that $0 < \mathrm{Re}(a_1) < \mathrm{Re}(a_2)$, $\omega_1, \omega_2, x \in \mathbb{R}^+$, and $a, t \in \mathbb{R}$ with $t > a$. The following asymptotic behaviours hold:*

$$E_{(a_1,a_2),\lambda}\big(-\omega_1\, x(t-a)^{a_1}, -\omega_2(t-a)^{a_2}\big) \sim \frac{\big(-\omega_1\, x(t-a)^{a_1}\big)^{-1}}{\Gamma\big(\lambda - a_1\big)}, \quad x \to +\infty \tag{90}$$

$$E_{(a_1,a_2),\lambda}\big(-\omega_1\, x(t-a)^{a_1}, -\omega_2(t-a)^{a_2}\big) \sim E_{a_2,\lambda}\big(-\omega_2(t-a)^{a_2}\big), \quad x \to 0^+. \tag{91}$$

Proof. Using the asymptotic expansion (27) and the definition of the three-parameter ML function in (26), we have, for $\omega, x \in \mathbb{R}^+$:

$$E^{\beta_3}_{\beta_1,\beta_2}\big(-\omega x\big) \;\sim\; \big(\omega x\big)^{-\beta_3} \sum_{k=0}^{\infty} \frac{(\beta_3)_k(-1)^k}{k!\,\Gamma\big(-\beta_1 k + \beta_2 - \beta_1\beta_3\big)} (\omega x)^{-k}, \quad x \to +\infty. \tag{92}$$

Comparing the terms in (92), we obtain

$$E^{\beta_3}_{\beta_1,\beta_2}\big(-\omega x\big) \;\sim\; \frac{\big(\omega x\big)^{-\beta_3}}{\Gamma\big(\beta_2 - \beta_1\beta_3\big)}, \quad x \to +\infty. \tag{93}$$

By (41) and using (93), we obtain

$$\begin{aligned} E_{(a_1,a_2),\lambda}\big(-\omega_1\, x(t-a)^{a_1}, -\omega_2(t-a)^{a_2}\big) &= \sum_{p=0}^{+\infty} \big(-\omega_2(t-a)^{a_2}\big)^p E^{p+1}_{a_1,\lambda+a_2 p}\big(-\omega_1\, x(t-a)^{a_1}\big) \\ &\sim \sum_{p=0}^{+\infty} \big(-\omega_2(t-a)^{a_2}\big)^p \frac{\big(-\omega_1\, x(t-a)^{a_1}\big)^{-(p+1)}}{\Gamma\big(\lambda + a_2 p - a_1(p+1)\big)}, \quad x \to +\infty. \end{aligned} \tag{94}$$

The leading term in (94) yields (90).

By (26) and considering the leading term, we have

$$E^{\beta_3}_{\beta_1,\beta_2}\big(-\omega x\big) \sim \frac{1}{\Gamma\big(\beta_2\big)}, \quad x \to 0^+. \tag{95}$$

Again, by (41) and using (95), we obtain

$$\begin{aligned} E_{(a_1,a_2),\lambda}\big(-\omega_1\,(t-a)^{a_1} x, -\omega_2\,(t-a)^{a_2}\big) &= \sum_{p=0}^{+\infty} \big(-\omega_2\,(t-a)^{a_2}\big)^p E^{p+1}_{a_1,\lambda+a_2 p}\big(-\omega_1\,(t-a)^{a_1} x\big) \\ &\sim E_{a_2,\lambda}\big(-\omega_2\,(t-a)^{a_2}\big), \quad x \to 0^+ \end{aligned} \tag{96}$$

which corresponds to (91). □

Remark 4.5. *The previous result differs from (38) and (39) because the asymptotic behaviours of (90) and (91) only account for the case where the first input variable of the bivariate ML function approaches zero or infinity, while the second input variable remains fixed. In contrast, (38) and (39) consider the simultaneous asymptotic behavior of both input variables.*

The analysis of the solutions continues under the assumption that $c_1 \neq 0$ and $\hat{f}(\kappa, t) = \hat{g}(\kappa, t) = 0$ in (70), which also applies to $\hat{\phi}_1$ due to symmetry. Additionally, we assume that the initial conditions (61) and (60) are real constants independent of κ, i.e.,

$$(i)\;\; I_t^{(1-\mu_2)\left(1-\frac{\alpha_2}{2}\right)}\, \hat{\phi}_1(\kappa, 0^+) = \gamma_{2,1} \qquad (ii)\;\; I_t^{(1-\mu_2)\left(1-\frac{\alpha_2}{2}\right)}\, \hat{\phi}_2(\kappa, 0^+) = \gamma_{2,2} \tag{97}$$

$$(iii)\;\; I_t^{(1-\mu_1)\left(1-\frac{\alpha_1}{2}\right)}\, \hat{\phi}_1(\kappa, 0^+) = \gamma_{1,1} \qquad (iv)\;\; I_t^{(1-\mu_1)\left(1-\frac{\alpha_1}{2}\right)}\, \hat{\phi}_2(\kappa, 0^+) = \gamma_{1,2}, \tag{98}$$

where $\gamma_{i,j} \in \mathbb{R}$ and $i, j \in \{1, 2\}$.

Applying (90) in Theorem 4.4 to each term in (70), we deduce that

$$\left\|\hat{\phi}_2(\kappa,t)\right\| \lesssim \left(|\gamma_{2,1}|\,\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)\right)} + |\gamma_{1,2}|\,c_1\,\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)\right)}\right)\frac{1}{c_0\,\|\kappa\|}, \quad \|\kappa\|\to\infty \tag{99}$$

if $\gamma_{2,1}\neq 0$ or $\gamma_{1,2}\neq 0$. Thus, $\hat{\phi}_2$ tends to zero as $\|\kappa\|$ grows large. For small $\|\kappa\|$, using (91) to each term in (70), we obtain:

$$\begin{aligned}\hat{\phi}_2(\kappa,t) \sim\ & \gamma_{2,1}\,c_1\,t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2-\frac{\alpha_1}{2}-1}\,E_{\alpha_2-\alpha_1,\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2-\frac{\alpha_1}{2}}(z_2)\\ &+\gamma_{2,2}\,t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}-1}\,E_{\alpha_2-\alpha_1,\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}}(z_2)\\ &-\gamma_{1,1}\,c_1\,t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}-1}\,E_{\alpha_2-\alpha_1,\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}}(z_2)\\ &+\gamma_{1,2}\,c_1^2\,t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2-\frac{\alpha_1}{2}-1}\,E_{\alpha_2-\alpha_1,\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2-\frac{\alpha_1}{2}}(z_2), \quad \|\kappa\|\to 0^+\end{aligned} \tag{100}$$

where $z_2=-c_1^2\,t^{\alpha_2-\alpha_1}$ and at least one of $\gamma_{2,2}$, $\gamma_{2,1}$, $\gamma_{1,1}$, and $\gamma_{1,2}$ is nonzero. If $\gamma_{2,1}=\gamma_{1,2}=0$, the following asymptotic behaviours are obtained:

$$\left\|\hat{\phi}_2(\kappa,t)\right\| \lesssim \left(|\gamma_{2,2}|\,\frac{t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)-\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)-\frac{\alpha_2}{2}\right)} + |\gamma_{1,1}|\,c_1\,\frac{t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)-\frac{\alpha_2}{2}-1}}{\Gamma\left(\mu_2\left(1-\frac{\alpha_2}{2}\right)-\frac{\alpha_2}{2}\right)}\right)\frac{1}{c_0^2\,\|\kappa\|^2}, \quad \|\kappa\|\to\infty \tag{101}$$

and

$$\begin{aligned}\hat{\phi}_2(\kappa,t) \sim\ & \gamma_{2,2}\,t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}-1}\,E_{\alpha_2-\alpha_1,\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}}(z_2)\\ &-\gamma_{1,1}\,c_1\,t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}-1}\,E_{\alpha_2-\alpha_1,\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}}(z_2), \quad \|\kappa\|\to 0^+.\end{aligned} \tag{102}$$

Finally, considering the asymptotic behaviour of $\hat{\phi}_2$ for both large and small $\|\kappa\|$, we obtain the following sufficient conditions for the L_p-integrability of $\hat{\phi}_2$ with respect to the variable κ.

Theorem 4.6. *For $c_1\neq 0$ and $\hat{f}(\kappa,t)=\hat{g}(\kappa,t)=0$, we have:*

1. *If at least one of $\gamma_{2,2},\gamma_{2,1}$, and $\gamma_{1,2}$ is nonzero and $p>n$ then $\hat{\phi}_2(\kappa,\cdot)\in L_p(\mathbb{R}^n)$;*
2. *If $\gamma_{2,2}=\gamma_{1,2}=0$ and $p>\frac{n}{2}$ then $\hat{\phi}_2(\kappa,\cdot)\in L_p(\mathbb{R}^n)$.*

For the Caputo case (i.e., $\mu_1=\mu_2=1$), we have $\gamma_{2,1}=\gamma_{1,1}=\gamma_1$ and $\gamma_{2,2}=\gamma_{1,2}=\gamma_2$. To avoid trivial solutions, γ_1 and γ_2 cannot both be zero. Applying (90) and (91) to each term in (74), we obtain the following asymptotic behaviours for $\hat{\phi}_2$:

$$\left\|\hat{\phi}_2(\kappa,t)\right\| \lesssim \left(|\gamma_1|\,\frac{t^{-\frac{\alpha_2}{2}}}{\Gamma\left(1-\frac{\alpha_2}{2}\right)} + |\gamma_2|\,c_1\,\frac{t^{-\frac{\alpha_1}{2}}}{\Gamma\left(1-\frac{\alpha_2}{2}\right)}\right)\frac{1}{c_0\,\|\kappa\|}, \quad \|\kappa\|\to\infty \tag{103}$$

and

$$\hat{\phi}_2(\kappa,t)\sim \gamma_2\,E_{\alpha_2-\alpha_1,1}(z_2)+\gamma_1\,c_1^2\,t^{\alpha_2-\alpha_1}\,E_{\alpha_2-\alpha_1,\alpha_2-\alpha_1}(z_2), \quad \|\kappa\|\to 0^+. \tag{104}$$

From (103) and (104), we deduce the following sufficient condition for the L_p-integrability of $\hat{\phi}_2$ with respect to the variable κ:

Theorem 4.7. *In the Caputo case, if $p>n$ then $\hat{\phi}_2(\kappa,\cdot)\in L_p(\mathbb{R}^n)$.*

The asymptotic behaviour and L_p-integrability condition for $\hat{\phi}_1$ can be derived analogously.

4.4. Graphical representations of the solutions in the Fourier-time domain

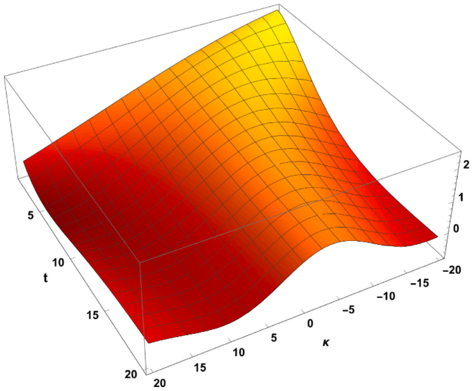
This section presents plots of $\hat{\phi}_1$ and $\hat{\phi}_2$ for the case where $n=1$ (one spatial variable), $c_0=c_1=1$, and $\hat{f}(\kappa,t)=\hat{g}(\kappa,t)=0$, representing the homogeneous case. These plots, generated using the software MATHEMATICA, illustrate the magnitude of $\hat{\phi}_1$ and $\hat{\phi}_2$ as functions of κ and t over a specific range of values.

The plots show the continuity and differentiability of the solutions in the Fourier-time domain. As expected, the solutions approach zero as t and $|\kappa|$ increase, which aligns with the asymptotic behaviour deduced for $\hat{\phi}_2$ in Section 4.3, specifically the decay as $t\to+\infty$ and $|\kappa|\to\infty$, and, by symmetry, for $\hat{\phi}_1$.

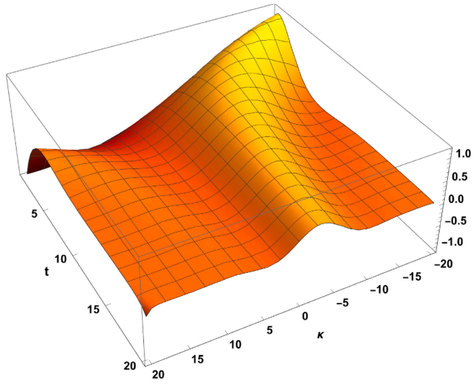
Next, we present some plots for various combinations of $\mu_1=\mu_2=0.25, 0.50, 0.75, 1$ and fractional orders $\alpha_1=0.5, 0.75$ and $\alpha_2=1.25, 1.5, 1.75$. These specific values are chosen to illustrate the behavior of the solutions for different types of fractional derivatives and different fractional orders. Figs. 1–3

From the plots, we observe changes in the rate of decay, such as the steepness of the decay, and the overall shape of the solutions, such as the smoothness and the presence of oscillations (potentially exhibiting more oscillations with higher fractional orders).

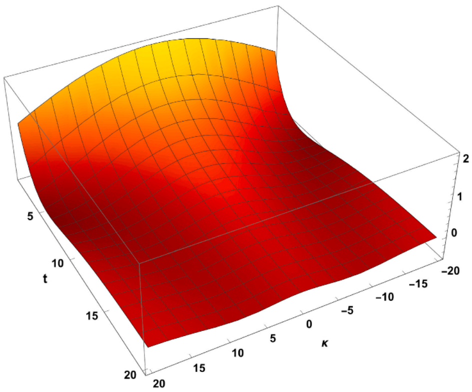
Finally, we present some plots in the Caputo case $\mu_1=\mu_2=1$ and some combinations of α_1 and α_2.



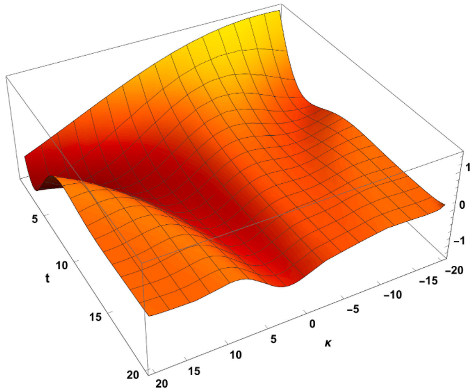
(a) $\widehat{\phi}_2$ with $\alpha_1 = 0.75$, $\alpha_2 = 1.50$, and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = \widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$



(b) $\widehat{\phi}_2$ with $\alpha_1 = 0.75$, $\alpha_2 = 1.50$, $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = 0$ and $\widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$



(c) $\widehat{\phi}_1$ with $\alpha_1 = 0.25$, $\alpha_2 = 1.25$, and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = \widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$



(d) $\widehat{\phi}_1$ with $\alpha_1 = 0.25$, $\alpha_2 = 1.50$, $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = 1$ and $\widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 0$

Fig. 1. Plots of $\widehat{\phi}_2$ and $\widehat{\phi}_1$ when $\mu_1 = \mu_2 = 0$ (Riemann-Liouville case).

4.5. Solutions in the space-time domain

To obtain the expressions of the solutions of ϕ_1 and ϕ_2 in the space-time domain, we apply the inverse Fourier transform to $\widehat{\phi}_1$ and $\widehat{\phi}_2$. This requires using the results established in Section 3, which provides the necessary Fourier transform pairs for the bivariate Mittag-Leffler functions involved in the solutions. Applying the Fourier pairs (40) and (47) to (70), we obtain the following result.

Theorem 4.8. *The solution ϕ_2 of the system (56)-(57), under the conditions (58)–(59), is expressed as the following sum of convolution integrals involving bivariate Fox H-functions:*

$$\begin{aligned}\phi_2(x,t)n =& \int_{\mathbb{R}^n} \big(G_1(x-z,t) + G_2(x-z,t)\big)\, h_{2,1}(z)\, dz + \int_{\mathbb{R}^n} G_3(x-z,t)\, h_{2,2}(z)\, dz \\ &+ \int_{\mathbb{R}^n} G_4(x-z,t)\, h_{1,1}(z)\, dz + \int_{\mathbb{R}^n} \big(G_5(x-z,t) + G_6(x-z,t)\big)\, h_{1,2}(z)\, dz \\ &+ \int_{\mathbb{R}^n}\int_0^t \big(G_7(x-z,t-w) + G_8(x-z,t-w)\big)\, f(z,w)\, dw\, dz + \int_{\mathbb{R}^n}\int_0^t G_9(x-z,t-w)\, g(z,w)\, dw\, dz. \end{aligned} \tag{105}$$

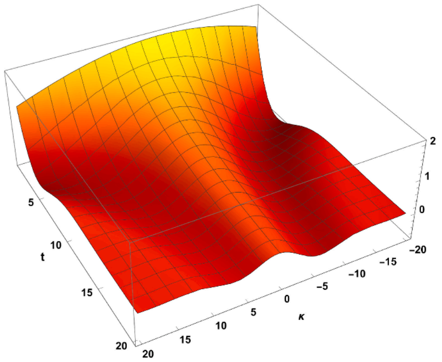
The functions G_j, $j = 1,3,4,5,7,9$ and G_k, $k = 2,6,8$ are defined by:

$$G_j(x,t) = \frac{\gamma_j}{\pi^{\frac{n}{2}}\, \|x\|^n}\, t^{\lambda_j - 1}\, H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c|c} z_1 & (0;1,1);\big(1-\frac{n}{2},1\big);\ ---\\ z_2 & \big(1-\lambda_j;\alpha_2,\alpha_2-\alpha_1\big);\ ---;(0,1)\end{array}\right] \tag{106}$$

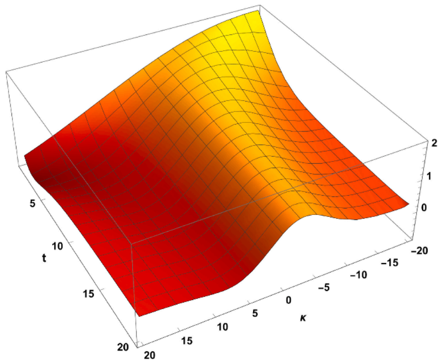
$$G_k(x,t) = \frac{-2i\,\gamma_k}{\pi^{\frac{n}{2}}}\, \frac{x}{\|x\|^{n+2}}\, t^{\lambda_k - 1}\, H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c|c} z_1 & (0;1,1);\big(-\frac{n}{2},1\big);\ ---\\ z_2 & \big(1-\lambda_k;\alpha_2,\alpha_2-\alpha_1\big);\ ---;(0,1)\end{array}\right], \tag{107}$$

where $z_1 = 4c_0^2\, \|x\|^{-2}\, t^{\alpha_2}$, $z_2 = c_1^2\, t^{\alpha_2-\alpha_1}$, and the parameters γ_j, λ_j, γ_k, λ_k are given by:

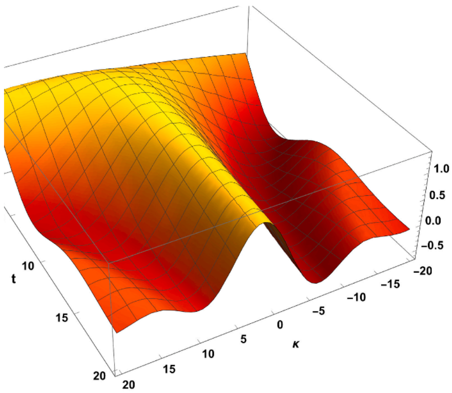
- $\gamma_1 = c_1,\ \lambda_1 = \mu_2\big(1-\frac{\alpha_2}{2}\big) + \alpha_2 - \frac{\alpha_1}{2}$,



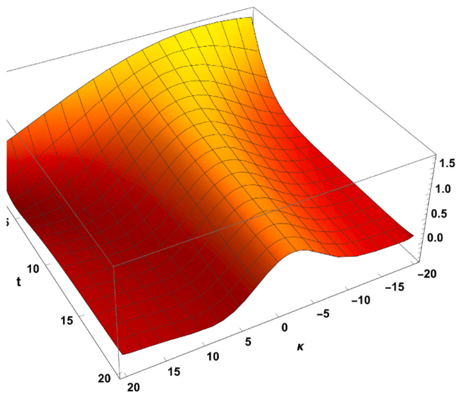
(a) $\widehat{\phi}_1$ with $\alpha_1 = 0.5$, $\alpha_2 = 1.5$, $\mu_1 = \mu_2 = 0.25$, and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = \widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$



(b) $\widehat{\phi}_2$ with $\alpha_1 = 0.5$, $\alpha_2 = 1.25$, $\mu_1 = \mu_2 = 0.50$, and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = \widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$



(c) $\widehat{\phi}_1$ with $\alpha_1 = 0.75$, $\alpha_2 = 1.75$, $\mu_1 = \mu_2 = 0.50$, and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = 0$ and $\widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$



(d) $\widehat{\phi}_2$ with $\alpha_1 = 0.50$, $\alpha_2 = 1.25$, $\mu_1 = \mu_2 = 0.75$, $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = 1$ and $\widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 0$

Fig. 2. Plots of $\widehat{\phi}_2$ and $\widehat{\phi}_1$ when $\mu_1, \mu_2 \in]0,1[$ (Intermediate case).

- $\gamma_2 = c_0,\ \lambda_2 = \mu_2\left(1 - \frac{\alpha_2}{2}\right) + \alpha_2,$
- $\gamma_3 = 1,\ \lambda_3 = \mu_2\left(1 - \frac{\alpha_2}{2}\right) + \frac{\alpha_2}{2},$
- $\gamma_4 = -c_1,\ \lambda_4 = \mu_1\left(1 - \frac{\alpha_1}{2}\right) + \frac{\alpha_2}{2},$
- $\gamma_5 = c_1^2,\ \lambda_5 = \mu_1\left(1 - \frac{\alpha_1}{2}\right) + \alpha_2 - \frac{\alpha_1}{2},$
- $\gamma_6 = c_0\, c_1,\ \lambda_6 = \mu_1\left(1 - \frac{\alpha_1}{2}\right) + \alpha_2,$
- $\gamma_7 = c_1,\ \lambda_7 = \alpha_2 - \frac{\alpha_1}{2},$
- $\gamma_8 = c_0,\ \lambda_8 = \alpha_2,$
- $\gamma_9 = -1,\ \lambda_7 = \frac{\alpha_2}{2}.$

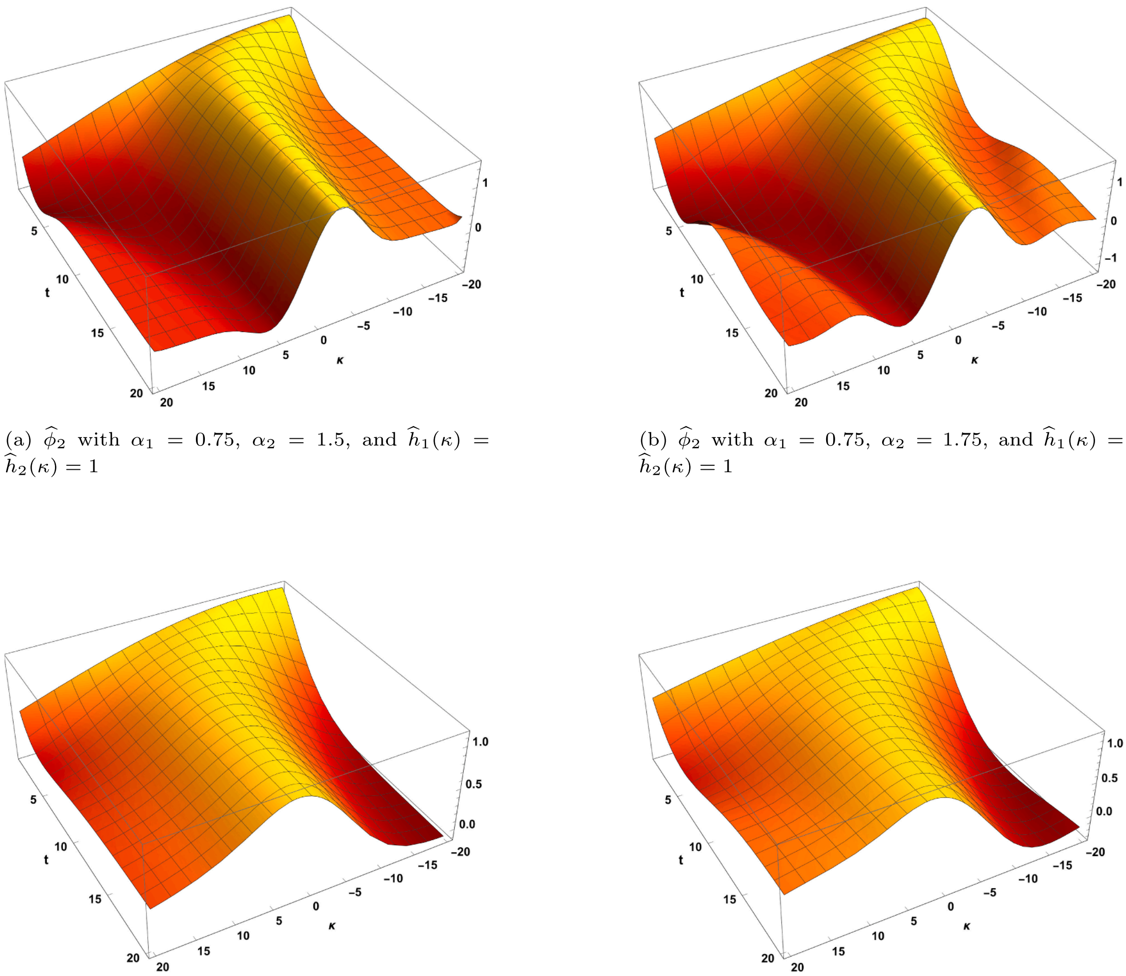
Corollary 4.9. *In the Caputo case ($\mu_1 = \mu_2 = 1$), the solution ϕ_2 of the system (71), subject to the initial conditions (72), is given by:*

$$\begin{aligned}\phi_2(x,t) = & \int_{\mathbb{R}^n} G_2(x-z,t)\, h_1(z)\, dz + \int_{\mathbb{R}^n} \left(G_3(x-z,t) + G_5(x-z,t) + G_6(x-z,t)\right) h_2(z)\, dz \\ & + \int_{\mathbb{R}^n} \int_0^t \left(G_7(x-z,t-w) + G_8(x-z,t-w)\right) f(z,w)\, dw\, dz + \int_{\mathbb{R}^n} \int_0^t G_9(x-z,t-w)\, g(z,w)\, dw\, dz.\end{aligned}$$

If $c_1 = 0$, that is, when considering the generalized time-fractional wave equation, the second variable vanishes, i.e., $z_2 = 0$. Consequently, the solution depends only on the variable $z_1 = 0$, in accordance with (23), making the solution simpler.

Corollary 4.10. *If $c_1 = 0$, the solution ϕ_2 reduces to:*

$$\begin{aligned}\phi_2(x,t) = & \int_{\mathbb{R}^n} G_2(x-z,t)\, h_{2,1}(z)\, dz + \int_{\mathbb{R}^n} G_3(x-z,t)\, h_{2,2}(z)\, dz \\ & + \int_{\mathbb{R}^n} \int_0^t G_8(x-z,t-w)\, f(z,w)\, dw\, dz + \int_{\mathbb{R}^n} \int_0^t G_9(x-z,t-w)\, g(z,w)\, dw\, dz. \qquad (108)\end{aligned}$$



(a) $\widehat{\phi}_2$ with $\alpha_1 = 0.75$, $\alpha_2 = 1.5$, and $\widehat{h}_1(\kappa) = \widehat{h}_2(\kappa) = 1$

(b) $\widehat{\phi}_2$ with $\alpha_1 = 0.75$, $\alpha_2 = 1.75$, and $\widehat{h}_1(\kappa) = \widehat{h}_2(\kappa) = 1$

(c) $\widehat{\phi}_1$ with $\alpha_1 = 0.25$, $\alpha_2 = 1.25$, and $\widehat{h}_1(\kappa) = \widehat{h}_2(\kappa) = 1$

(d) $\widehat{\phi}_1$ with $\alpha_1 = 0.25$, $\alpha_2 = 1.50$, and $\widehat{h}_1(\kappa) = \widehat{h}_2(\kappa) = 1$

Fig. 3. Plots of $\widehat{\phi}_2$ and $\widehat{\phi}_1$ when $\mu_1 = \mu_2 = 1$ (Caputo case).

The functions G_j, $j = 3, 9$ and G_k, $k = 2, 8$ reduce to Fox H-functions of one variable:

$$G_j(x,t) = \frac{\gamma_j}{\pi^{\frac{n}{2}}\,\|x\|^n}\, t^{\lambda_j - 1}\, H^{0,2}_{2,1}\left[\, z_1 \,\middle|\, \begin{matrix} (0,1), \left(1-\frac{n}{2}, 1\right) \\ (1-\lambda_j, \alpha_2) \end{matrix} \right], \tag{109}$$

$$G_k(x,t) = \frac{-2i\,\gamma_k}{\pi^{\frac{n}{2}}}\, \frac{x}{\|x\|^{n+2}}\, t^{\lambda_k - 1}\, H^{0,2}_{2,1}\left[\, z_1 \,\middle|\, \begin{matrix} (0,1), \left(-\frac{n}{2}, 1\right) \\ (1-\lambda_k, \alpha_2) \end{matrix} \right], \tag{110}$$

where $z_1 = 4c_0^2\,\|x\|^{-2}\, t^{\alpha_2}$ and the parameters γ_j, λ_j, γ_k, λ_k are as defined in Theorem 4.8.

The solution ϕ_1 can be obtained by using the symmetry relations (68) and (75).

To conclude this section, we present series representation for G_j and G_k based on the double Mellin-Barnes type integral representation (20) of the bivariate Fox H-function. Assuming that n is odd to handle sequences of simples poles and $f(x,t) = g(x,t) = 0$, and adapting the calculations presented in [35, Sec. 4] in a straightforward manner, we obtain the following series representations for (106) and (107):

$$\begin{aligned} G_j(x,t) = {} & \frac{\gamma_j}{4c_0^2\,\pi^{\frac{n}{2}}\,\|x\|^{n+2}}\, t^{\lambda_j - \alpha_2 - 1} \times \sum_{p=0}^{+\infty} \sum_{q=0}^{\frac{n-3-2p}{2}} \frac{\Gamma\left(\frac{n}{2} - 1 - p - q\right)}{\Gamma\left(\lambda_j - \alpha_2 - \alpha_1 p - \alpha_2 q\right)\, p!\, q!} \left(-\frac{c_1^2\,\|x\|^2}{4c_0^2\, t^{\alpha_1}}\right)^p \left(-\frac{\|x\|^2}{4c_0^2\, t^{\alpha_2}}\right)^q \\ & + \frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}\,\gamma_j}{(4c_0^2\pi)^{\frac{n}{2}}}\, t^{\lambda_j - \frac{\alpha_2 n}{2} - 1} \times \sum_{p=0}^{+\infty} \sum_{r=0}^{+\infty} \frac{\left(1 - \frac{r+n}{2}\right)_p}{\Gamma\left(\lambda_j - \frac{\alpha_2 n}{2} + (\alpha_2 - \alpha_1)p - \frac{\alpha_2}{2} r\right) \left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}\, p!\, r!} \left(-c_1^2\, t^{\alpha_2 - \alpha_1}\right)^p \left(-\frac{\|x\|}{c_0\, t^{\frac{\alpha_2}{2}}}\right)^r \end{aligned} \tag{111}$$

and

$$G_k(x,t) = \frac{-i\,\gamma_k}{2\,\pi^{\frac{n}{2}}}\,\frac{x}{\|x\|^{n+4}}\,t^{\lambda_k-\alpha_2-1} \times \sum_{p=0}^{+\infty}\sum_{q=0}^{\frac{n-3-2p}{2}} \frac{\Gamma\left(\frac{n}{2}-p-q\right)}{\Gamma\left(\lambda_k-\alpha_2-\alpha_1 p-\alpha_2 q\right)\,p!\,q!}\left(-\frac{c_1^2\,\|x\|^2}{4c_0^2\,t^{\alpha_1}}\right)^p\left(-\frac{\|x\|^2}{4c_0^2\,t^{\alpha_2}}\right)^q + \frac{i\,(-1)^{\frac{n-1}{2}}\sqrt{\pi}\,\gamma_k}{c_0\,(4c_0^2\pi)^{\frac{n}{2}}}\,\frac{x}{\|x\|}\,t^{\lambda_k-\frac{\alpha_2(n+1)}{2}-1} \times \sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty}\frac{\left(\frac{1-r-n}{2}\right)_p}{\Gamma\left(\lambda_k-\frac{\alpha_2(n+1)}{2}+(\alpha_2-\alpha_1)p-\frac{\alpha_2}{2}r\right)\left(1+\frac{r}{2}\right)_{\frac{n-1}{2}}\,p!\,r!}\left(-c_1^2\,t^{\alpha_2-\alpha_1}\right)^p\left(-\frac{\|x\|}{c_0\,t^{\frac{\alpha_2}{2}}}\right)^r. \tag{112}$$

The absolute convergence of series (111) and (112) is guaranteed by applying the ratio test. This is done following Horn's technique, a method for analyzing the convergence of multiple hypergeometric series. Specifically, applying [35, Lem. 8.1] demonstrates that both series converge absolutely for all $(x,t)\in\mathbb{R}^n\times\mathbb{R}^+$ and $n\in\mathbb{N}$. Furthermore, considering formula (128) in [35], the preceding double series can be interpreted as generalized Lauricella series, which are multivariate extensions of hypergeometric series.

Remark 4.11. *For the generalized time-fractional wave equation case, corresponding to $c_1=0$, meaning that the term with the derivative of order $\alpha_1/2$ vanishes, the series in (111) and (112) degenerate into single sums/series, as only the term $p=0$ remains. Hence, we obtain:*

$$G_j(x,t) = \frac{\gamma_j}{4c_0^2\,\pi^{\frac{n}{2}}\,\|x\|^{n+2}}\,t^{\lambda_j-\alpha_2-1}\sum_{q=0}^{\frac{n-3}{2}}\frac{\Gamma\left(\frac{n}{2}-1-q\right)}{\Gamma\left(\lambda_j-\alpha_2-\alpha_2 q\right)\,q!}\left(-\frac{\|x\|^2}{4c_0^2\,t^{\alpha_2}}\right)^q + \frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}\,\gamma_j}{(4c_0^2\pi)^{\frac{n}{2}}}\,t^{\lambda_j-\frac{\alpha_2 n}{2}-1}\sum_{r=0}^{+\infty}\frac{1}{\Gamma\left(\lambda_j-\frac{\alpha_2 n}{2}-\frac{\alpha_2}{2}r\right)\left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}\,r!}\left(-\frac{\|x\|}{c_0\,t^{\frac{\alpha_2}{2}}}\right)^r \tag{113}$$

and

$$G_k(x,t) = \frac{-i\,\gamma_k}{2\,\pi^{\frac{n}{2}}}\,\frac{x}{\|x\|^{n+4}}\,t^{\lambda_k-\alpha_2-1}\sum_{q=0}^{\frac{n-3}{2}}\frac{\Gamma\left(\frac{n}{2}-q\right)}{\Gamma\left(\lambda_k-\alpha_2-\alpha_2 q\right)\,q!}\left(-\frac{\|x\|^2}{4c_0^2\,t^{\alpha_2}}\right)^q + \frac{i\,(-1)^{\frac{n-1}{2}}\sqrt{\pi}\,\gamma_k}{c_0\,(4c_0^2\pi)^{\frac{n}{2}}}\,\frac{x}{\|x\|}\,t^{\lambda_k-\frac{\alpha_2(n+1)}{2}-1}\sum_{r=0}^{+\infty}\frac{1}{\Gamma\left(\lambda_k-\frac{\alpha_2(n+1)}{2}-\frac{\alpha_2}{2}r\right)\left(1+\frac{r}{2}\right)_{\frac{n-1}{2}}\,r!}\left(-\frac{\|x\|}{c_0\,t^{\frac{\alpha_2}{2}}}\right)^r. \tag{114}$$

4.6. Behaviours of the solutions in the space-time domain

In this section, we investigate the behaviour of the solutions with respect to both spatial and temporal variables. Here, the notation $f(x)=\mathcal{O}(g(x))$ as $x\to x_0$ denotes the existence of a positive real number M such that $|f(x)|\le M\,|g(x)|$. Considering (22), we obtain the following behaviour of the functions G_j and G_k given by (106) and (107):

$$|G_j(x,t)| = \mathcal{O}\left(t^{\lambda_j-\alpha_2\frac{n}{2}-1}\right) \qquad \text{and} \qquad \|G_k(x,t)\| = \mathcal{O}\left(\|x\|\,t^{\lambda_k-\alpha_2\left(1+\frac{n}{2}\right)-1}\right) \tag{115}$$

as $\min\{|z_1|,|z_2|\}\to\infty$, i.e., $\min\{t^{\alpha_2}\|x\|^{-2}, t^{\alpha_2-\alpha_1}\}\to\infty$. Moreover, by (21) we have

$$|G_j(x,t)| = \mathcal{O}\left(\frac{t^{\lambda_j-1}}{\|x\|^n}\right) \qquad \text{and} \qquad \|G_k(x,t)\| = \mathcal{O}\left(\frac{t^{\lambda_k-1}}{\|x\|^{n+1}}\right). \tag{116}$$

as $\max\{|z_1|,|z_2|\}\to 0$, i.e., $\max\{t^{\alpha_2}\|x\|^{-2}, t^{\alpha_2-\alpha_1}\}\to 0$.

4.6.1. Behavior for large and small values of t

We now examine the behaviour of ϕ_2 given by (105), with $c_1\neq 0$, and $f(x,t)=g(x,t)=0$, assuming that all the initial conditions (58)-(59) are equal to $\delta(x)=\prod_{i=1}^n\delta(x_i)$. Consequentely, ϕ_2 reduces to the sum of the fundamental solutions $G_1,\ldots,G_6$. When $t\to+\infty$ with $t^{\alpha_2}\gg\|x\|^2$, then $|z_1|=4c_0\,\|x\|^{-2}\,t^{\alpha_2}$ and $|z_2|=c_1^2\,t^{\alpha_2-\alpha_1}$ both tend to ∞. Then, by (115), we obtain:

$$\|\phi_2(\cdot,t)\| \le \mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{1-n}{2}\alpha_2-1}\right)+\mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2\left(1-\frac{n}{2}\right)-\frac{\alpha_1}{2}-1}\right)+\mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)-\frac{n}{2}\alpha_2-1}\right) + \mathcal{O}\left(t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2\left(1-\frac{n}{2}\right)-\frac{\alpha_1}{2}-1}\right)+\mathcal{O}\left(t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{1-n}{2}\alpha_2-1}\right)+\mathcal{O}\left(t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)-\frac{n}{2}\alpha_2-1}\right). \tag{117}$$

Comparing the exponents of t in (117), we find:

$$\|\phi_2(\cdot,t)\| = \begin{cases} \mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2\left(1-\frac{n}{2}\right)-\frac{\alpha_1}{2}-1}\right) & \text{if } 0 \le \mu_1 < \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} \\ \mathcal{O}\left(t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2\left(1-\frac{n}{2}\right)-\frac{\alpha_1}{2}-1}\right) & \text{if } \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} < \mu_1 \le 1 \end{cases}. \tag{118}$$

When $\mu_1 = \frac{\mu_2(2-\alpha_2)}{2-\alpha_1}$, i.e., $\mu_1\left(1-\frac{\alpha_1}{2}\right) = \mu_2\left(1-\frac{\alpha_2}{2}\right)$, we have:

$$\|\phi_2(\cdot,t)\| = \mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2\left(1-\frac{n}{2}\right)-\frac{\alpha_1}{2}-1}\right). \tag{119}$$

Given the negative exponents of t in (119) and (118), we conclude that $\|\phi_2(\cdot,t)\|$ decays to zero as $t \to +\infty$. If any of the initial conditions are identically zero, the behaviour of ϕ_2 may change. For instance, if $h_{1,1}(x) = h_{1,2}(x) = 0$, then the dependence on μ_1 disappears, leading to:

$$\|\phi_2(\cdot,t)\| = \mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2\left(1-\frac{n}{2}\right)-\frac{\alpha_1}{2}-1}\right) \to 0, \qquad \text{as } t \to +\infty. \tag{120}$$

Similarly, if $h_{2,1}(x) = h_{2,2}(x) = 0$, the dependence on μ_2 disappears, resulting in:

$$\|\phi_2(\cdot,t)\| = \mathcal{O}\left(t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2\left(1-\frac{n}{2}\right)-\frac{\alpha_1}{2}-1}\right) \to 0, \qquad \text{as } t \to +\infty. \tag{121}$$

We now turn our attention to the behaviour of ϕ_2 as $t \to 0^+$, assuming that $t^{\alpha_2} \ll \|x\|^2$. This assumption means that the time variable is approaching zero faster than the spatial variable. Applying (116) to the fundamental solutions $G_1, \ldots, G_6$, we obtain:

$$\begin{aligned}\|\phi_2(\cdot,t)\| \le{}& \mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}-1}\right) + \mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2-\frac{\alpha_1}{2}-1}\right) + \mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\alpha_2-1}\right) \\ &+ \mathcal{O}\left(t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2-\frac{\alpha_1}{2}-1}\right) + \mathcal{O}\left(t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}-1}\right) + \mathcal{O}\left(t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\alpha_2-1}\right).\end{aligned} \tag{122}$$

Comparing the exponents of t in (122), we find:

$$\|\phi_2(\cdot,t)\| = \begin{cases} \mathcal{O}\left(t^{\mu_1\left(1-\frac{\alpha_1}{2}\right)+\frac{\alpha_2}{2}-1}\right) & \text{if } 0 \le \mu_1 < \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} \\ \mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}-1}\right) & \text{if } \frac{\mu_2(2-\alpha_2)}{2-\alpha_1} < \mu_1 \le 1 \end{cases}. \tag{123}$$

When $\mu_1 = \frac{\mu_2(2-\alpha_2)}{2-\alpha_1}$, that is, $\mu_1\left(1-\frac{\alpha_1}{2}\right) = \mu_2\left(1-\frac{\alpha_2}{2}\right)$, we obtain:

$$\|\phi_2(\cdot,t)\| = \mathcal{O}\left(t^{\mu_2\left(1-\frac{\alpha_2}{2}\right)+\frac{\alpha_2}{2}-1}\right). \tag{124}$$

While the exponents of t in (123) and in (124) are negative, indicating that the right-hand sides tend to infinity for small values of t, a more comprehensive analysis is needed to fully understand the behaviour of ϕ_2 when $t \to 0^+$.

The behaviour of $\hat{\phi}_1$ exhibits analogous characteristics and can be deduced through similar arguments.

4.6.2. Asymptotic behaviour for large and small values of $\|x\|$

We now analyse the behaviour of ϕ_2 for large and small values of $\|x\|$, assuming $c_1 \neq 0$ and $f(x,t) = g(x,t) = 0$ in (105). Additionally, we assume that the initial conditions (58) and (58) equal to $\delta(x) = \prod_{j=1}^{n} \delta(x_j)$.

When $\|x\| \to \infty$ and $t \to 0^+$, we have that $z_1 = 4c_0\,\|x\|^{-2}\,t^{\alpha_2}$ and $z_2 = c_1^2\,t^{\alpha_2-\alpha_1}$ both tend to zero. Applying (116) to $G_1, \ldots, G_6$ in (106)-(107), we obtain:

$$\left\|\phi_2(x,\cdot)\right\| \le \mathcal{O}\left(\frac{1}{\|x\|^n}\right) + \mathcal{O}\left(\frac{1}{\|x\|^{n+1}}\right) = \mathcal{O}\left(\frac{1}{\|x\|^n}\right) \tag{125}$$

which implies that ϕ_2 tends to zero as $\|x\|$ grows large and t decreases to zero.

We now examine the behavior of ϕ_2 as $\|x\| \to 0$ and $t \to +\infty$, which implies that z_1 and z_2 tend to infinity. Applying (115) to $G_1, \ldots, G_6$ yields:

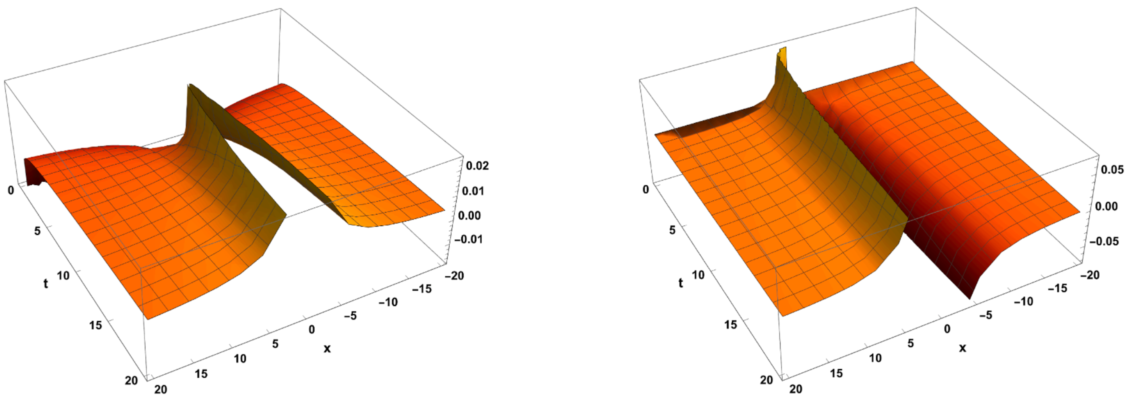
$$\left\|\phi_2(x,\cdot)\right\| \le \mathcal{O}(1) + \mathcal{O}(\|x\|) = \mathcal{O}(1). \tag{126}$$

The function ϕ_1 has the same behaviour in the x-variable as ϕ_2.

4.7. Graphical representation of the solutions in the space-time domain

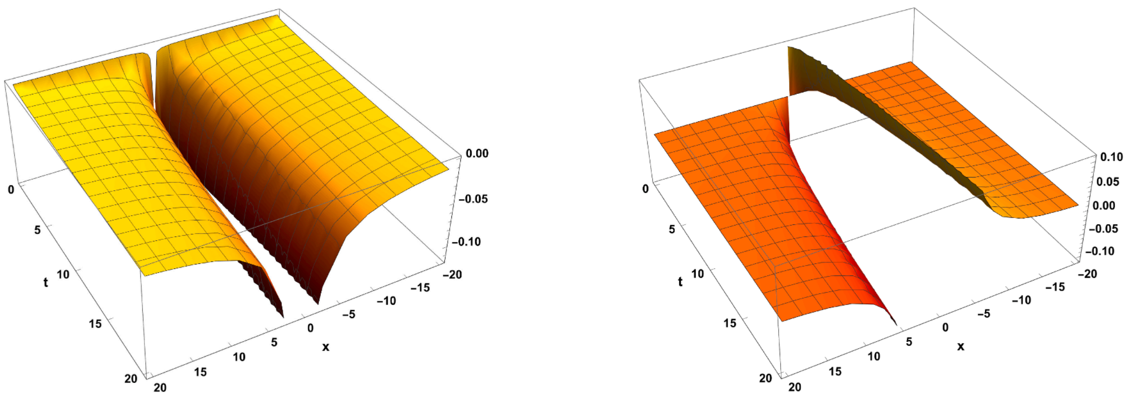
In this section we present some graphical representations of ϕ_1 and ϕ_2 when $n = 1$, $c_0 = c_1 = 1$, and $f(x,t) = g(x,t) = 0$ in (105)–(107), generated using the software Mathematica.

Analysis of Figs. 4–6 shows that the real component is an even function and exhibits a discontinuity at $x = 0$. This discontinuity arises from the nature of the fundamental solutions. Regarding the imaginary part, while the discontinuity at $x = 0$ persists, the functions displays odd symmetry. The observed behaviours of both components align with the predictions derived in Sections 4.6.1 and 4.6.2 for ϕ_2 (and, by symmetry, for ϕ_1), namely, that the functions tend to towards zero as both t and $|x|$ increases.



(a) Real part   (b) Imaginary part

Fig. 4. Plots of the real and imaginary parts of ϕ_1 with $\alpha_1 = 0.25$, $\alpha_2 = 1.50$, $h_{2,1}(x) = h_{2,2}(x) = \delta(x)$, $h_{1,1}(x) = h_{1,2}(x) = 0$, and $\mu_1 = \mu_2 = 0$ (Riemann-Liouville case).



(a) Real part   (b) Imaginary part

Fig. 5. Plots of the real and imaginary parts of ϕ_1 with $\alpha_1 = 0.75$, $\alpha_2 = 1.75$, $h_{2,1}(x) = h_{2,2}(x) = 0$, $h_{1,1}(x) = h_{1,2}(x) = \delta(x)$, and $\mu_1 = \mu_2 = 0.50$ (Intermediate Case).

5. The integer order case

In this section, we analyse the behaviour of our solutions for the particular case where $\alpha_2 = 2$ and $\alpha_1 = 1$, corresponding to the classical telegraph equation. This special case allows us to compare our fractional solutions with well-known integer-order solutions.

When $c_1 \neq 0$, we still encounter, after the application of the Dirac factorization method, fractional derivatives of order $\frac{1}{2}$, while when $c_1 = 0$ we deal only with time-derivatives of integer order. The expressions for these particular cases are obtained by substituting $\alpha_2 = 2$ and $\alpha_1 = 1$ into the previously deduced general expressions and applying simplifications.

5.1. Solutions in the Fourier-time domain

We first analyse the case $c_1 \neq 0$.

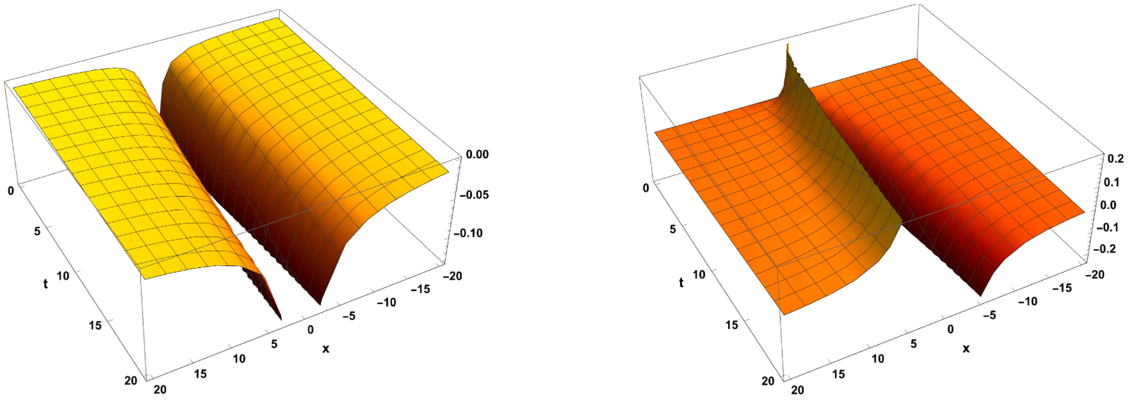
Theorem 5.1. *For $c_1 \neq 0$, the solution $\widehat{\phi}_2(\kappa, t)$ of the following Fourier-time system*

$$\begin{cases} \partial_t \widehat{\phi}_1(\kappa,t) + c_1 \, {}^H\partial_t^{\frac{1}{2},\mu_1} \widehat{\phi}_2(\kappa,t) + c_0\, \kappa\, \widehat{\phi}_2(\kappa,t) = \widehat{f}(\kappa,t) \\ -\partial_t \widehat{\phi}_2(\kappa,t) + c_1 \, {}^H\partial_t^{\frac{1}{2},\mu_1} \widehat{\phi}_1(\kappa,t) - c_0\, \kappa\, \widehat{\phi}_1(\kappa,t) = \widehat{g}(\kappa,t) \end{cases}$$

subject to the initial conditions

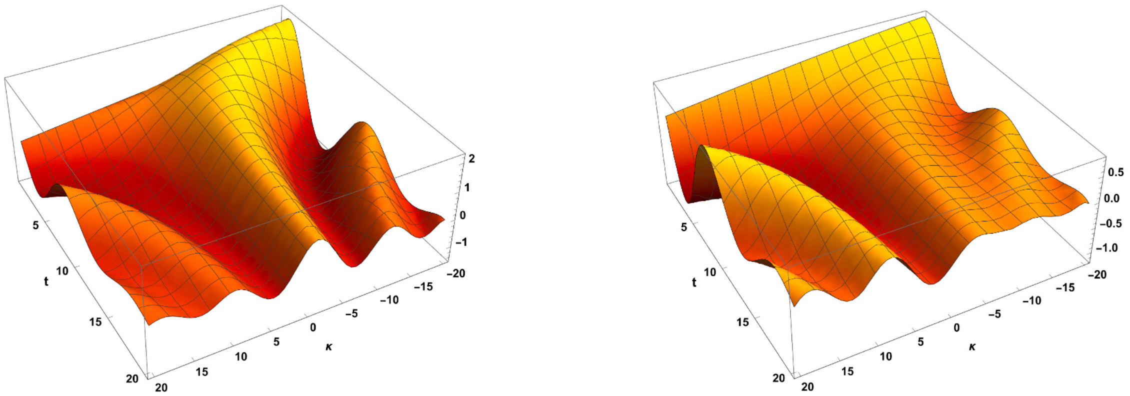
$$(i)\ \ \widehat{\phi}_1(\kappa,0^+) = \widehat{h}_{2,1}(\kappa) \qquad (iii)\ \ \widehat{\phi}_2(\kappa,0^+) = \widehat{h}_{2,2}(\kappa) \tag{127}$$

$$(ii)\ \ I_t^{\frac{1-\mu_1}{2}} \widehat{\phi}_1(\kappa,0^+) = \widehat{h}_{1,1}(\kappa) \qquad (iv)\ \ I_t^{\frac{1-\mu_1}{2}} \widehat{\phi}_2(\kappa,0^+) = \widehat{h}_{1,2}(\kappa) \tag{128}$$



(a) Real part (b) Imaginary part

Fig. 6. Plots of the real and imaginary parts of ϕ_2 with $\alpha_1 = 0.75$, $\alpha_2 = 1.75$, $h_1(x) = h_2(x) = \delta(x)$, and $\mu_1 = \mu_2 = 1$ (Caputo case).



(a) $\widehat{\phi}_2$ with $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = \widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$ (b) $\widehat{\phi}_1$ with $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = 1$ and $\widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 0$

Fig. 7. Plots of $\widehat{\phi}_2$ and $\widehat{\phi}_1$ when $\mu_1 = \mu_2 = 0$ (Riemann-Liouville case).

is given by

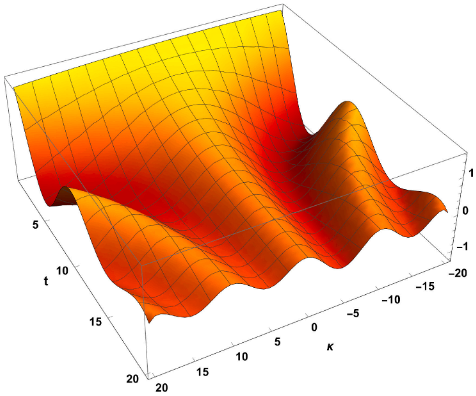
$$\begin{aligned}\widehat{\phi}_2(\kappa,t) = {} & \left(\sum_{j=0}^{1}(-c_0)^{1-j}\, c_1^j\, \kappa^{1-j}\, t^{1-\frac{j}{2}}\, E_{(2,1),2-\frac{j}{2}}(z_1,z_2)\right)\widehat{h}_{2,1}(\kappa) + E_{(2,1),1}(z_1,z_2)\,\widehat{h}_{2,2}(\kappa)\\ & - c_1\, t^{\frac{\mu_1}{2}}\, E_{(2,1),\frac{\mu_1}{2}+1}(z_1,z_2)\,\widehat{h}_{1,1}(\kappa) + \left(\sum_{j=0}^{1}(-c_0)^{1-j}\, c_1^{1+j}\, \kappa^{1-j}\, t^{\frac{\mu_1}{2}+1-\frac{j}{2}}\, E_{(2,1),\frac{\mu_1}{2}+2-\frac{j}{2}}(z_1,z_2)\right)\widehat{h}_{1,2}(\kappa),\\ & + \left(\sum_{j=0}^{1}(-c_0)^{1-j}\, c_1^j\, \kappa^{1-j}\, t^{1-\frac{j}{2}}\, E_{(2,1),2-\frac{j}{2}}(z_1,z_2)\right) *_t \widehat{f}(\kappa,t) - E_{(2,1),1}(z_1,z_2) *_t \widehat{g}(\kappa,t)\end{aligned} \tag{129}$$

where $z_1 = -c_0^2\, \|\kappa\|^2\, t^2$ and $z_2 = -c_1^2\, t$, and the convolution $*_t$ is defined by (15). Similarly, we can obtain the expression for $\widehat{\phi}_1$ considering the symmetries described in (68).

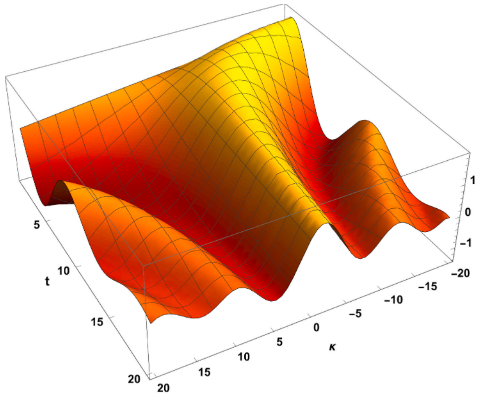
The following plots illustrate the solutions $\widehat{\phi}_1$ and $\widehat{\phi}_2$ for the integer order case, with $c_0 = c_1 = 1$ and $\widehat{f}(\kappa,t) = \widehat{g}(\kappa,t) = 0$, generated using the software MATHEMATICA.

Analysis of Figs. 7–9 shows that the solutions are continuous and differentiable in the Fourier-time domain. The asymptotic behaviour is consistent with the predictions presented in Sections 4.3.1 and 4.3.2 (upon substituting $\alpha_1 = 1$ and $\alpha_2 = 2$ into the results therein) for large and small values of t and $\|\kappa\|$, respectively.

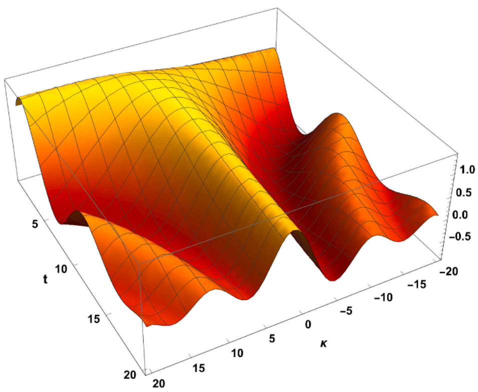
We now consider the case where $c_1 = 0$, corresponding to the time-fractional wave equation. This particular value of c_1 significantly simplifies the solution, reducing the number of terms, and the bivariate ML function reduces to a two-parameter ML function, as shown in (31). Furthermore, by considering the specific cases of the ML function outlined in (25), we arrive at the following result:



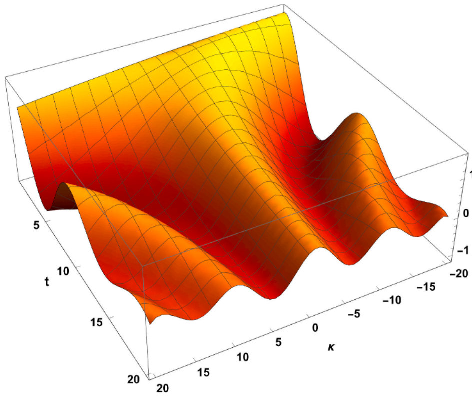
(a) $\widehat{\phi}_1$ with $\mu_1 = \mu_2 = 0$ and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = \widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$



(b) $\widehat{\phi}_2$ with $\mu_1 = \mu_2 = 0.50$ and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = \widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$

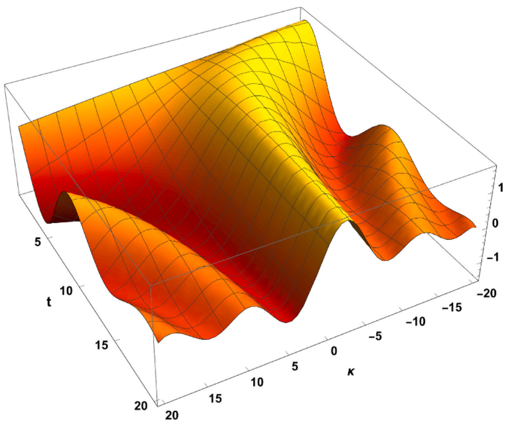


(c) $\widehat{\phi}_1$ with $\mu_1 = \mu_2 = 0.50$ and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = 0$ and $\widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 1$

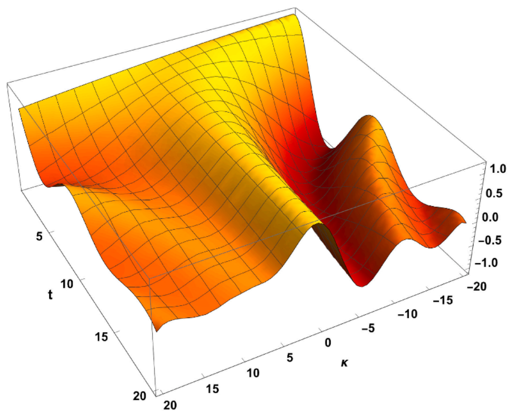


(d) $\widehat{\phi}_2$ with $\mu_1 = \mu_2 = 0.75$ and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = 1$ and $\widehat{h}_{1,1}(\kappa) = \widehat{h}_{1,2}(\kappa) = 0$

Fig. 8. Plots of $\widehat{\phi}_2$ and $\widehat{\phi}_1$ when $\mu_1, \mu_2 \in]0,1[$ (Intermediate case).



(a) $\widehat{\phi}_2$ with $\widehat{h}_1(\kappa) = \widehat{h}_2(\kappa) = 1$



(b) $\widehat{\phi}_1$ with $\widehat{h}_1(\kappa) = \widehat{h}_2(\kappa) = 1$

Fig. 9. Plots of $\widehat{\phi}_2$ and $\widehat{\phi}_1$ when $\mu_1 = \mu_2 = 1$ (Caputo case).

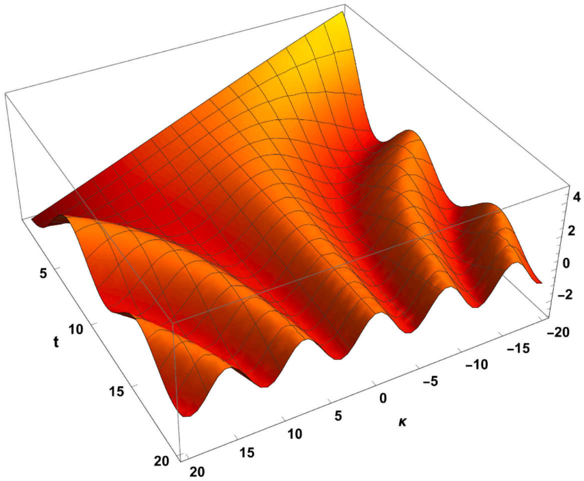


Fig. 10. Plot of $\widehat{\phi}_2$ when $\alpha_2 = 2$, $\alpha_1 = 1$, and $c_1 = 0$.

Corollary 5.2. *If $c_1 = 0$ in Theorem 5.1, then $\widehat{\phi}_2$ reduces to*

$$\widehat{\phi}_2(\kappa, t) = -\widehat{h}_{2,1}(\kappa)\, \frac{\kappa}{\|\kappa\|^2}\, \sin(c_0\, \|\kappa\|\, t) + \widehat{h}_{2,2}(\kappa)\, \cos(c_0\, \|\kappa\|\, t) - \frac{\kappa}{\|\kappa\|^2}\, \widehat{f}(\kappa, t) *_t \sin(c_0\, \|\kappa\|\, t). \tag{130}$$

Fig. 10 presents a graphical representation of (130) for $n = 1$, $c_0 = 1$, $\widehat{f}(\kappa, t) = 0$, and $\widehat{h}_{2,1}(\kappa) = \widehat{h}_{2,2}(\kappa) = 1$.

In this case, the solution $\widehat{\phi}_2$ exhibits odd symmetry with respect to the κ-variable and demonstrates sinusoidal behaviour. Given that $c_1 = 0$, there is no influence or perturbation arising from the derivative of order $\frac{1}{2}$.

In the space-time domain, we derive explicit expression of our solutions in terms of sums of convolution integrals involving bivariate Fox H-functions. Furthermore, we present series representation of our solutions.

Theorem 5.3. *The solution ϕ_2 of the system (56)–(57), subject to the conditions (58)-(59) with $\alpha_1 = 1$ and $\alpha_2 = 2$, is given by (105) with $z_1 = 4c_0^2\, \|x\|^{-2}\, t^2$ and $z_2 = c_1^2\, t$. The functions G_j and G_k are defined, for $j = 1, 3, 4, 5, 7, 9$ and $k = 2, 6, 8$ respectively, as follows:*

$$G_j(x, t) = \frac{\gamma_j}{\pi^{\frac{n}{2}}\, \|x\|^n}\, t^{\lambda_j - 1}\, H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle| \begin{array}{c} (0; 1, 1); \left(1 - \frac{n}{2}, 1\right); --- \\ \left(1 - \lambda_j; 2, 1\right); ---; (0, 1) \end{array}\right] \tag{131}$$

$$G_k(x, t) = \frac{-2i\, \gamma_k}{\pi^{\frac{n}{2}}}\, \frac{x}{\|x\|^{n+2}}\, t^{\lambda_k - 1}\, H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c} z_1 \\ z_2 \end{array}\middle| \begin{array}{c} (0; 1, 1); \left(-\frac{n}{2}, 1\right); --- \\ \left(1 - \lambda_k; 2, 1\right); ---; (0, 1) \end{array}\right] \tag{132}$$

with parameters $\gamma_j, \lambda_j, \gamma_k, \lambda_k$ given by

- $\gamma_1 = c_1,\ \lambda_1 = \frac{3}{2}$,
- $\gamma_2 = c_0,\ \lambda_2 = 2$,
- $\gamma_3 = 1,\ \lambda_3 = 1$,
- $\gamma_4 = -c_1,\ \lambda_4 = \frac{\mu_1}{2} + 1$,
- $\gamma_5 = c_1^2,\ \lambda_5 = \frac{\mu_1}{2} + \frac{3}{2}$,
- $\gamma_6 = c_0\, c_1,\ \lambda_6 = \frac{\mu_1}{2} + 2$,
- $\gamma_7 = c_1,\ \lambda_7 = \frac{3}{2}$,
- $\gamma_8 = c_0,\ \lambda_8 = 2$,
- $\gamma_9 = -1,\ \lambda_9 = 1$.

Using (23), we obtain the following corollary.

Corollary 5.4. *If $c_1 = 0$, then the solution ϕ_2 is given by (108), where the functions G_j, $j = 3, 9$ and G_k, $k = 2, 8$, reduce to Fox H-functions of one variable:*

$$G_j(x, t) = \frac{\gamma_j}{\pi^{\frac{n}{2}}\, \|x\|^n}\, t^{\lambda_j - 1}\, H^{0,2}_{2,1}\left[\ z_1\ \middle| \begin{array}{c} (0, 1), \left(1 - \frac{n}{2}, 1\right) \\ (1 - \lambda_j, \alpha_2) \end{array}\right] \tag{133}$$

and

$$G_k(x, t) = \frac{-2i\, \gamma_k}{\pi^{\frac{n}{2}}}\, \frac{x}{\|x\|^{n+2}}\, t^{\lambda_k - 1}\, H^{0,2}_{2,1}\left[\ z_1\ \middle| \begin{array}{c} (0, 1), \left(-\frac{n}{2}, 1\right) \\ (1 - \lambda_k, \alpha_2) \end{array}\right], \tag{134}$$

where $z_1 = 4c_0^2\, \|x\|^{-2}\, t^2$ and the parameters γ_j, λ_j, γ_k, λ_k are defined in Theorem 5.3.

Series representations of the space-time solutions for the odd-dimensional case can be derived by substituting $\alpha_1 = 1$ and $\alpha_2 = 2$ into (111)–(114).

Table 1
Solutions for other systems of two-term time-fractional diffusion equations of Dirac type obtained from different choices of the triplet (j,k,l) of Pauli matrices (see (53)).

Triplet	System	A	B	ϕ_1^*	ϕ_2^*
$(1,3,2)$	$\begin{cases} \left({}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} - i\,c_0\,\partial_x\right)\phi_2^*(x,t) + c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\phi_1^*(x,t) = 0 \\ \left({}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + i\,c_0\,\partial_x\right)\phi_1^*(x,t) - c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\phi_2^*(x,t) = 0 \end{cases}$	$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$\frac{\phi_2-\phi_1}{2}$	$\frac{\phi_1+\phi_2}{2}$
$(2,1,3)$	$\begin{cases} \left(-i\,{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\right)\phi_2^*(x,t) + c_0\,\partial_x\,\phi_1^*(x,t) = 0 \\ \left(i\,{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\right)\phi_1^*(x,t) - c_0\,\partial_x\,\phi_2^*(x,t) = 0 \end{cases}$	$\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$	$-\frac{\phi_1+i\,\phi_2}{2}$	$\frac{i\,\phi_1+\phi_2}{2}$
$(3,2,1)$	$\begin{cases} \left(-i\,c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1} + c_0\,\partial_x\right)\phi_2^*(x,t) + {}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\,\phi_1^*(x,t) = 0 \\ \left(i\,c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1} + c_0\,\partial_x\right)\phi_1^*(x,t) - {}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\,\phi_2^*(x,t) = 0 \end{cases}$	$\begin{bmatrix} 0 & 1 \\ -i & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -i \\ -1 & 0 \end{bmatrix}$	$-i\,\phi_2$	$-\phi_1$
$(1,2,3)$	$\begin{cases} \left({}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} - i\,c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\right)\phi_2^*(x,t) + c_0\,\partial_x\,\phi_1^*(x,t) = 0 \\ \left({}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + i\,c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\right)\phi_1^*(x,t) - c_0\,\partial_x\,\phi_2^*(x,t) = 0 \end{cases}$	$\begin{bmatrix} i & 1 \\ i & -1 \end{bmatrix}$	$\begin{bmatrix} -\frac{i}{2} & -\frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$	$-i\,\frac{\phi_1+\phi_2}{2}$	$\frac{\phi_1-\phi_2}{2}$
$(2,3,1)$	$\begin{cases} \left(-i\,{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + c_0\,\partial_x\right)\phi_2^*(x,t) + c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\phi_1^*(x,t) = 0 \\ \left(i\,{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + c_0\,\partial_x\right)\phi_1^*(x,t) - c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\phi_2^*(x,t) = 0 \end{cases}$	$\begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{i}{2} & -\frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{bmatrix}$	$-\frac{i\,\phi_1+\phi_2}{2}$	$\frac{-i\,\phi_1+\phi_2}{2}$

6. Other systems

We have shown how to solve the factorized system (53) for the triplet $(j,k,l)=(3,1,2)$. Other factorizations of (50) are feasible. For instance, for the triplet $(j,k,l)=(1,3,2)$, we obtain the following matricial equation:

$$\left[\sigma_1\,{}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + c_1\,\sigma_3\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1} + c_0\,\sigma_2\,\partial_x\right]\Phi^*(x,t) = 0, \tag{135}$$

where $\Phi^*(x,t) = \left[\phi_1^*(x,t),\ \phi_2^*(x,t)\right]^T$. This yields the following coupled system of two-term time-fractional diffusion Dirac-type equations:

$$\begin{cases} \left({}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} - i\,c_0\,\partial_x\right)\phi_2^*(x,t) + c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\phi_1^*(x,t) = 0 & (136) \\ \left({}^H\partial_t^{\frac{\alpha_2}{2},\mu_2} + i\,c_0\,\partial_x\right)\phi_1^*(x,t) - c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\phi_2^*(x,t) = 0. & (137) \end{cases}$$

In this coupled system, the terms $c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\phi_1^*$ and $c_1\,{}^H\partial_t^{\frac{\alpha_1}{2},\mu_1}\,\phi_2^*$ modify the time-fractional transport equations of the wave in a non-dissipative medium. This contrasts with the system (54)–(55) where the subdiffusion process is perturbed by the terms ${}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\,\phi_1$ and ${}^H\partial_t^{\frac{\alpha_2}{2},\mu_2}\,\phi_2$. The solution of the system (136)–(137) is related with the homogeneous solution of (54)–(55]) by:

$$\phi_1^* = \frac{\phi_2-\phi_1}{2} \qquad \text{and} \qquad \phi_2^* = \frac{\phi_1+\phi_2}{2}.$$

This is readily apparent, as there exist invertible matrices A and B such that $A\sigma_1 B = \sigma_3$, $A\sigma_2 B = \sigma_2$, and $A\sigma_3 B = \sigma_1$. Such matrices are given, e.g., by

$$A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Table 1 summarizes the homogeneous systems, the invertible matrices A and B, and the solutions for all triplets distinct from the initial triplet $(3,1,2)$.

The previous systems can also be solved in the more general setting of the so-called ψ−fractional calculus. This calculus provides a broader framework for fractional differentiation and integration, allowing for greater flexibility in modeling complex phenomena. We now recall the definition of ψ-fractional integral and ψ-fractional Hilfer derivative presented in [30].

Definition 6.1 (cf. [30, Def. 4]). *Let $[a,b]$ be a finite or infinite interval on the real line $\mathbb{R}$ and $\alpha > 0$. Also, let ψ be an increasing and positive monotone function on (a,b). The left Riemann-Liouville fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by*

$$\left(I_{a^+}^{\alpha;\psi} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(w)\,(\psi(t)-\psi(w))^{\alpha-1}\,f(w)\,dw,\ t > a. \tag{138}$$

Definition 6.2 (cf. [30, Def. 7]). *Let $\alpha > 0$, $m = \lfloor \alpha \rfloor + 1$, $I = [a,b]$ be a finite or infinite interval on the real line and $f, \psi \in C^m[a,b]$ two functions such that ψ is a positive monotone increasing function and $\psi'(t) \neq 0$, for all $t \in I$. The ψ-Hilfer left fractional derivative ${}^H\mathbb{D}_{t,a^+}^{\alpha,\mu;\psi}$ of order α and type $\mu \in [0,1]$ is defined by*

$$\left({}^H D_{a^+}^{\alpha,\mu;\psi} f\right)(t) = I_{a^+}^{\mu(m-\alpha);\psi}\left(\frac{1}{\psi'(t)}\,\frac{d}{dt}\right)^m I_{a^+}^{(1-\mu)(m-\alpha);\psi} f(t). \tag{139}$$

Application of transmutation relations of ψ-fractional calculus, as detailed in [7,12], yields the solution of the system (54)–(55) with ψ-Hilfer derivatives as $\phi_1(x, \psi(t))$ and $\phi_2(x, \psi(t))$. This notation indicates that the solutions are expressed in terms of the function $\psi(t)$, which included in the kernel of the ψ-fractional derivative, effectively transforming the time variable.

CRediT authorship contribution statement

M. Ferreira: Writing – review & editing, Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization; **M. M. Rodrigues:** Writing – review & editing, Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization; **N. Vieira:** Writing – review & editing, Writing – original draft, Validation, Software, Methodology, Investigation, Formal analysis, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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