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Eigenfunctions of the Time-fractional Telegraph Equation of Distributed Order

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Abstract In this work, the eigenfunction problem for the time-fractional telegraph operator of distributed order in $\mathbb{R}^n \times \mathbb{R}^+$ is considered. By employing the technique of the Fourier, Laplace and Mellin transforms, an integral representation of the eigenfunctions involving the Fox H-function is obtained.

INTRODUCTION

Fractional partial differential equations with distributed order have been studied over the past decades. One reason of the interest is the relation of these equations with physical processes involving times-scales, for example, fractional kinetics, the Cauchy problem of time-fractional diffusion-wave, generalized time-fractional diffusion, time-fractional reaction-diffusion, fractional sub-diffusion equations, and continuous random walk processes. For a general overview of fractional equations of distributed order we refer [3, 4, 5].

One interesting topic in the analysis of fractional partial differential equations is the study of the existence of eigenvalues and the correspondent eigenfunctions. The knowledge of eigenvalues and eigenfunctions plays an important role in the study of the existence of solutions of nonlinear perturbations of the fractional partial differential operators.

The aim of this paper is to obtain an integral representation of the eigenfunctions of the time-fractional telegraph operator of distributed order in $\mathbb{R}^n \times \mathbb{R}^+$. This integral representation involves Fox H-functions, and it is deduced via a combination of the Laplace, Fourier, and Mellin transform. For the inversion of the Laplace transform the classical Titchmarsh's theorem plays an important role.

PRELIMINARIES

Let $a,b \in \mathbb{R}$ with $a < b$, and ${}^C D^\gamma_{a^+}$ the left Caputo fractional derivative of order $\gamma > 0$ on $[a,b] \subset \mathbb{R}$, defined by (see [2])

$$\left({}^C D^\gamma_{a^+} f\right)(x) = \frac{1}{\Gamma(m-\gamma)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\gamma-m+1}}\, dt, \qquad x > a, \tag{1}$$

where $m = [\gamma]+1$ and $[\gamma]$ means the integer part of γ. This definition of fractional derivative can be naturally extended to $\mathbb{R}^n$ considering partial fractional integrals and derivatives (see Chapter 5 in [6]).

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In this work some integral transforms are used, namely, the Laplace and the Fourier transforms. The Laplace transform of a real valued function $f(t)$ is defined by (see [2]) $\mathscr{L}\{f(t)\}(\mathbf{s}) = \widetilde{f}(\mathbf{s}) = \int_0^{+\infty} e^{-\mathbf{s}t} f(t)\, dt$, with $\mathrm{Re}(\mathbf{s}) \in \mathbb{C}$, and when it is applied to (1) leads to (see formula (5.3.3) in [2])

$$\mathscr{L}\left\{{}^{C}D_{a^+}^{\gamma} f(t)\right\}(\mathbf{s}) = \mathbf{s}^{\gamma}\widetilde{f}(\mathbf{s}) - \sum_{j=0}^{m-1} f^{(j)}(a)\,\mathbf{s}^{\gamma-j-1}, \qquad m = [\gamma]+1. \tag{2}$$

Concerning the inverse Laplace transform of functions involving a branch point, we have the theorem from Titchmarsh (see [7]).

Theorem 1 *Let $\widetilde{f}(\mathbf{s})$ be an analytic function which has a branch cut on the real negative semiaxis, such that $\widetilde{f}(\mathbf{s}) = O(1)$, when $|\mathbf{s}| \to +\infty$, and $\widetilde{f}(\mathbf{s}) = O\left(\frac{1}{|\mathbf{s}|}\right)$, when $|\mathbf{s}| \to 0$, for any sector $|\arg(\mathbf{s})| < \pi - \eta$, where $0 < \eta < \pi$. Then the inverse Laplace transform of $\widetilde{f}(\mathbf{s})$ is given by $f(t) = \mathscr{L}^{-1}\left\{\widetilde{f}(\mathbf{s})\right\}(t) = -\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\,\mathrm{Im}\left(\widetilde{f}(re^{i\pi})\right) dr$, where $\mathrm{Im}(\cdot)$ denotes the imaginary part.*

The n-dimensional Fourier transform of a function $f(x)$ of $x \in \mathbb{R}^n$ is defined by (see [2]) $\mathscr{F}\{f(x)\}(\kappa) = \widehat{f}(\kappa) = \int_{\mathbb{R}^n} e^{i\kappa\cdot x} f(x)\, dx$, with $\kappa \in \mathbb{R}^n$, while the corresponding inverse Fourier transform is given by the formula

$$f(x) = \mathscr{F}^{-1}\left\{\widehat{f}(\kappa)\right\}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot\kappa} f(\kappa)\, d\kappa, \qquad x \in \mathbb{R}^n. \tag{3}$$

For the n-dimensional Laplace operator $\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ we have

$$\mathscr{F}\{\Delta f(x)\}(\kappa) = -|\kappa|^2\,\mathscr{F}\{f(x)\}(\kappa). \tag{4}$$

The Fox H-function $H_{p,q}^{m,n}$ is defined, via a Mellin-Barnes type integral in the form (see [1]), by

$$H_{p,q}^{m,n}\left[z \;\middle|\; \begin{matrix} (a_1,\ \alpha_1),\ldots,(a_p,\ \alpha_p) \\ \\ (b_1,\ \beta_1),\ldots,(b_q,\ \beta_q) \end{matrix} \right] = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\ \prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\ \prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}\, z^{-s}\, ds, \tag{5}$$

where $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}^+$, for $i = 1,\ldots,p$ and $j = 1,\ldots,q$, and $\mathscr{C}$ is a suitable contour in the complex plane separating the poles of the two factors in the numerator (see [1]).

TIME-FRACTIONAL TELEGRAPH EQUATION OF DISTRIBUTED ORDER

Let us consider the following eigenfunction problem for the time-fractional telegraph equation of distributed order

$$\int_1^2 b_2(\beta)\left[{}_{0^+}^{C}\partial_t^{\beta} u(x,t)\right] d\beta + \int_0^1 b_1(\alpha)\left[{}_{0^+}^{C}\partial_t^{\alpha} u(x,t)\right] d\alpha - \Delta_x u(x,t) = \lambda\, u(x,t), \tag{6}$$

for given order-density functions $b_2(\beta) > 0$ and $b_1(\alpha) > 0$, subject to the following initial and boundary conditions

$$u(x,0) = \delta(x) = \prod_{i=1}^{n}\delta(x_i), \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad \lim_{|x|\to+\infty} u(x,t) = 0, \quad \int_1^2 b_2(\beta)\, d\beta = C_2, \quad \int_0^1 b_1(\alpha)\, d\alpha = C_1, \tag{7}$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$, Δ_x is the classical Laplace operator in $\mathbb{R}^n$, the partial time-fractional derivatives of order $\beta \in [1,2]$ and $\alpha \in [0,1]$ are in the Caputo sense and given by (1), $\lambda \in \mathbb{R}^+$, and $C_1, C_2 \in \mathbb{R}^+$. The positive constants C_1 and C_2 can be taken as 1 if we want to assume the normalization condition for the integral. Let us start applying in (6) the Laplace transform with respect to the variable $t \in \mathbb{R}^+$ and the $n-$dimensional Fourier transform with respect to the variable $x \in \mathbb{R}^n$. Taking into account relations (2) and (4), the initial conditions in (7), and making straightforward calculations we obtain

$$\widehat{\widetilde{u}}(\kappa,\mathbf{s}) = \frac{\mathbf{s}B_2(\mathbf{s}) + \mathbf{s}B_1(\mathbf{s}) + \lambda\,\mathbf{s}}{\mathbf{s}^2\left(B_2(\mathbf{s}) + B_1(\mathbf{s}) + |\kappa|^2\right)}, \tag{8}$$

where

$$B_2(\mathbf{s}) = \int_1^2 b_2(\beta)\mathbf{s}^\beta \, d\beta - \lambda, \qquad B_1(\mathbf{s}) = \int_0^1 b_1(\alpha)\mathbf{s}^\alpha \, d\alpha. \tag{9}$$

Let us now invert the Laplace and Fourier transforms in order to obtain our solution in the time-space domain. Let us consider the following auxiliar functions in the Laplace domain

$$\widehat{\widehat{u}}_1(\kappa,\mathbf{s}) = \frac{B_2(\mathbf{s})+B_1(\mathbf{s})}{\mathbf{s}\left(B_2(\mathbf{s})+B_1(\mathbf{s})+|\kappa|^2\right)}, \qquad \widehat{\widehat{u}}_2(\kappa,\mathbf{s}) = \frac{1}{\mathbf{s}\left(B_2(\mathbf{s})+B_1(\mathbf{s})+|\kappa|^2\right)}. \tag{10}$$

Applying Theorem 1 for the inversion of the Laplace transform, we obtain

$$\widehat{u}_1(\kappa,t) = -\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\,\mathrm{Im}\left(\widehat{\widehat{u}}_1\left(\kappa, re^{i\pi}\right)\right) dr, \qquad \widehat{u}_2(\kappa,t) = -\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\,\mathrm{Im}\left(\widehat{\widehat{u}}_2\left(\kappa, re^{i\pi}\right)\right) dr. \tag{11}$$

In order to simplify (11), we need to evaluate the imaginary parts of the functions $\widehat{\widehat{u}}_j\left(\kappa, re^{i\pi}\right)$, $j=1,2$, along the ray $\mathbf{s}=re^{i\pi}$, with $r>0$, which is the branch cut of the function $\mathbf{s}^\alpha$. In this sense, for $j=1,2$, by writing

$$B_j\left(re^{i\pi}\right) = \rho_j\left(\cos(\gamma_j\pi)+i\sin(\gamma_j\pi)\right) \Longrightarrow \begin{cases} \rho_j = \rho_j(r) = \left|B_j\left(re^{i\pi}\right)\right| \\ \gamma_j = \gamma_j(r) = \dfrac{1}{\pi}\arg\left(B_j\left(re^{i\pi}\right)\right) \end{cases}, \tag{12}$$

we obtain, after straightforward calculations, the following expressions for the imaginary part of the functions $\widehat{\widehat{u}}_j$, $j=1,2$

$$\mathrm{Im}\left\{\widehat{\widehat{u}}_1\left(\kappa, re^{i\pi}\right)\right\} = K_1(|\kappa|,r) = \frac{-B\,|\kappa|^2}{r\left[\left(A+|\kappa|^2\right)^2+B^2\right]}, \qquad \mathrm{Im}\left\{\widehat{\widehat{u}}_2\left(\kappa, re^{i\pi}\right)\right\} = K_2(|\kappa|,r) = \frac{B}{r\left[\left(A+|\kappa|^2\right)^2+B^2\right]}, \tag{13}$$

where $A=\rho_2\cos(\gamma_2\pi)+\rho_1\cos(\gamma_1\pi)$ and $B=\rho_2\sin(\gamma_2\pi)+\rho_1\sin(\gamma_1\pi)$. Applying the inverse Laplace transform to (8) and taking into account expressions (11), and (13), we obtain

$$\widehat{u}(\kappa,t) = -\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\left[K_1(|\kappa|,r)+\lambda K_2(|\kappa|,r)\right] dr. \tag{14}$$

For the inversion of the Fourier transform, taking into account (3), we obtain

$$u(x,t) = -\mathscr{F}^{-1}\left\{\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\left[K_1(|\kappa|,r)+\lambda K_2(|\kappa|,r)\right] dr\right\}(x,t). \tag{15}$$

Making use of the following formula presented in [6] for the inverse Fourier transform

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot\kappa}\,\varphi(|\kappa|)\,d\kappa = \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\varphi(w)\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|\,w)\,dw, \tag{16}$$

and since we are dealing with radial functions in κ, (15) can be rewritten as

$$u(x,t) = -\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\underbrace{\frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\left[K_1(w,r)+\lambda K_2(w,r)\right] w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|\,w)\,dw}_{\mathbf{I}}\,dr. \tag{17}$$

The inner integral in (17) was already calculated in [8] using the Mellin transform (for more details see Section 3) and has the following representation in terms of Mellin-Barnes integrals

$$\mathbf{I} = \frac{B\left(A^2+B^2\right)^{-\frac{1}{2}}}{r\,\pi^{\frac{n-3}{2}}\,(2\,|x|)^n\sin(\psi)}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1+\frac{s}{2}\right)\Gamma(n-s)\,\Gamma\left(-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{\psi s}{2\pi}\right)\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\Gamma\left(1-\frac{\psi s}{2\pi}\right)}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right)^{-s} ds$$

$$-\lambda\,\frac{B\left(A^2+B^2\right)^{-1}}{r\,\pi^{\frac{n-3}{2}}\,(2\,|x|)^n\sin(\psi)}\,\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(n-s)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(-\frac{\psi}{\pi}+\frac{\psi s}{2\pi}\right)\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\Gamma\left(1+\frac{\psi}{\pi}-\frac{\psi s}{2\pi}\right)}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right)^{-s} ds,$$

where $\psi = \arccos\left(\frac{A}{\sqrt{A^2+B^2}}\right)$. The previous expression is equivalent, by (5), to the following expression in terms of Fox H-functions (see formula (58) in [8])

$$\mathbf{I} = \frac{B\left(A^2+B^2\right)^{-\frac{1}{2}}}{r\,\pi^{\frac{n-3}{2}}\,(2\,|x|)^n\,\sin(\psi)}\,H^{1,2}_{4,3}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\,\middle|\,\begin{array}{c}(1-n,1),\left(1,\frac{1}{2}\right),\left(0,\frac{1}{2}\right),\left(0,\frac{\psi}{2\pi}\right)\\ \left(1,\frac{1}{2}\right),\left(\frac{1-n}{2},\frac{1}{2}\right),\left(0,\frac{\psi}{2\pi}\right)\end{array}\right]$$

$$-\lambda\,\frac{B\left(A^2+B^2\right)^{-1}}{r\,\pi^{\frac{n-3}{2}}\,(2\,|x|)^n\,\sin(\psi)}\,H^{0,2}_{3,2}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\,\middle|\,\begin{array}{c}(1-n,1),\left(0,\frac{1}{2}\right),\left(-\frac{\psi}{\pi},\frac{\psi}{2\pi}\right)\\ \left(\frac{1-n}{2},\frac{1}{2}\right),\left(-\frac{\psi}{\pi},\frac{\psi}{2\pi}\right)\end{array}\right]. \quad (18)$$

From (18) we obtain the following result.

Theorem 2 *The eigenfunctions of the time-fractional telegraph operator of distributed order are given by*

$$u(x,t) = \frac{-1}{\pi^{\frac{n-1}{2}}\,(2\,|x|)^n}\int_0^{+\infty}\frac{B\left(A^2+B^2\right)^{-\frac{1}{2}}\,e^{-rt}}{r\sin(\psi)}\left[\mathscr{H}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right)-\lambda\left(A^2+B^2\right)^{-\frac{1}{2}}\,\mathscr{H}^*\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right)\right]dr, \quad (19)$$

where the functions $\mathscr{H}$ *and* $\mathscr{H}^*$ *are expressed in terms of the following Fox H-functions*

$$\mathscr{H}\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right) = H^{1,2}_{4,3}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\,\middle|\,\begin{array}{c}(1-n,1),\left(1,\frac{1}{2}\right),\left(0,\frac{1}{2}\right),\left(0,\frac{\psi}{2\pi}\right)\\ \left(1,\frac{1}{2}\right),\left(\frac{1-n}{2},\frac{1}{2}\right),\left(0,\frac{\psi}{2\pi}\right)\end{array}\right],$$

$$\mathscr{H}^*\left(\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\right) = H^{0,2}_{3,2}\left[\frac{\left(A^2+B^2\right)^{-\frac{1}{4}}}{|x|}\,\middle|\,\begin{array}{c}(1-n,1),\left(0,\frac{1}{2}\right),\left(-\frac{\psi}{\pi},\frac{\psi}{2\pi}\right)\\ \left(\frac{1-n}{2},\frac{1}{2}\right),\left(-\frac{\psi}{\pi},\frac{\psi}{2\pi}\right)\end{array}\right].$$

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