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A scaling analysis in the SIRI epidemiological model

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For the spatial stochastic epidemic reinfection model SIRI, where susceptibles S can become infected I, then recover and remain only partial immune against reinfection R, we determine the phase transition lines using pair approximation for the moments derived from the master equation. We introduce a scaling argument that allows us to determine analytically an explicit formula for these phase transition lines and prove rigorously the heuristic results obtained previously.

Keywords: stochastic processes; reinfection model; pair approximation; phase transition lines

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1. Introduction

In [10], we presented the phase transition lines between no-growth and compact growth, between compact growth and annular growth and between no-growth and annular growth in pair approximation for a reinfection model, called SIRI. It describes a susceptible, infected, recovered epidemic process, SIR, with additional partial reinfection, a transition from partially immune recovered R to the infected class I, hence the name SIRI. This model is a simplified version of general multi-strain models, as, e.g., described in [3,4], where after an initial infection, immunity against one strain only gives partial immunity against a genetically close mutant strain. For a recent investigation of reinfection models in the biological context, see for example [9]. In the physics literature, models with partial immunization have also found wide interest [1,5] due to their critical behaviour connecting directed percolation and dynamic percolation.

In particular, the findings of two thresholds in reinfection models [3] have sparked discussions around the so-called reinfection threshold. The geometric interpretation in the spatial set-up as given in [5] for their partial immunity models has helped us to understand the two thresholds, found in the biological mean field models to correspond to the transitions from no-growth to annular growth and from annular growth to compact growth [10]. The existence of these two

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thresholds in systems with the possibility of reinfection with partial immunity, as known from, e.g., influenza, malaria and tuberculosis (see [3], and personal communication G. Gomes, Oeiras), has consequences for disease control and vaccination policies. The accurate matching of the models used in biology with those used in physics is vital for the mentioned applications and is given in [10] and the present study, in which the most interesting phase transition line is rigorously proven (the second phase transition having trivial form of a straight line in parameter space).

We compare the especially tricky phase transition line between no-growth and annular growth with simulations [10]. In pair approximation, these critical points have different values for the SIS system [7] and the SIR system [6], as opposed to the mean field values which are the same for SIS and SIR. In the present article, we prove the analytical formula of the transition line presented in [10] between no-growth and annular growth. The proof of this analytical formula of the transition line has a difficulty far beyond the calculation of the other transition lines. For the analytic calculations, we have to introduce a scaling argument that, in particular, allows us to overcome the difficulty arising from an indetermination occurring in the proof. This scaling argument consists of computing the ratio $\langle R\rangle^*/\langle I\rangle^*$ of the average of the recovered individuals $\langle R\rangle^*$ against the infected individuals $\langle I\rangle^*$, observing that several terms along the computation vanish.

In Section 2, we describe the spatial epidemic model for reinfection, the SIRI model, in its stochastic description. We present the dynamic equations for the expectation values of the total number of infected, recovered and for the pairs $\langle SI\rangle^*$, $\langle RI\rangle^*$ and $\langle SR\rangle^*$ closed via the pair approximation as given in [10]. In Section 3, we prove the analytical formula of the transition line between no-growth and annular growth using a scaling argument.

2. The SIRI epidemic model

To describe reinfection in a simple epidemic model, we investigate an extension on classical SIS or SIR models extending to the SIRI model [10]. We consider the following transitions between host classes for N individuals being either susceptible S, infected I by a disease or recovered R

$$S + I \xrightarrow{\beta} I + I,$$
$$I \xrightarrow{\gamma} R,$$
$$R + I \xrightarrow{\tilde{\beta}} I + I,$$
$$R \xrightarrow{\alpha} S,$$

resulting in the master equation (11) for variables S_i, I_i and $R_i \in \{0, 1\}$, $i = 1, 2, \ldots, N$, for N individuals, with constraint $S_i + I_i + R_i = 1$, see Equation (8). The term ‘master equation’ has been used for different types of equations in stochastic systems, see historic remarks in [11, p. 97 f.] to clarify the terminology.

The first infection $S + I \xrightarrow{\beta} I + I$ occurs with an infection rate β, whereas after recovery with rate γ, the respective host becomes resistant up to a possible reinfection $R + I \xrightarrow{\tilde{\beta}} I + I$ with a reinfection rate $\tilde{\beta}$. Hence, the recovered are only partially immunized. For further analysis of possible stationary states, we include a transition from recovered to susceptibles α, which might be simply due to demographic effects (or very slow waning immunity for some diseases). We will later consider the limit of vanishing or very small α. In the case of demography that would be in the order of inverse 70 years, whereas for the basic epidemic processes like first infection β, we would expect inverse a few weeks. We consider that the N individuals live on a regular lattice, where each corner has the same number, Q, of edges.

2.1. *The ODEs for the moments*

In [10], we presented the following ordinary differential equations (ODEs) for the first moments $\langle S\rangle$, $\langle I\rangle$ and $\langle R\rangle$ (see Equation (6)) and for the second moments $\langle SS\rangle$, $\langle II\rangle$, $\langle RR\rangle$, $\langle SI\rangle$, $\langle SR\rangle$ and $\langle IR\rangle$ (see Equation (7)) in pair approximation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle I\rangle = \beta\,\langle SI\rangle - \gamma\langle I\rangle + \tilde{\beta}\,\langle RI\rangle, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle R\rangle = \gamma\langle I\rangle - \alpha\langle R\rangle - \tilde{\beta}\,\langle RI\rangle, \tag{2}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle SI\rangle &= \alpha\langle RI\rangle - (\gamma+\beta)\,\langle SI\rangle + \beta(Q-1)\,\langle SI\rangle - \beta\frac{Q-1}{Q}\frac{(2\langle SI\rangle + \langle SR\rangle)\cdot\langle SI\rangle}{N-\langle I\rangle-\langle R\rangle}\\ &\quad + \tilde{\beta}\frac{Q-1}{Q}\frac{\langle SR\rangle\,\langle RI\rangle}{\langle R\rangle},\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle RI\rangle &= \gamma\,(Q\langle I\rangle - \langle SI\rangle) - (\alpha+2\gamma+\tilde{\beta})\,\langle RI\rangle + \beta\frac{Q-1}{Q}\frac{\langle SR\rangle\quad\langle SI\rangle}{N-\langle I\rangle-\langle R\rangle}\\ &\quad + \tilde{\beta}\frac{Q-1}{Q}\frac{(Q\langle R\rangle - \langle SR\rangle - 2\langle RI\rangle)\cdot\langle RI\rangle}{\langle R\rangle},\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle SR\rangle &= \gamma\langle SI\rangle + \alpha\,(Q\langle R\rangle - 2\langle SR\rangle - \langle RI\rangle)\\ &\quad - \beta\frac{Q-1}{Q}\frac{\langle SR\rangle\quad\langle SI\rangle}{N-\langle I\rangle-\langle R\rangle} - \tilde{\beta}\frac{Q-1}{Q}\frac{\langle RI\rangle\,\langle SR\rangle}{\langle R\rangle}.\end{aligned} \tag{5}$$

For a detailed discussion of the pair approximation applied to the SIRI model, see Section 2.4. We recall that the expectation value of the total number of infected hosts $\langle I\rangle$ at a given time t is

$$\begin{aligned}\langle I\rangle(t) &= \sum_{\mathrm{SIR}}\left(\sum_{i=1}^{N} I_i\right) p(S_1, I_1, R_1, S_2, \ldots, R_N, t)\\ &= \sum_{i=1}^{N}\sum_{\mathrm{SIR}} I_i\; p(S_1, I_1, R_1, S_2, \ldots, R_N, t)\\ &= \sum_{i=1}^{N}\langle I_i\rangle(t),\end{aligned} \tag{6}$$

where $\sum_{\mathrm{SIR}}$ denotes the sum $\sum_{S_1=0}^{1}\sum_{I_1=0}^{1}\sum_{R_1=0}^{1}\sum_{S_2=0}^{1}\cdots\sum_{R_N=0}^{1}$, and $p(S_1, I_1, R_1, S_2, \ldots, R_N, t)$ is the probability of the state $S_1, I_1, R_1, S_2, \ldots, R_N$ that occurs at time t, given by the master equation for the SIRI model (Section 2.2).

Similarly to the first moment, the second moment $\langle SI\rangle$ of the expectation value of two individuals neighbours in which one is susceptible and one is infected is the pair given by

$$\langle SI\rangle(t) = \sum_{\mathrm{SIR}}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\; S_i I_j\right) p(S_1, I_1, R_1, \ldots, R_N, t) = \sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\,\langle S_i I_j\rangle(t). \tag{7}$$

The other first and second moments are defined similarly. These are dynamic variables, e.g., $\langle I\rangle(t)$, and the stationary values will be denoted by $\langle I\rangle^*$, $\langle R\rangle^*$, etc.

2.2. *From the master equation to the dynamics of the moments*

We are going to determine the ODEs for the first moments $\langle S\rangle$, $\langle I\rangle$ and $\langle R\rangle$ (see Equation (16)), and for the second moments $\langle SS\rangle$, $\langle II\rangle$, $\langle RR\rangle$, $\langle SI\rangle$, $\langle SR\rangle$ and $\langle IR\rangle$ (see Equation (24)) using the master equation that we present below. We use the term master equation as specified in [11], especially in application to chemical reactions. For spatial systems, our approach corresponds to the master equation as used by Glauber [2] for an Ising spin dynamics. There the spin variable at each lattice site σ_i can take values -1 or $+1$, whereas our state variables, e.g., I_i, can take 0 or 1. Whereas Glauber fixed the transition rates to obtain the desired stationary states to give the original Ising model, we fix the transition rates due to the infection dynamics (in the way chemical reactions are described [11]). But the structure of our master equation has the same formal form as the one used in the spatial set-up by Glauber [2].

Let $p(S_1, I_1, R_1, S_2, I_2, R_2, \dots, R_N, t)$ be the probability of the state $S_1, I_1, R_1, S_2, I_2, R_2, \dots, R_N$ that occurs at time t. Let $J_{i,j} \in \{0, 1\}$ be the elements of the $N \times N$ adjacency matrix J, symmetric and with zero diagonal elements, that describes the neighbouring structure of the individuals: if $J_{i,j} = 1$, then the individual i is a neighbour of j and if $J_{i,j} = 0$, then the individual i is not a neighbour of j. Following Glauber [2], the master equation for the SIRI model gives the time evolution of the probability $p(S_1, I_1, R_1, S_2, I_2, R_2, \dots, R_N, t)$ with respect to the underlying regular grid describing the spatial interactions of the model:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} p(S_1, I_1, R_1, S_2, I_2, R_2, ..., R_N, t)\\
&\quad= \sum_{i=1}^{N} \beta \left(\sum_{j=1}^{N} J_{ij} I_j \right) (1 - S_i) p(S_1, I_1, R_1, ..., 1 - S_i, 1 - I_i, R_i ..., R_N, t)\\
&\qquad+ \sum_{i=1}^{N} \gamma (1 - I_i) p(S_1, I_1, R_1, \dots, S_i, 1 - I_i, 1 - R_i \dots, R_N, t)\\
&\qquad+ \sum_{i=1}^{N} \tilde{\beta} \left(\sum_{j=1}^{N} J_{ij} I_j \right) (1 - R_i) p(S_1, I_1, R_1, \dots, S_i, 1 - I_i, 1 - R_i \dots, R_N, t)\\
&\qquad+ \sum_{i=1}^{N} \alpha (1 - R_i) p(S_1, I_1, R_1, \dots, 1 - S_i, I_i, 1 - R_i \dots, R_N, t)\\
&\qquad- \sum_{i=1}^{N} \left[\beta \left(\sum_{j=1}^{N} J_{ij} I_j \right) S_i + \gamma I_i + \tilde{\beta} \left(\sum_{j=1}^{N} J_{ij} I_j \right) R_i + \alpha R_i \right]\\
&\qquad\cdot p(\cdots S_i, I_i, R_i \cdots).
\end{aligned} \tag{8}$$

The expectation value of the marginal quantity I_i is defined by

$$\begin{aligned}
\langle I_i\rangle(t) &= \sum_{S_1=0}^{1} \sum_{I_1=0}^{1} \sum_{R_1=0}^{1} \sum_{S_2=0}^{1} \cdots \sum_{R_N=0}^{1} I_i \; p(S_1, I_1, R_1, S_2, \dots, R_N, t)\\
&= \sum_{\mathrm{SIR}} I_i \; p(S_1, I_1, R_1, S_2, \dots, R_N, t),
\end{aligned}$$

where $\sum_{\text{SIR}}$ denotes the sum $\sum_{S_1=0}^{1}\sum_{I_1=0}^{1}\sum_{R_1=0}^{1}\sum_{S_2=0}^{1}\cdots\sum_{R_N=0}^{1}$, and its dynamics is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle I_i\rangle &= \sum_{\text{SIR}} I_i\ \frac{\mathrm{d}}{\mathrm{d}t}p(S_1, I_1, R_1, S_2, \ldots, R_N)\\ &= A + B + C + D + E,\end{aligned} \tag{9}$$

with

$$\begin{aligned}
A &= \sum_{\text{SIR}} I_i\ \sum_{k=1}^{N}\beta\left(\sum_{j=1}^{N} J_{kj}I_j\right)(1-S_k)\ p(\ldots, 1-S_k, 1-I_k, R_k, \ldots),\\
B &= \sum_{\text{SIR}} I_i\ \sum_{k=1}^{N}\gamma\ (1-I_k)\ p(\ldots, S_k, 1-I_k, 1-R_k, \ldots),\\
C &= \sum_{\text{SIR}} I_i\ \sum_{k=1}^{N}\tilde{\beta}\left(\sum_{j=1}^{N} J_{kj}I_j\right)(1-R_k)\ p(\ldots, S_k, 1-I_k, 1-R_k, \ldots),\\
D &= \sum_{\text{SIR}} I_i\ \sum_{k=1}^{N}\alpha\ (1-R_k)\ p(\ldots, 1-S_k, I_k, 1-R_k, \ldots),\\
E &= -\sum_{\text{SIR}} I_i\ \sum_{k=1}^{N}\left[\beta\left(\sum_{j=1}^{N} J_{kj}I_j\right)S_k + \gamma I_k + \tilde{\beta}\left(\sum_{j=1}^{N} J_{kj}I_j\right)R_k + \alpha R_k\right]\\
&\quad \cdot p(\ldots, S_k, I_k, R_k, \ldots).
\end{aligned}$$

Making a change of variables, we observe, for any expression f, that

$$\begin{aligned}&\sum_{I_k=0}^{1}\sum_{R_k=0}^{1} f(S_k, I_k, R_k)\ p(S_1, ..., S_k, 1-I_k, 1-R_k, \ldots, R_N)\\ &= \sum_{I_k=0}^{1}\sum_{R_k=0}^{1} f(S_k, 1-I_k, 1-R_k)p(S_1, \ldots, S_k, I_k, R_k, \ldots, R_N).\end{aligned} \tag{10}$$

Hence, we have $A = A_1 + A_2$, where

$$\begin{aligned}
A_1 &= \sum_{\text{SIR}} I_i \sum_{k=1\wedge k\neq i}^{N}\beta\left(\sum_{j=1}^{N} J_{kj}I_j\right)S_k\ p(\ldots, S_k, I_k, R_k, \ldots),\\
A_2 &= \sum_{\text{SIR}}(1-I_i)\beta\left(\sum_{j=1}^{N} J_{ij}I_j\right)S_i\ p(\ldots, S_i, I_i, R_i, \ldots).
\end{aligned}$$

Similarly, we have $B = B_1 + B_2$, where

$$\begin{aligned}
B_1 &= \sum_{\text{SIR}} I_i \sum_{k=1\wedge k\neq i}^{N}\gamma\ I_k\ p(\ldots, S_k, I_k, R_k, \ldots),\\
B_2 &= \sum_{\text{SIR}}(1-I_i)\gamma\ I_i\ p(\ldots, S_i, I_i, R_i, \ldots),
\end{aligned}$$

$C = C_1 + C_2$, where

$$C_1 = \sum_{\text{SIR}} I_i \sum_{k=1 \wedge k \neq i}^{N} \tilde{\beta} \left(\sum_{j=1}^{N} J_{kj} I_j \right) R_k \; p(\ldots, S_k, I_k, R_k, \ldots),$$

$$C_2 = \sum_{\text{SIR}} (1 - I_i) \tilde{\beta} \left(\sum_{j=1}^{N} J_{ij} I_j \right) R_i \, p(\ldots, S_i, I_i, R_i, \ldots),$$

$D = D_1 + D_2$, where

$$D_1 = \sum_{\text{SIR}} I_i \sum_{k=1 \wedge k \neq i}^{N} \alpha \; R_k \; p(\ldots, S_k, I_k, R_k, \ldots),$$

$$D_2 = \sum_{\text{SIR}} I_i \alpha R_i \, p(\ldots, S_i, I_i, R_i, \ldots),$$

and $E = E_1 + E_2$, where

$$E_1 = -\sum_{\text{SIR}} I_i \sum_{k=1 \wedge k \neq i}^{N} \left[\beta \left(\sum_{j=1}^{N} J_{kj} I_j \right) S_k + \gamma I_k + \tilde{\beta} \left(\sum_{j=1}^{N} J_{kj} I_j \right) R_k + \alpha R_k \right] \cdot p(\ldots, S_k, I_k, R_k, \ldots),$$

$$E_2 = -\sum_{\text{SIR}} I_i \left[\beta \left(\sum_{j=1}^{N} J_{ij} I_j \right) S_i + \gamma I_i + \tilde{\beta} \left(\sum_{j=1}^{N} J_{ij} I_j \right) R_i + \alpha R_i \right] \cdot p(\ldots, S_i, I_i, R_i, \ldots).$$

We note that $A_1 + B_1 + C_1 + D_1 + E_1 = 0$. Observing that $I_i \cdot (1 - I_i) = 0$, we obtain that $B_2 = 0$. Since one individual can not stay in more than one state, we obtain that $I_i \cdot R_i = 0$ and $I_i \cdot S_i = 0$. Therefore, $D_2 = 0$ and

$$\begin{aligned} E_2 &= -\gamma \sum_{\text{SIR}} I_i^2 \; p(\ldots, S_i, I_i, R_i, \ldots) \\ &= -\gamma \sum_{\text{SIR}} I_i \; p(\ldots, S_i, I_i, R_i, \ldots) \\ &= -\gamma \; \langle I_i \rangle. \end{aligned} \tag{11}$$

Hence, Equation (9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle I_i \rangle = A_2 + C_2 + E_2. \tag{12}$$

Observing that $(1 - I_i) \cdot S_i = S_i - I_i \cdot S_i = S_i$, we get

$$\begin{aligned} A_2 &= \beta \sum_{\text{SIR}} \sum_{j=1}^{N} J_{ij} \; I_j S_i \; p(\ldots, S_i, I_i, R_i, \ldots) \\ &= \beta \sum_{j=1}^{N} J_{ij} \; \langle I_j S_i \rangle. \end{aligned} \tag{13}$$

Since $(1 - I_i) \cdot R_i = R_i - I_i \cdot R_i = R_i$, we obtain that

$$\begin{aligned} C_2 &= \tilde{\beta} \sum_{\mathrm{SIR}} \sum_{j=1}^{N} J_{ij} I_j R_i \; p(\ldots, S_i, I_i, R_i, \ldots) \\ &= \tilde{\beta} \sum_{j=1}^{N} J_{ij} \; \langle I_j R_i \rangle. \end{aligned} \tag{14}$$

Applying the formulas in Equations (11), (13) and (14) to Equation (12), we obtain the dynamics of $\langle I_i \rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle I_i \rangle = \beta \sum_{j=1}^{N} J_{ij} \; \langle I_j S_i \rangle + \tilde{\beta} \sum_{j=1}^{N} J_{ij} \; \langle I_j R_i \rangle - \gamma \; \langle I_i \rangle. \tag{15}$$

The expectation value of the total number of infected hosts at a given time is defined in Equation (6). Hence, by Equation (15), it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle I \rangle &= \sum_{i=1}^{N} \frac{\mathrm{d}}{\mathrm{d}t} \; \langle I_i \rangle \\ &= \beta \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} \; \langle I_j S_i \rangle + \tilde{\beta} \sum_{i=1}^{N} \sum_{j=1}^{N} J_{ij} \langle I_j R_i \rangle - \gamma \sum_{i=1}^{N} \langle I_i \rangle \\ &= \beta \langle SI \rangle + \tilde{\beta} \; \langle RI \rangle - \gamma \langle I \rangle, \end{aligned}$$

where $\langle SI \rangle$ is defined in Equation (7) and $\langle RI \rangle$ is the pair given by

$$\langle RI \rangle(t) = \sum_{\mathrm{SIR}} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \; J_{ij} \; R_i I_j \right) \; p(S_1, I_1, R_1, \ldots, R_N, t).$$

Doing a similar reasoning for the mean total number of susceptible and recovered hosts, we obtain the following ODEs for the first moments

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle S \rangle &= \alpha \langle R \rangle - \beta \; \langle SI \rangle, \\ \frac{\mathrm{d}}{\mathrm{d}t} \langle I \rangle &= \beta \; \langle SI \rangle - \gamma \langle I \rangle + \tilde{\beta} \; \langle RI \rangle, \\ \frac{\mathrm{d}}{\mathrm{d}t} \langle R \rangle &= \gamma \langle I \rangle - \alpha \langle R \rangle - \tilde{\beta} \; \langle RI \rangle. \end{aligned} \tag{16}$$

Now, we have to compute the ODEs for the second moments. We will present the details for the dynamics of the pair $\langle SI \rangle$ defined in Equation (7) and the ODEs for the order moments follow similarly.

The local expectation value $\langle S_i I_j\rangle(t)$ is defined by

$$\langle S_i I_j\rangle(t) = \sum_{\text{SIR}} S_i I_j \; p(S_1, I_1, R_1, S_2, \dots, R_N, t),$$

where $\sum_{\text{SIR}}$ denotes the sum $\sum_{S_1=0}^{1} \sum_{I_1=0}^{1} \sum_{R_1=0}^{1} \sum_{S_2=0}^{1} \cdots \sum_{R_N=0}^{1}$, and its dynamics is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle S_i I_j\rangle &= \sum_{\text{SIR}} \; S_i I_j \; \frac{\mathrm{d}}{\mathrm{d}t} p(S_1, I_1, R_1, S_2, \dots, R_N) \\ &= A + B + C + D + E, \end{aligned} \tag{17}$$

with

$$\begin{aligned}
A &= \sum_{\text{SIR}} \; S_i I_j \; \sum_{l=1}^{N} \beta \left(\sum_{k=1}^{N} J_{lk} I_k \right) (1 - S_l) \; p(\dots, 1 - S_l, 1 - I_l, R_l, \dots) \\
B &= \sum_{\text{SIR}} \; S_i I_j \; \sum_{l=1}^{N} \gamma \; (1 - I_l) \; p(\dots, S_l, 1 - I_l, 1 - R_l, \dots) \\
C &= \sum_{\text{SIR}} \; S_i I_j \; \sum_{l=1}^{N} \tilde{\beta} \left(\sum_{k=1}^{N} J_{lk} I_k \right) (1 - R_l) \; p(\dots, S_l, 1 - I_l, 1 - R_l, \dots) \\
D &= \sum_{\text{SIR}} \; S_i I_j \; \sum_{l=1}^{N} \alpha \; (1 - R_l) \; p(\dots, 1 - S_l, I_l, 1 - R_l, \dots) \\
E &= -\sum_{\text{SIR}} \; S_i I_j \; \sum_{l=1}^{N} \left[\beta \left(\sum_{k=1}^{N} J_{lk} I_k \right) S_l + \gamma I_l + \tilde{\beta} \left(\sum_{k=1}^{N} J_{lk} I_k \right) R_l + \alpha R_l \right] \\
&\quad \cdot \; p(\dots, S_l, I_l, R_l, \dots).
\end{aligned}$$

Hence, making a change of variables as in Equation (10) we have $A = A_1 + A_2 + A_3$, where

$$\begin{aligned}
A_1 &= \sum_{\text{SIR}} \; S_i I_j \sum_{l=1 \wedge l \neq i \wedge l \neq j}^{N} \beta \left(\sum_{k=1}^{N} J_{lk} I_k \right) S_l \; p(\dots, S_l, I_l, R_l, \dots), \\
A_2 &= \sum_{\text{SIR}} \; (1 - S_i) \, I_j \; \beta \left(\sum_{k=1}^{N} J_{ik} I_k \right) S_i \; p(\dots, S_i, I_i, R_i, \dots), \\
A_3 &= \sum_{\text{SIR}} \; S_i (1 - I_j) \; \beta \left(\sum_{k=1}^{N} J_{jk} I_k \right) S_j \; p(\dots, S_j, I_j, R_j, \dots).
\end{aligned}$$

Similarly, we have $B = B_1 + B_2 + B_3$, where

$$\begin{aligned}
B_1 &= \sum_{\text{SIR}} \; S_i I_j \sum_{l=1 \wedge l \neq i \wedge l \neq j}^{N} \gamma \; I_l \; p(\dots, S_l, I_l, R_l, \dots), \\
B_2 &= \sum_{\text{SIR}} \; S_i I_j \; \gamma \; I_i \; p(\dots, S_i, I_i, R_i, \dots), \\
B_3 &= \sum_{\text{SIR}} \; S_i (1 - I_j) \; \gamma \; I_j \; p(\dots, S_j, I_j, R_j, \dots),
\end{aligned}$$

$C = C_1 + C_2 + C_3$, where

$$C_1 = \sum_{\text{SIR}} S_i I_j \sum_{l=1 \wedge l \neq i \wedge l \neq j}^{N} \tilde{\beta} \left(\sum_{k=1}^{N} J_{lk} I_k \right) R_l \; p(\ldots, S_l, I_l, R_l, \ldots),$$

$$C_2 = \sum_{\text{SIR}} S_i I_j \; \tilde{\beta} \left(\sum_{k=1}^{N} J_{ik} I_k \right) R_i \; p(\ldots, S_i, I_i, R_i, \ldots),$$

$$C_3 = \sum_{\text{SIR}} S_i \left(1 - I_j\right) \; \tilde{\beta} \left(\sum_{k=1}^{N} J_{jk} I_k \right) R_j \; p(\ldots, S_j, I_j, R_j, \ldots),$$

$D = D_1 + D_2 + D_3$, where

$$D_1 = \sum_{\text{SIR}} S_i I_j \sum_{l=1 \wedge l \neq i \wedge l \neq j}^{N} \alpha \; R_l \; p(\ldots, S_l, I_l, R_l, \ldots),$$

$$D_2 = \sum_{\text{SIR}} (1 - S_i) I_j \; \alpha \; R_i \; p(\ldots, S_i, I_i, R_i, \ldots),$$

$$D_3 = \sum_{\text{SIR}} S_i I_j \; \alpha \; R_j \; p(\ldots, S_j, I_j, R_j, \ldots)$$

and $E = E_1 + E_2 + E_3$, where

$$E_1 = -\sum_{\text{SIR}} S_i I_j \sum_{l=1 \wedge l \neq i \wedge l \neq j}^{N} \left[\beta \left(\sum_{k=1}^{N} J_{lk} I_k \right) S_l + \gamma I_l + \tilde{\beta} \left(\sum_{k=1}^{N} J_{lk} I_k \right) R_l + \alpha R_l \right] \cdot p(\ldots, S_l, I_l, R_l, \ldots),$$

$$E_2 = -\sum_{\text{SIR}} S_i I_j \left[\beta \left(\sum_{k=1}^{N} J_{ik} I_k \right) S_i + \gamma I_i + \tilde{\beta} \left(\sum_{k=1}^{N} J_{ik} I_k \right) R_i + \alpha R_i \right] \cdot p(\ldots, S_i, I_i, R_i, \ldots),$$

$$E_3 = -\sum_{\text{SIR}} S_i I_j \left[\beta \left(\sum_{k=1}^{N} J_{jk} I_k \right) S_j + \gamma I_j + \tilde{\beta} \left(\sum_{k=1}^{N} J_{jk} I_k \right) R_j + \alpha R_j \right] \cdot p(\ldots, S_j, I_j, R_j, \ldots).$$

We start to note that $A_1 + B_1 + C_1 + D_1 + E_1 = 0$. Observing that the state variables are in the set $\{0; 1\}$ and therefore, e.g., $(1 - S_i) \cdot S_i = 0$, we have that $A_2 = 0$ and $B_3 = 0$. We also observe that, e.g., $S_i \cdot I_i = 0$, because one individual cannot stay in more than one state. Hence, $B_2 = 0$, $C_2 = 0$, $D_3 = 0$,

$$\begin{aligned} E_2 &= -\sum_{\text{SIR}} S_i^2 I_j \; \beta \left(\sum_{k=1}^{N} J_{ik} I_k \right) \cdot p(\ldots, S_i, I_i, R_i, \ldots) \\ &= -\beta \sum_{k=1}^{N} J_{ik} \sum_{\text{SIR}} S_i I_j I_k \cdot p(\ldots, S_i, I_i, R_i, \ldots) \\ &= -\beta \sum_{k=1}^{N} J_{ik} \, \langle S_i I_j I_k \rangle, \end{aligned} \quad (18)$$

and

$$\begin{aligned}
E_3 &= -\sum_{\text{SIR}} S_i I_j^2 \; \gamma \cdot p(\ldots, S_j, I_j, R_j, \ldots) \\
&= -\gamma \sum_{\text{SIR}} S_i I_j \cdot p(\ldots, S_j, I_j, R_j, \ldots) \\
&= -\gamma \; \langle S_i I_j \rangle. \qquad (19)
\end{aligned}$$

Therefore, Equation (17) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle S_i I_j \rangle = A_3 + C_3 + D_2 + E_2 + E_3. \qquad (20)$$

Observing that $S_i(1 - I_j)S_j = S_i S_j - S_i I_j S_j = S_i S_j$, we obtain that

$$\begin{aligned}
A_3 &= \beta \sum_{\text{SIR}} S_i S_j \left(\sum_{k=1}^{N} J_{jk} I_k \right) \; p(\ldots, S_j, I_j, R_j, \ldots) \\
&= \beta \sum_{k=1}^{N} J_{jk} \sum_{\text{SIR}} S_i S_j I_k \; p(\ldots, S_j, I_j, R_j, \ldots) \\
&= \beta \sum_{k=1}^{N} J_{jk} \langle S_i S_j I_k \rangle. \qquad (21)
\end{aligned}$$

Similarly, since $S_i(1 - I_j)R_j = S_i R_j$, we have

$$\begin{aligned}
C_3 &= \sum_{\text{SIR}} S_i R_j \; \tilde{\beta} \left(\sum_{k=1}^{N} J_{jk} I_k \right) \; p(\ldots, S_j, I_j, R_j, \ldots) \\
&= \tilde{\beta} \sum_{k=1}^{N} J_{jk} \sum_{\text{SIR}} S_i R_j I_k \; p(\ldots, S_j, I_j, R_j, \ldots) \\
&= \tilde{\beta} \sum_{k=1}^{N} J_{jk} \langle S_i R_j I_k \rangle, \qquad (22)
\end{aligned}$$

and with the relation $(1 - S_i) I_j R_i = R_i I_j$, we also have

$$\begin{aligned}
D_2 &= \sum_{\text{SIR}} R_i I_j \; p(\ldots, S_i, I_i, R_i, \ldots) \\
&= \alpha \; \langle R_i I_j \rangle. \qquad (23)
\end{aligned}$$

Applying Equations (18), (19), (21)–(23) in the dynamics of $\langle S_i I_j \rangle$ given in Equation (20), we obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \langle S_i I_j \rangle = {} & \beta \sum_{k=1}^{N} J_{jk} \; \langle S_i S_j I_k \rangle + \tilde{\beta} \sum_{k=1}^{N} J_{jk} \langle S_i R_j I_k \rangle + \alpha \langle R_i I_j \rangle \\
& - \beta \sum_{k=1}^{N} J_{ik} \; \langle S_i I_j I_k \rangle - \gamma \; \langle S_i I_j \rangle.
\end{aligned}$$

Hence, by Equation (7) we obtain for the dynamics of the pair $\langle SI\rangle$ the ODE

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle SI\rangle &= \sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\langle S_i I_j\rangle \\ &= \beta \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} J_{ij}J_{jk}\,\langle S_i S_j I_k\rangle + \tilde{\beta}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} J_{ij}J_{jk}\,\langle S_i R_j I_k\rangle \\ &\quad + \alpha \sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\,\langle R_i I_j\rangle - \beta \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} J_{ij}J_{ik}\,\langle S_i I_j I_k\rangle - \gamma \sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\,\langle S_i I_j\rangle \\ &= \beta\,\langle SSI\rangle + \tilde{\beta}\,\langle SRI\rangle + \alpha\langle RI\rangle - \beta\,\langle ISI\rangle - \gamma\langle SI\rangle,\end{aligned}$$

where the triples appear, e.g.,

$$\begin{aligned}\langle SRI\rangle(t) &= \sum_{\mathrm{SIR}}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} J_{ij}J_{jk}\; S_i R_j I_k\right)\; p(S_1, I_1, R_1, \dots, R_N, t) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} J_{ij}J_{jk}\,\langle S_i R_j I_k\rangle,\end{aligned}$$

and $\langle I_i S_j I_k\rangle$ is the local expectation value.

Doing similar computations for the others pairs, we obtain for the dynamics of the second moments the following ODEs:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle SS\rangle &= 2\alpha\langle RS\rangle - 2\beta\,\langle SSI\rangle,\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle II\rangle &= 2\beta\,\langle ISI\rangle - 2\gamma\langle II\rangle + 2\tilde{\beta}\,\langle IRI\rangle,\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle RR\rangle &= 2\gamma\langle IR\rangle - 2\tilde{\beta}\,\langle RRI\rangle - 2\alpha\langle RR\rangle,\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle SI\rangle &= \beta\,\langle SSI\rangle + \tilde{\beta}\,\langle SRI\rangle - \gamma\langle SI\rangle - \beta\,\langle ISI\rangle + \alpha\langle RI\rangle,\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle RS\rangle &= \gamma\langle SI\rangle - \beta\,\langle RSI\rangle - \tilde{\beta}\,\langle SRI\rangle + \alpha\langle RR\rangle - \alpha\langle RS\rangle,\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle RI\rangle &= \gamma\langle II\rangle + \beta\,\langle RSI\rangle + \tilde{\beta}\,\langle RRI\rangle - \gamma\langle IR\rangle - \tilde{\beta}\,\langle IRI\rangle - \alpha\langle RI\rangle.\end{aligned} \tag{24}$$

2.3. *Balance equations for means and pairs*

As in [10], we explain how to use the balance equations to reduce the number of equations in the ODEs for the first moments (see Equation (16)) and second moments (see Equation (24)) to the five equations presented in Equations (1)–(5). The balance equations are also used in Section 2.4 to find a closed form of the ODEs for the moments.

From $S_i + I_i + R_i = 1$, it follows immediately that for the means

$$\langle S\rangle + \langle I\rangle + \langle R\rangle = N \tag{25}$$

holds, and from this that

$$\frac{\mathrm{d}}{\mathrm{d}t}N = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle S\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\langle I\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\langle R\rangle \tag{26}$$

also holds. A check of the results of the dynamics Equation (16) is to insert the three equations into Equation (26) and verify the sum to be equal to zero. In this case, it can be confirmed by eye immediately. For the pair dynamics in all variables S, I and R, however, the check of the balance is not so obvious. The balance equation is now, again for regular lattices,

$$\langle SS\rangle + \langle II\rangle + \langle RR\rangle + 2\langle SI\rangle + 2\langle SR\rangle + 2\langle IR\rangle = N \cdot Q, \tag{27}$$

which can be obtained by explicitly expressing all terms including variable S in terms of the independent variables I and R, hence

$$\langle SR\rangle + \langle IR\rangle + \langle RR\rangle = Q\langle R\rangle, \tag{28}$$

etc. The pair balance dynamics is now

$$\frac{\mathrm{d}}{\mathrm{d}t}(N \cdot Q) = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\langle SS\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\langle II\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\langle RR\rangle + 2\frac{\mathrm{d}}{\mathrm{d}t}\langle SI\rangle + 2\frac{\mathrm{d}}{\mathrm{d}t}\langle SR\rangle + 2\frac{\mathrm{d}}{\mathrm{d}t}\langle IR\rangle, \tag{29}$$

which is exactly fulfilled by the ODE system for the pair dynamics given in Equation (24). From these balance equations, we can reduce the ODE system for total expectation values and for pair expectation values to five independent variables $\langle I\rangle$, $\langle R\rangle$, $\langle SI\rangle$, $\langle RI\rangle$ and $\langle SR\rangle$.

2.4. *Pair approximation*

As in [10], to obtain approximate expressions for the triples appearing in the equation system (24) in terms of pairs we will do some assumptions in the model. For a detailed discussion of the pair approximation in general, see [6–8]. We will study the pair approximation for the triples $\langle RSI\rangle$ and $\langle IRI\rangle$. The approximations for the other triples follow similarly.

We consider only the true triples $\widetilde{\langle IRI\rangle}$ in which the last site of, e.g., infected is not identical to the first; hence with the definition

$$\widetilde{\langle IRI\rangle} = \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1,k\neq i}^{N} J_{ij}J_{jk}\langle I_i R_j I_k\rangle, \tag{30}$$

we have

$$\langle IRI\rangle = \widetilde{\langle IRI\rangle} + \langle RI\rangle \tag{31}$$

when the local variable at site k, here I_k, is of the same type as the one in i, here I_i and simply

$$\langle RSI\rangle = \widetilde{\langle RSI\rangle} \tag{32}$$

when the local variable at site k, now I_k, is different from the one in i, now R_i. For triples, which are by nature just pairs, i.e., with $i = k$, we have locally $\langle I_i R_j I_k\rangle = \langle I_i^2 R_j\rangle = \langle I_i R_j\rangle$, since $I_i \in \{0, 1\}$, so they should be counted as pairs, i.e., given by Equation (31), whereas in

Equation (32) for $i = k$ we have $\langle R_i S_j I_k \rangle = \langle R_i S_j I_i \rangle = 0$, since $R_i, I_i \in \{0, 1\}$ and $S_i + I_i + R_i = 1$. The difference between $\widetilde{\langle IRI \rangle}$ and $\langle IRI \rangle$ does first appear in the triples, since in the pairs the diagonal of the adjacency matrix is zero, avoiding the eventual double counting of singlets.

Then we consider all the possible combinations, where sums over the adjacency matrix only come to play, $\sum_{j=1}^{N} J_{ij} = Q_i$. These indicate the number of neighbours to a lattice site i, and from now on we will only consider regular lattices (later the square lattice with periodic boundary conditions). Hence we can assume that all Q_i are equal, i.e., $Q = Q_i$ for all i.

The pair approximation yields

$$\langle RSI \rangle = \widetilde{\langle RSI \rangle} \approx \frac{Q-1}{Q} \cdot \frac{\langle RS \rangle \cdot \langle SI \rangle}{\langle S \rangle}, \tag{33}$$

obtained from an analogue for the Bayesian formula for conditional probabilities applied to the local expectation values

$$\langle R_i S_j I_k \rangle \approx \frac{\langle R_i S_j \rangle \cdot \langle S_j I_k \rangle}{\langle S_j \rangle} \tag{34}$$

and a spatial homogeneity argument, namely

$$\langle S_j I_k \rangle \approx \langle S_i I_j \rangle \approx \frac{\langle SI \rangle}{NQ}, \tag{35}$$

and

$$\langle S_j \rangle \approx \frac{\langle S \rangle}{N}. \tag{36}$$

For the triple $\langle IRI \rangle$, the pair approximation is given by

$$\langle IRI \rangle = \widetilde{\langle IRI \rangle} + \langle RI \rangle \approx \frac{Q-1}{Q} \cdot \frac{\langle RI \rangle^2}{\langle R \rangle} + \langle RI \rangle. \tag{37}$$

With expressions like the one in Equations (33) and (37) and using the balance equations, we obtain the closed equation system for the dynamics of the moments presented in Equations (1)–(5).

3. Analytic expression of the phase transition line

Let

$$E = \alpha + \gamma Q + \tilde{\beta}, \tag{38}$$

$$F = D + \sqrt{D^2 + 4\alpha(Q-1)E}, \tag{39}$$

where

$$D = \gamma Q - \tilde{\beta}(Q-1) - \alpha(Q-2). \tag{40}$$

We observe for the transition line between no-growth and annular growth that the stationary value $\langle I \rangle^*$ and also the stationary value $\langle R \rangle^*$ tend to zero, but their ratio stays finite. Hence, we conclude the following lemma:

LEMMA 3.1 *The scaling limit of* $\langle R \rangle^* / \langle I \rangle^*$ *when* $\langle I \rangle^*$ *tends to zero is given by*

$$\lim_{\langle I \rangle^* \to 0} \frac{\langle R \rangle^*}{\langle I \rangle^*} = \frac{\gamma F}{2\alpha E}. \tag{41}$$

Proof We consider the equilibrium manifold of the ODE system given by Equations (1)–(5). We use Equations (1), (2) and (5) to compute $\langle SI\rangle^*$, $\langle RI\rangle^*$ and $\langle SR\rangle^*$, and we replace their values in Equations (3) and (4) giving the following two implicit equations:

$$Q\langle I\rangle^* - \frac{\widetilde{\beta}+2\gamma}{\widetilde{\beta}}\left(\langle I\rangle^* - \frac{\alpha}{\gamma}\langle R\rangle^*\right) + \left((Q-1)\langle I\rangle^* + \frac{\alpha}{\gamma}\langle R\rangle^*\right)$$
$$\cdot\left(1 - 2\frac{N-\langle I\rangle^* - \langle R\rangle^* - \alpha/\beta Q\langle R\rangle^*}{Q/Q-1\,(N-\langle I\rangle^*-\langle R\rangle^*)+\langle R\rangle^*} - \frac{2}{\widetilde{\beta}Q}\frac{\gamma\langle I\rangle^* - \alpha\langle R\rangle^*}{\langle R\rangle^*}\right) = 0 \tag{42}$$

and

$$Q\langle I\rangle^* - \frac{\alpha\,(\beta+2\gamma)}{\beta\gamma}\langle R\rangle^* - \frac{\widetilde{\beta}+2\gamma}{\widetilde{\beta}}\left(\langle I\rangle^* - \frac{\alpha}{\gamma}\langle R\rangle^*\right)$$
$$+\frac{\alpha}{\gamma}(Q-1)\langle R\rangle^*\left(1 - \frac{2\alpha\langle R\rangle^*}{\beta Q\,(N-\langle I\rangle^*-\langle R\rangle^*)}\right) \tag{43}$$
$$+(Q-1)\left(\langle I\rangle^* - \frac{\alpha}{\gamma}\langle R\rangle^*\right)\left(1 - \frac{2\,(\gamma\langle I\rangle^* - \alpha\langle R\rangle^*)}{\widetilde{\beta}Q\langle R\rangle^*}\right) = 0.$$

We start proving that if $\langle I\rangle^*$ tends to zero, then $\langle R\rangle^*$ also converges to zero. We observe that the function in Equation (42) is continuous at $\langle I\rangle^* = 0$, and its value is

$$\frac{\widetilde{\beta}+2\gamma}{\widetilde{\beta}}\frac{\alpha}{\gamma}\langle R\rangle^* + \frac{\alpha}{\gamma}\langle R\rangle^*\left(1 - 2\frac{N-\langle R\rangle^* - (\alpha/\beta Q)\langle R\rangle^*}{(Q/(Q-1))(N-\langle R\rangle^*)+\langle R\rangle^*} + \frac{2\alpha}{\widetilde{\beta}Q}\right) = 0. \tag{44}$$

Hence, we obtain that $\langle R\rangle^* = 0$ or

$$\langle R\rangle^* = -\frac{\beta Q\left(\widetilde{\beta}+\gamma Q+\alpha\right)}{\beta\widetilde{\beta}Q(Q-2)+\widetilde{\beta}\alpha(Q-1)-\beta(\gamma Q+\alpha)}. \tag{45}$$

But the value presented in Equation (45) is not a solution of Equation (43) for $\langle I\rangle^* = 0$. Hence, the stationary state $\langle R\rangle^*$ converges to zero when $\langle I\rangle^*$ tends to zero. Let

$$\begin{aligned} N_{1,2} &= -\alpha\,\tilde{\beta}\,\gamma\,Q,\\ N_{0,3} &= \tilde{\beta}\,\alpha\,\left(-\gamma Q + \alpha(Q-1)\right),\\ N_{0,2} &= \alpha\,\tilde{\beta}\,\gamma\,NQ, \end{aligned} \tag{46}$$

and

$$\begin{aligned} D_{3,0} &= \gamma^2(Q-1),\\ D_{2,1} &= -\tilde{\beta}\gamma\,Q(Q-1) - 2\,\alpha\gamma\,(Q-1) + 2\gamma^2(Q-1),\\ D_{1,2} &= \alpha^2(Q-1) - \tilde{\beta}\gamma\,Q(Q-1) - 3\,\alpha\gamma\,Q + 2\,\alpha\gamma + \gamma^2 Q,\\ D_{0,3} &= \alpha^2(Q-1) - \alpha\gamma\,Q,\\ D_{2,0} &= -\gamma^2 N(Q-1),\\ D_{1,1} &= N\gamma(2\,\alpha\,(Q-1) + \tilde{\beta}\,Q(Q-1) - \gamma Q),\\ D_{0,2} &= N\alpha\,(\gamma Q - \alpha(Q-1)). \end{aligned} \tag{47}$$

Solving Equation (43) in order to isolate the parameter β, we obtain that

$$\beta(\tilde{\beta}) = \frac{N_\beta(\langle I\rangle^*, \langle R\rangle^*, \tilde{\beta})}{D_\beta(\langle I\rangle^*, \langle R\rangle^*, \tilde{\beta})}, \tag{48}$$

where N_β is given by

$$N_\beta(\langle I\rangle^*, \langle R\rangle^*, \tilde{\beta}) = N_{1,2}\langle I\rangle^*\langle R\rangle^{*2} + N_{0,3}\langle R\rangle^{*3} + N_{0,2}\langle R\rangle^{*2}, \tag{49}$$

and D_β is given by

$$\begin{aligned} D_\beta(\langle I\rangle^*, \langle R\rangle^*, \tilde{\beta}) &= D_{3,0}\langle I\rangle^{*3} + D_{2,1}\langle I\rangle^{*2}\langle R\rangle^* + D_{1,2}\langle I\rangle^*\langle R\rangle^{*2} \\ &\quad + D_{0,3}\langle R\rangle^{*3} + D_{2,0}\langle I\rangle^{*2} + D_{1,1}\langle I\rangle^*\langle R\rangle^* + D_{0,2}\langle R\rangle^{*2}. \end{aligned} \tag{50}$$

Substituting in Equation (42) the expression for β given in Equation (48), we obtain

$$\frac{N(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta})}{D(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta})} = 0, \tag{51}$$

where the denominator is given by

$$\begin{aligned} D(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta}) &= \tilde{\beta} Q\langle R\rangle^* \left(\gamma QN - \langle R\rangle^*(\gamma Q - \alpha(Q-1)) - \gamma Q\langle I\rangle^*\right) \\ &\quad \cdot (\langle R\rangle^* - Q(N - \langle I\rangle^*)), \end{aligned} \tag{52}$$

and the numerator is given by

$$\begin{aligned} N(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta}) &= C_{4,0}\langle I\rangle^{*4} + C_{3,1}\langle I\rangle^{*3}\langle R\rangle^* + C_{2,2}\langle I\rangle^{*2}\langle R\rangle^{*2} + C_{1,3}\langle I\rangle^*\langle R\rangle^{*3} \\ &\quad + C_{0,4}\langle R\rangle^{*4} + C_{3,0}\langle I\rangle^{*3} + C_{2,1}\langle I\rangle^{*2}\langle R\rangle^* + C_{1,2}\langle I\rangle^*\langle R\rangle^{*2} \\ &\quad + C_{0,3}\langle R\rangle^{*3} + C\langle I\rangle^{*2} + B\langle I\rangle^*\langle R\rangle^* + A\langle R\rangle^{*2}, \end{aligned} \tag{53}$$

with

$$A = -2\alpha N^2 Q^2(\alpha + \gamma Q + \tilde{\beta}), \tag{54}$$

$$B = 2\gamma N^2 Q^2(\gamma Q - \tilde{\beta}(Q-1) - \alpha(Q-2)), \tag{55}$$

$$C = 2\gamma^2 N^2 Q^2(Q-1). \tag{56}$$

The other coefficients $C_{i,j}$ of the numerator are not presented here, because we will not use them in the future computations. We are going to find the limit of the ratio $\langle R\rangle^*/\langle I\rangle^*$, when $\langle I\rangle^*$ tends to zero, such that

$$N(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta}) = 0 \tag{57}$$

is satisfied. Dividing Equation (57) by $\langle I\rangle^{*2}$ and furthermore defining the ratio of recovered over infected $\langle V\rangle^* = \langle R\rangle^*/\langle I\rangle^*$, we obtain that

$$\begin{aligned} &C_{4,0}\langle I\rangle^{*2} + C_{3,1}\langle I\rangle^*\langle R\rangle^* + C_{2,2}\langle R\rangle^{*2} + C_{1,3}\langle V\rangle^*\langle R\rangle^{*2} + C_{0,4}\langle V\rangle^{*2}\langle R\rangle^{*2} \\ &\quad + C_{3,0}\langle I\rangle^* + C_{2,1}\langle R\rangle^* + C_{1,2}\langle V\rangle^*\langle R\rangle^* + C_{0,3}\langle V\rangle^{*2}\langle R\rangle^* \\ &\quad + C + B\langle V\rangle^* + A\langle V\rangle^{*2} = 0. \end{aligned} \tag{58}$$

When $\langle I \rangle^*$ tends to zero, we already proved that $\langle R \rangle^*$ converges to zero. Hence, from Equation (58), we obtain

$$A\langle V \rangle^{*2} + B\langle V \rangle^* + C = 0, \tag{59}$$

in the limiting case when $\langle I \rangle^*$ tends to 0. Therefore, there are two solutions $\langle V \rangle^*_{1,2}$ for $\langle V \rangle^*$ given by

$$\langle V \rangle^*_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{60}$$

Since $C = 2\gamma^2 N^2 Q^2 (Q-1) > 0$ and $A = -2\alpha N^2 Q^2 (\alpha + \gamma Q + \tilde{\beta}) < 0$, we conclude that $-4AC > 0$ and so $B^2 - 4AC > B^2$. Hence, Equation (59) has a unique positive solution

$$\langle V \rangle^* = \frac{-B - \sqrt{B^2 - 4\,A\,C}}{2A}. \tag{61}$$

Inserting into Equation (61) the expressions of A, B and C presented in Equations (54)–(56), we obtain Equation (41). ■

Now we will use the value of the ratio $\langle R \rangle^*/\langle I \rangle^*$ at criticality to obtain the analytic expression of the phase transition line.

Let

$$G(\tilde{\beta}) = \gamma \tilde{\beta} Q \cdot F^2, \tag{62}$$

and

$$H(\tilde{\beta}) = 2(2\alpha(Q-1) + \tilde{\beta} Q(Q-1) - \gamma Q) \cdot E \cdot F + (\gamma Q - \alpha(Q-1)) \cdot F^2 - 4\alpha(Q-1) \cdot E^2, \tag{63}$$

where E and F are defined in Equations (38) and (39), respectively.

THEOREM 3.2 *Let* $\alpha > 0$. *The phase transition line* $\beta(\tilde{\beta}) = \beta_c(\tilde{\beta}, \alpha, \gamma, Q, N)$ *between no-growth and annular growth for the spatial epidemic SIRI model in pair approximation is given by*

$$\beta(\tilde{\beta}) = \frac{G(\tilde{\beta})}{H(\tilde{\beta})}, \tag{64}$$

with $0 \le \tilde{\beta} \le \gamma/(Q-1)$.

Proof We observe that Equation (48) can be rewritten in terms of $\langle I \rangle^*$, $\langle R \rangle^*$ and $\langle V \rangle^* = \langle R \rangle^*/\langle I \rangle^*$ as follows:

$$\beta(\tilde{\beta}) = \frac{L_1\ \langle R \rangle^* + N_{0,2}\langle V \rangle^{*2}}{D_{3,0}\langle I \rangle^* + L_2\langle R \rangle^* + D_{2,0} + D_{1,1}\langle V \rangle^* + D_{0,2}\langle V \rangle^{*2}}, \tag{65}$$

where

$$L_1 = N_{1,2}\langle V \rangle^* + N_{0,3}\langle V \rangle^{*2}, \tag{66}$$

$$L_2 = D_{2,1} + D_{1,2}\langle V \rangle^* + D_{0,3}\langle V \rangle^{*2}, \tag{67}$$

and the coefficients $N_{i,j}$ and $D_{i,j}$ are presented in Equations (46) and (47), respectively. The phase transition curve follows from Equation (65) by letting $\langle I \rangle^*$ tends to zero. Under this limit, Equation (65) reduces to

$$\beta(\tilde{\beta}) = \frac{N_{0,2}\langle V\rangle^{*2}}{D_{2,0} + D_{1,1}\langle V\rangle^{*} + D_{0,2}\langle V\rangle^{*2}}. \tag{68}$$

Hence, the numerator of Equation (68) is given by

$$\begin{aligned} N_{0,2}\langle V\rangle^{*2} &= \alpha\,\tilde{\beta}\,\gamma\,NQ\frac{\gamma^2 F^2}{4\,\alpha^2 E^2} \\ &= \tilde{\beta}\,\gamma^3\,NQ\frac{F^2}{4\,\alpha E^2}, \end{aligned} \tag{69}$$

and the denominator is given by

$$\begin{aligned} &-\gamma^2 N(Q-1) + N\gamma(2\,\alpha\,(Q-1) + \tilde{\beta}\,Q(Q-1) - \gamma\,Q)\frac{\gamma\,F}{2\,\alpha E} \\ &\quad + N\alpha\,(\gamma\,Q - \alpha(Q-1))\frac{\gamma^2\,F^2}{4\,\alpha^2 E^2} \\ &= \gamma^2 N\left(-(Q-1) + (2\,\alpha\,(Q-1) + \tilde{\beta}\,Q(Q-1) - \gamma\,Q)\frac{F}{2\,\alpha E}\right. \\ &\quad \left. + (\gamma\,Q - \alpha(Q-1))\frac{F^2}{4\,\alpha E^2}\right). \end{aligned} \tag{70}$$

Dividing Equation (69) by Equation (70), we obtain the explicit formula for the phase transition curve of the SIRI model, which can be written as given in Equation (64). ■

This completes the expression for the critical curve $\beta(\tilde{\beta})$ for the general α and γ case. When α tend to 0, we obtain the following expression for $\beta(\tilde{\beta})$:

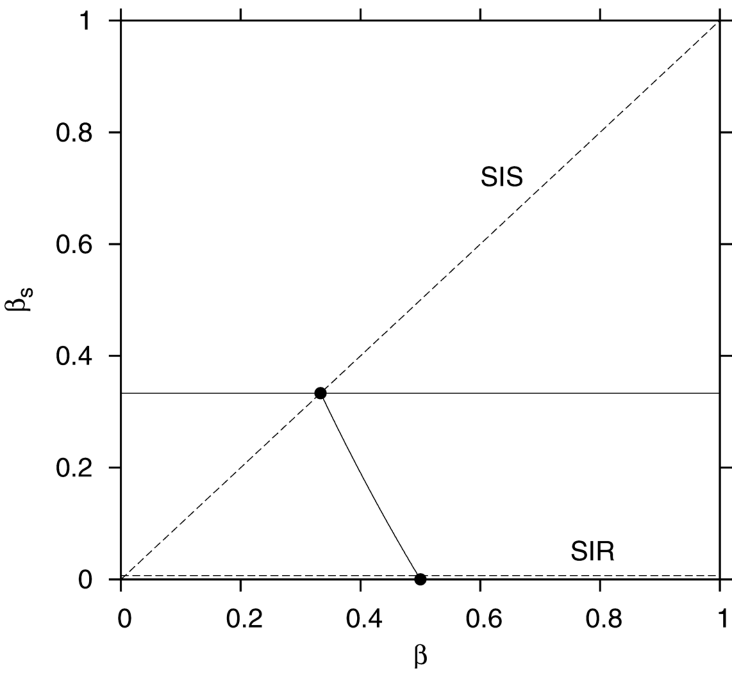

Figure 1. The phase transition line between no-growth and annular growth determined from the analytic solution in the limiting case when α tends to zero, which is explicitly given in Equation (71). The horizontal transition line of the SIRI limiting case when $\alpha = 0$ and the phase transition points of SIS and SIR limiting cases are also presented as calculated in [10]. The SIS and SIR limiting cases are given by dashed lines. (Parameters $Q = 4$ appropriate for spatial two-dimensional systems and $\gamma = 1$ were used.)

COROLLARY 3.3 *In the limit when α tends to zero, the phase transition line between no-growth and annular growth $\beta(\tilde{\beta})$ for the spatial epidemic SIRI model is given by*

$$\lim_{\alpha\to 0}\beta(\tilde{\beta}) = \frac{\gamma^2 Q - \gamma\tilde{\beta}(Q-1)}{\gamma Q(Q-2) + \tilde{\beta}(Q-1)}. \tag{71}$$

In Figure 1, we show the horizontal transition line corresponding in the left-hand side to transition from no-growth to compact growth and in the right-hand side to transition from annular growth to compact growth [10], and the obliqual line is the phase transition between no-growth and annular growth as determined in Corollary 3.3. The intersection of these two lines is the phase transition for the SIS model and the intersection of the obliqual line with the horizontal axis is the phase transition line for the SIR model.

4. Summary

We have computed the analytic expression of the phase transition line between no-growth and annular growth for the spatial reinfection SIRI model using pair approximation for the moments derived from the master equation. We have introduced a scaling argument that allowed us to determine analytically an explicit formula for the phase transition line between no-growth and annular growth. This scaling argument consisted in computing the ratio $\langle R\rangle^*/\langle I\rangle^*$ of the average of the recovered individuals $\langle R\rangle^*$ against the infected individuals $\langle I\rangle^*$.

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