JUNE 27–29, 2022
ISEP · PORTO, PORTUGAL

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Bibliographic Information

Book Title

International Conference on Mathematical Analysis and Applications in Science and Engineering – Book of Extended Abstracts

Series Editors

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Publication Year: 2022

Publisher

ISEP | P.PORTO

Book ISBN & DOI

978-989-53496-3-0

https://doi.org/10.34630/20734

Conference Name

ICMA^{2}SC'2022

Dates & Location

June 27th-29th 2022, School of Engineering, Polytechnic of Porto, Porto, Portugal

International Conference on Mathematical Analysis and Applications in Science and Engineering
ICMA^{2}SC'22
ISEP Porto-Portugal, June 27 - July 1, 2022

On a regular ψ-fractional Sturm-Liouville problem

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Abstract: In this short paper, we consider a ψ-fractional Sturm-Liouville eigenvalue problem by using left ψ-Caputo and right ψ-Riemann-Liouville fractional derivatives. We study the main properties of the eigenfunctions and the eigenvalues of the associated fractional boundary problem.

keywords: ψ-fractional derivatives; ψ-fractional Sturm-Liouville problem; Eigenvalues; Eigenfunctions.

MSC2010: 34B24; 26A33; 34L10.

1 Introduction

The classical regular Sturm-Liouville (S-L) problem is associated with the real second-order linear ordinary differential equation of the form:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y(x) = \lambda\,w(x)\,y(x),$$

where $p(x), w(x) > 0$, and $p(x), p'(x), q(x), w(x)$ are continuous functions on (a, b). The function $w(x)$, sometimes denoted by $r(x)$, is called the weight or density function. The unknown function $y(x)$ is continuous and differentiable on (a, b). In addition, $y(x)$ is required to satisfy some boundary conditions of the form $c_1 y(a) + c_2 y'(a) = 0$ and $d_1 y(b) + d_2 y'(b) = 0$, with $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. Finding the values of λ, known as eigenvalues, for which there exists a non-trivial solution $y(x)$, called eigenfunction associated to λ, constitutes the classical S-L problem. Fractional versions of this problem were already considered in the literature (see e.g. [3, 4] and references therein). Motivated by the unification given by the ψ-fractional calculus we study a regular ψ-fractional S-L problem.

2 Preliminaries

In this section, we introduce the concepts related to the ψ-fractional calculus necessary to this work (for more details see [5] and references therein).

Definition 1 *(cf. [5]) Let (a,b) be a finite or infinite interval on the real line $\mathbb{R}$ and $\alpha > 0$. Also let ψ be a monotone increasing and positive function on (a,b), having a continuous derivative ψ' in (a,b). The left- and right-sided Riemnann-Liouville (RL) fractional integrals of a function f with respect to another function ψ on (a,b) are given respectively by*

$$I_{a^+}^{\alpha;\psi} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_a^x \psi'\left(t\right) \left(\psi\left(x\right) - \psi\left(t\right)\right)^{\alpha-1} f\left(t\right) \, dt, \qquad x > a, \tag{1}$$

$$I_{b^-}^{\alpha;\psi} f\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)} \int_x^b \psi'\left(t\right) \left(\psi\left(t\right) - \psi\left(x\right)\right)^{\alpha-1} f\left(t\right) \, dt, \qquad x < b. \tag{2}$$

Assuming further that $f, \psi \in C^n\left(a,b\right)$, where $n = [\alpha] + 1$, and $\psi'\left(x\right) \neq 0$, for all $x \in \left(a,b\right)$, the corresponding inverse operators, i.e., the left- and right-sided ψ-RL fractional derivatives of order $\alpha > 0$, are defined respectively by

$$D_{a^+}^{\alpha;\psi} f(x) = \left(\frac{1}{\psi'\left(x\right)} \frac{d}{dx}\right)^n I_{a^+}^{n-\alpha;\psi} f\left(x\right) \quad \text{and} \quad D_{b^-}^{\alpha;\psi} f(x) = \left(-\frac{1}{\psi'\left(x\right)} \frac{d}{dx}\right)^n I_{b^-}^{n-\alpha;\psi} f\left(x\right). \tag{3}$$

Moreover, the left- and right-sided ψ-Caputo fractional derivatives of order $\alpha > 0$, are defined respectively by

$${}^C D_{a^+}^{\alpha;\psi} f(x) = I_{a^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'\left(x\right)} \frac{d}{dx}\right)^n f\left(x\right) \quad \text{and} \quad {}^C D_{b^-}^{\alpha;\psi} f(x) = I_{b^-}^{n-\alpha;\psi} \left(-\frac{1}{\psi'\left(x\right)} \frac{d}{dx}\right)^n f\left(x\right). \tag{4}$$

Using the concepts of ψ-RL and ψ-Caputo fractional derivatives we can encompass in one definition several fractional derivatives known in the literature In [5], the authors listed several fractional derivatives that can be obtained for specific choices of the function ψ such as Caputo, RL, Hadamard, Katugampola, Chen, Jumarie, Prabhakar, Erdélyi-Kober, Weyl, among others.

Proposition 1 *Let $\alpha, \beta > 0$ and $f \in C([a,b])$. Then the following relations hold*

$$\begin{aligned} & I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} = I_{a^+}^{\alpha+\beta;\psi}, && I_{b^-}^{\alpha;\psi} I_{b^-}^{\beta;\psi} = I_{b^-}^{\alpha+\beta;\psi}, \\ & D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} f(x) = f(x), && D_{b^-}^{\alpha;\psi} I_{b^-}^{\alpha;\psi} f(x) = f(x), \\ & {}^C D_{a^+}^{\alpha;\psi} I_{a^+}^{\alpha;\psi} f(x) = f(x), && {}^C D_{b^-}^{\alpha;\psi} I_{b^-}^{\alpha;\psi} f(x) = f(x). \end{aligned}$$

Moreover, for $f \in C^n([a,b])$, $n = [\alpha] + 1$, and putting $f_\psi^{[k]}(x) = \left(\frac{1}{\psi'(x)} \frac{d}{dx}\right)^k f(x)$, we have

$$\begin{aligned} {}^C D_{a^+}^{\alpha;\psi} f(x) &= D_{a^+}^{\alpha;\psi} \left[f(x) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} \left(\psi(x) - \psi(a)\right)^k \right], \\ I_{a^+}^{\alpha;\psi} \, {}^C D_{a^+}^{\alpha;\psi} f(x) &= f(x) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a)}{k!} \left(\psi(x) - \psi(a)\right)^k. \end{aligned}$$

Proposition 2 *For $\alpha \in (0,1)$, the fractional differential operators (3) and (4) satisfy the following ψ-fractional integration by parts:*

$$\int_a^b f(x) D_{a^+}^{\alpha;\psi} g(x) \psi'(x) \, dx = \int_a^b {}^C D_{b^-}^{\alpha;\psi} f(x) g(x) \psi'(x) \, dx + f(x) I_{a^+}^{1-\alpha;\psi} g(x) \Big|_{x=a}^{x=b}, \tag{5}$$

$$\int_a^b f(x) D_{b^-}^{\alpha;\psi} g(x) \psi'(x) \, dx = \int_a^b {}^C D_{a^+}^{\alpha;\psi} f(x) g(x) \psi'(x) \, dx - f(x) I_{b^-}^{1-\alpha;\psi} g(x) \Big|_{x=a}^{x=b}. \tag{6}$$

For more properties of these ψ-fractional operators see [5] and references therein.

3 Regular ψ-fractional Sturm-Liouville problem

We consider the following ψ-fractional S-L equation

$$-D_{b^-}^{\alpha;\psi}\left(p\,(x)\ {}^{C}D_{a^+}^{\alpha;\psi}f\,(x)\right)+q(x)f(x)=\lambda\,w(x)\,f\,(x) \tag{7}$$

subject to the following boundary conditions

$$c_1\,f\,(a)+c_2\,I_{b^-}^{1-\alpha;\psi}\left(p(x)\ {}^{C}D_{a^+}^{\alpha;\psi}f(x)\right)\Big|_{x=a}=0,\quad c_1^2+c_2^2\neq 0, \tag{8}$$

$$d_1\,f\,(b)+d_2\,I_{b^-}^{1-\alpha;\psi}\left(p(x)\ {}^{C}D_{a^+}^{\alpha;\psi}f(x)\right)\Big|_{x=b}=0,\quad d_1^2+d_2^2\neq 0, \tag{9}$$

where $\alpha\in(0,1)$, $x\in[a,b]$, $p(x),w(x)>0$, for all $x\in[a,b]$, and p,q,w are real-valued continuous functions in $[a,b]$. Our problem consists in finding the values of λ such that the boundary-value problem (7)-(9) has a non-trivial solution. We denote by $\mathcal{L}^{\alpha;\psi}$ the ψ-fractional S-L operator associated to problem (7) given by

$$\mathcal{L}^{\alpha;\psi}:=-D_{b^-}^{\alpha;\psi}\left(p(x)\,{}^{C}D_{a^+}^{\alpha;\psi}\right)+q(x).$$

As suggested by the ψ-fractional integration by parts of Proposition 2 we will work in the weighted space $L_1([a,b],\psi'(x)dx)$. Using (6) and the boundary conditions (8)-(9) one can prove the following theorem.

Theorem 3 *The eigenvalues of our ψ-fractional problem (7)-(9) are real. Moreover, the eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the sensity function $w(x)$ on $[a,b]$, that is*

$$\int_a^b f_{\lambda_1}(x)f_{\lambda_2}(x)w(x)\,\psi'(x)\,dx,\qquad \lambda_1\neq\lambda_2.$$

Our ψ-fractional S-L equation with boundary conditions can be written in integral form as the next proposition shows.

Proposition 4 *Let $\alpha>1/2$ and f_λ be an eigenfunction associated to the eigenvalue λ. On the space $C([a,b])$, the $\psi-$fractional S-L problem is equivalent to the integral equation*

$$f_\lambda(x)=-I_{a^+}^{\alpha;\psi}\Big(\frac{1}{p\,(x)}\,I_{b^-}^{\alpha;\psi}F_\lambda\,(f)\Big)+A\,(x)\int_a^b F_\lambda\,(f)\,dx+B\,(x)\left(I_{a^+}^{\alpha;\psi}\Big(\frac{1}{p\,(x)}\,I_{b^-}^{\alpha;\psi}F_\lambda\,(f)\Big)\right)\Bigg|_{x=b} \tag{10}$$

where the coefficients $A\,(x)$ and $B\,(x)$ are given by

$$A\,(x)\;=\frac{c_2}{\Delta}\left(d_2+d_1\Big(Y-\;I_{a^+}^{\alpha;\psi}\Big(\frac{(\psi\,(b)-\psi\,(x))^{\alpha-1}}{p\,(x)\;\Gamma\,(\alpha)}\Big)\Big)\right),$$

$$B\,(x)\;=\frac{d_1}{\Delta}\left(c_1\,I_{a^+}^{\alpha;\psi}\Big(\frac{(\psi\,(b)-\psi\,(x))^{\alpha-1}}{p\,(x)\;\Gamma\,(\alpha)}\Big)-c_2\right),$$

with $\Delta=c_1(d_1Y+d_2)-c_2d_1\neq 0$, $Y=I_{a^+}^{\alpha;\psi}\Big(\frac{(\psi(b)-\psi(x))^{\alpha-1}}{p(x)\,\Gamma(\alpha)}\Big)\Big|_{x=b}$, and $F_\lambda(f)=q(x)f_\lambda(x)-\lambda w(x)f_\lambda(x)$.

Sketch of the proof: Using composition rules we can write the equation (7) as follows:

$$D^{\alpha;\psi}_{b^-}\Big(p(x)\ {}^{C}D^{\alpha;\psi}_{a^+}\Big)\Big[f_\lambda(x) + I^{\alpha;\psi}_{a^+}\Big(\frac{1}{p(x)} I^{\alpha;\psi}_{b^-} F_\lambda(f)\Big)\Big] = 0. \tag{11}$$

On the space $C([a,b])$ a general solution of (11) is given by

$$f_\lambda(x) + I^{\alpha;\psi}_{a^+}\Big(\frac{1}{p(x)} I^{\alpha;\psi}_{b^-} F_\lambda(f)\Big) = \xi_1 + \xi_2\, I^{\alpha;\psi}_{a^+}\Big(\frac{(\psi(b)-\psi(x))^{\alpha-1}}{p(x)\,\Gamma(\alpha)}\Big), \quad \xi_1, \xi_2 \in \mathbb{R}. \tag{12}$$

From [3, Eq. (34)] the fractional integral that is multiplied by ξ_2 gives a continuous function on $[a,b]$ only for $1/2 < \alpha < 1$. From (12) and using composition rules we obtain:

$$I^{1-\alpha;\psi}_{b^-}\Big(p(x)\ {}^{C}D^{\alpha;\psi}_{a^+} f_\lambda(x)\Big) = \xi_2 - I^{1;\psi}_{b^-} F_\lambda(f). \tag{13}$$

Finally, from (13) and (12) and using the boundary conditions (8)-(9) we can relate the coefficients ξ_1, ξ_2 with the values $c_j, d_j, j = 1, 2$ in the boundary conditions to obtain (10).

Our final theorem gives the conditions under which the eigenfunctions exist and are unique. The proof follows similar arguments as the proof of [3, Thm. 9].

Theorem 5 *Let $\alpha > 1/2$ and assume that $\Delta \neq 0$. Then our ψ-fractional problem has unique continuous eigenfunctions f_λ if the fixed point condition holds*

$$\|q(x) + \lambda\omega(x)\| < \frac{m_p}{\|\varphi(x)\| + \|B(x)\|\varphi(b) + m_p\|A(x)\|(\psi(b) - \psi(a))},$$

where $\varphi(x) = I^{\alpha;\psi}_{a^+}\Big(\frac{(\psi(b)-\psi(x))^{\alpha-1}}{p(x)\,\Gamma(\alpha)}\Big)$ *and* $m_p = \min_{x\in[a,b]} |p(x)|$.

4 Conclusion

In this work we showed how to use the ψ-fractional calculus to study a ψ-fractional S-L problem.

Acknowledgments

The work of the authors was supported by Portuguese funds through CIDMA-Center for Research and Development in Mathematics and Applications, and FCT–Fundação para a Ciência e a Tecnologia, within projects UIDB/04106/2020 and UIDP/04106/2020. N. Vieira was also supported by FCT via the 2018 FCT program of Stimulus of Scientific Employment - Individual Support (Ref: CEECIND/01131/2018).

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