Hyperbolic Linear Canonical Transforms of Quaternion Signals and Uncertainty *

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Abstract

This paper is concerned with Linear Canonical Transforms (LCTs) associated with two-dimensional quaternion-valued signals defined in an open rectangle of the Euclidean plane endowed with a hyperbolic measure, which we call Quaternion Hyperbolic Linear Canonical Transforms (QHLCTs). These transforms are defined by replacing the Euclidean plane wave with a corresponding hyperbolic relativistic plane wave in one dimension multiplied by quadratic modulations in both the hyperbolic spatial and frequency domains, giving the hyperbolic counterpart of the Euclidean LCTs. We prove the fundamental properties of the partial QHLCTs and the right-sided QHLCT by employing hyperbolic geometry tools and establish main results such as the Riemann-Lebesgue Lemma, the Plancherel and Parseval Theorems, and inversion formulas. The analysis is carried out in terms of novel hyperbolic derivative and hyperbolic primitive concepts, which lead to the differentiation and integration properties of the QHLCTs. The results are applied to establish two quaternionic versions of the Heisenberg uncertainty principle for the right-sided QHLCT. These uncertainty principles prescribe a lower bound on the product of the effective widths of quaternion-valued signals in the hyperbolic spatial and frequency domains. It is shown that only hyperbolic Gaussian quaternion functions minimize the uncertainty relations.

Keywords: Quaternionic Analysis, Quaternion Hyperbolic Linear Canonical Transforms, Plancherel and Parseval Theorems, Riemann-Lebesgue Lemma, Heisenberg uncertainty principles.

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1 Introduction

The Linear Canonical Transform (LCT) is a four-parameter family of linear integral transforms which was first introduced in the 1970s by Collins [4] and Moshinsky et al. [24] (cf. Healy et al. [13] and the bibliography quoted there) and has found many applications in signal processing, design of filters, signal separation, pattern recognition, and optics [11, 12, 27, 29] (see also [28, 33] and further references given in these papers). The flexibility of choice on the four parameters makes the LCT a highly adaptable transformation and, in particular, allows us to recognize that the classical Fourier Transform (FT), the Fractional Fourier transform (FRFT), the Fresnel transform, scaling operations, and multiplication by chirp functions are merely particular cases of the LCT. The LCT has more degrees of freedom and is more flexible than the conventional FT and the FRFT but with similar computational costs as the FT. The LCT is used to analyze and measure chirp signals and obtain sampling theorems for certain types of non-bandlimited signals with nonlinear Fourier atoms [22]. The LCT was first introduced within the framework of Quaternionic Analysis in [17]; these transforms were called the Quaternion Linear Canonical Transforms (QLCTs). The authors investigated an uncertainty principle for the

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right-sided QLCT, which prescribes a lower bound on the product of the effective widths of quaternionic signals in the spatial and frequency domains. The same type of QLCT was employed in deriving a quaternionic version of the Bochner-Minlos Theorem by Kou and Morais [18]. Concerning further details and applications of the QLCTs, the reader is referred to Kou et al. [16, 19, 21].

In recent years, Petrov [25] defined the FT and the convolution of functions on the interval $(-1, 1)$ by employing the diffeomorphism between $\mathbb{R}$ and $(-1, 1)$. The main idea in developing the proposed transforms was to find applications in the study of differential and integro-differential type equations, including Prandtl, Tricomi, Lavrentjev-Bitsadze, and Laplace-Beltrami equations on the sphere [25, 26]. Recently, in [3, 6], the authors studied new aspects of the FT defined by Petrov, such as solvability and boundedness results for multipliers and convolution equations and the associated pseudo-differential calculus in the sense of Hörmander. These works have renewed interest in extending such types of transforms within the Quaternionic Analysis language. In [10], we generalized the transform explored in [25] by introducing the two-sided and right-sided quaternion hyperbolic FTs associated with the two-dimensional quaternion-valued signals defined in an open rectangle of the Euclidean plane endowed with a hyperbolic measure.

The motivation of the present paper is two-fold. On the one hand, it is of intrinsic interest to check whether the Euclidean QLCTs can be extended to non-Euclidean spaces using hyperbolic geometry tools. More precisely, we introduce the QLCTs on the spaces of quaternion-valued signals defined in an open rectangle of the Euclidean plane endowed with a hyperbolic measure. We will call these new transforms the Quaternion Hyperbolic Linear Canonical Transforms (QHLCTs). We replace the Euclidean kernels of the LCTs depending on three real parameters with two quaternionic hyperbolic factors, where two quaternion algebra generators take over the role of the imaginary unit. The QHLCTs will pave the ground for extending all known properties of the QLCTs in the standard 2D Euclidean case, thus unifying the rich canonical structure of Quaternionic Analysis on Euclidean and hyperbolic spaces. On the other hand, our work gives the fundamental tools to study hyperbolic quaternion-valued signals within Quaternionic Signal Analysis. This understanding can be the basis for more generalizations of the Euclidean Quaternionic Signal Analysis.

The contents of the paper are summarized in the following. The primary background material necessary for our investigation is contained in Section 2. The new hyperbolic derivative and primitive concepts and their properties are fundamental in the analysis. Using the hyperbolic geometry language, Section 3 presents the partial QHLCTs of two-dimensional quaternion-valued signals defined in an open rectangle of the Euclidean plane endowed with a hyperbolic measure. We prove the primary operational properties, the Riemann-Lebesgue Lemma, the Plancherel and Parseval Theorems, and an inversion formula for the partial QHLCTs, representing hyperbolic relativistic counterparts of the corresponding properties of the Euclidean partial QLCTs. In Section 4, we introduce the right-sided QHLCT and prove some of its fundamental properties, which form the basis for the later developments. The novel hyperbolic derivative and primitive notions lead to the differentiation and integration properties of the right-sided QHLCT. We further prove the Riemann-Lebesgue Lemma, the Plancherel and Parseval Theorems, and an inversion formula for the right-sided QHLCT. The results in [1, 2, 5, 7, 14, 15] of the different definitions of the Quaternion Fourier Transforms (QFTs) or QLCTs [17, 18, 19, 20] can be seen as particular cases of our achieved results as we highlight throughout the paper. The second part of the paper is devoted to applications. In Section 5, we establish two hyperbolic analogues of the Heisenberg uncertainty principle for the right-sided QHLCT, which give a lower bound on the product of the effective widths of quaternion-valued signals in the hyperbolic spatial and frequency domains. It is shown that only hyperbolic Gaussian quaternion functions minimize the uncertainty relations. Some discussions about the new results and future work related to the QHLCTs are drawn in Section 6.

2 Definitions and terminology

2.1 Einstein's special relativity in the real line

In one-dimensional special relativity, velocities x and y (whose magnitudes are given as fractions of the speed of light) do not add in the usual way. We have the following definition arising in the study of the algebraic structure of the open interval $(-t, t)$ for some $t > 0$ (cf. [31]).

Definition 2.1. *Let $t > 0$ be any positive constant and let $\mathbb{R}_t = (-t, t)$. We define the (Einstein) relativistic addition law by*

$$x \oplus y = \frac{x+y}{1+\frac{xy}{t^2}} \tag{1}$$

and the relativistic multiplication by real scalars by

$$\lambda \otimes x = t \tanh\left(\lambda \tanh^{-1}\left(\frac{x}{t}\right)\right) \tag{2}$$

for all $x, y \in \mathbb{R}_t$ and all $\lambda \in \mathbb{R}$.

It can be seen that the set $\mathbb{R}_t$, together with the operations $\oplus$ and $\otimes$, forms a vector space over the field of real numbers.

The addition (1) is both commutative and associative, so that it is a group operation. The additive identity for the vector space is the element $0 \in \mathbb{R}_t$, satisfying $0 \oplus x = x$ for all $x \in \mathbb{R}_t$ and the additive inverse for the element $x \in \mathbb{R}_t$ is $\ominus x \in \mathbb{R}_t$, such that $\ominus x \oplus x = 0$. (We shall note that in this case, $\ominus x = -x$.) Moreover, it follows that $x \ominus y = x \oplus (\ominus y)$. We observe that in the limit of large t, the interval $\mathbb{R}_t$ expands to the whole of the space $\mathbb{R}$, and as we see from (1) and (2), $\lim_{t\to+\infty} x \oplus y = x + y$ and $\lim_{t\to+\infty} \lambda \otimes x = \lambda x$ for all $x, y \in \mathbb{R}_t$ and all $\lambda \in \mathbb{R}$. In this way, the open interval $\mathbb{R}_t$ has an algebraic structure similar to $\mathbb{R}$, and in the limit $t \to +\infty$, the hyperbolic structure agrees with the Euclidean structure. There exists indeed an isomorphism between $(\mathbb{R}, +, \times)$ and $(\mathbb{R}_t, \oplus, \otimes)$ through the mapping $f(x) = t \tanh(x/t)$, $x \in \mathbb{R}$.

Einstein's addition law (1) in the one-dimensional case can also be deduced from the Möbius transformations of the complex unit disk, whose importance in the theory of gyrogroups has been well-known for over thirty years; see, e.g., [30, 31]. The algebraic properties of the Einstein and Möbius gyrogroups in the context of hyperbolic harmonic analysis can be found in [8, 9].

We draw the reader's attention to the formal similarities between the following concepts in the context of hyperbolic geometry and the corresponding usual ones in Euclidean geometry. (All results have already been proved in [10].)

Definition 2.2. *Let $t \in \mathbb{R}^+$, $f\colon \mathbb{R}_t \to \mathbb{R}$, and a an interior point of $\mathbb{R}_t$. We say that f is h-continuous at the point a if for any real number $\epsilon > 0$ there exists some real number $\delta > 0$ such that for every $x \in \mathbb{R}_t$ with $|x \ominus a| < \delta$, it holds that $|f(x) - f(a)| < \epsilon$.*

A natural definition of piecewise h-continuity is given as follows.

Definition 2.3. *Let $t \in \mathbb{R}^+$. A function $f\colon \mathbb{R}_t \to \mathbb{R}$ is said to be piecewise h-continuous in $\mathbb{R}_t$ if:*

(i) *the interval $\mathbb{R}_t$ can be subdivided into a finite number, say m, of intervals $(-t, a_1), (a_1, a_2), \ldots, (a_r, a_{r+1}), \ldots, (a_{m-1}, t)$, in each of which f is h-continuous;*

(ii) *f is finite at the end-points a_r, with $r = 1, \ldots, m-1$.*

To express condition (ii), we introduce the standard notation. If $\epsilon \to 0$ purely through positive values of ϵ, we say that $f(x \oplus \epsilon) \to f(x \oplus 0)$ whenever the limit exists. Similarly, if $\epsilon \to 0$ through negative values of ϵ only, we say that $f(x \oplus \epsilon) \to f(x \ominus 0)$. For $f(0 \oplus 0)$, we shall write $f(0^+)$, and for $f(0 \ominus 0)$, we shall write $f(0^-)$. Condition (ii) then states that at each end-point a_r, $f(a_r \oplus 0)$ and $f(a_r \ominus 0)$ should both be finite (though they need not be equal).

Another concept we require is that of a uniformly h-continuous function.

Definition 2.4. *Let $t \in \mathbb{R}^+$. A function $f\colon \mathbb{R}_t \to \mathbb{R}$ is uniformly h-continuous on $\mathbb{R}_t$ if for every real number $\epsilon > 0$ there exists some real number $\delta > 0$ such that for every $x, y \in \mathbb{R}_t$ with $|x \ominus y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.*

Definition 2.5. *Let $t \in \mathbb{R}^+$, $f\colon \mathbb{R}_t \to \mathbb{R}$, and let a be an interior point of $\mathbb{R}_t$. We say that f has an h-derivative (or is h-differentiable) at the point $x = a$ if the following limit*

$$\lim_{\epsilon\to 0} \frac{f(a \oplus \epsilon) - f(a)}{\epsilon} \tag{3}$$

exists and is finite. We call the h-derivative of f at $x = a$ to the limit value and denote it by $f'_h(a)$. If the h-derivative exists and is finite for all points x in $\mathbb{R}_t$, we denote the h-derivative of f by $f'_h(x)$ and say that f is h-differentiable at every point of $\mathbb{R}_t$.

We refer to the following relationship between the hyperbolic and the Euclidean derivatives and the operational properties of the h-derivative, which play a role in Sections 3 and 4.

Proposition 2.1 ([10]). *If $f\colon \mathbb{R}_t \to \mathbb{R}$ is h-differentiable, then*

$$f'_h(x) = f'(x)\Big(1 - \frac{x^2}{t^2}\Big), \tag{4}$$

where f' denotes the standard Euclidean derivative of f. Furthermore, the following properties hold:

1. $(f \pm g)'_h = f'_h \pm g'_h$,
2. $(fg)'_h = f'_h\, g + f\, g'_h$,
3. $\Big(\dfrac{f}{g}\Big)'_h = \dfrac{f'_h\, g - f\, g'_h}{g^2}, \quad g \neq 0$,
4. $(f \circ g)'_h = (f' \circ g) \times g'_h$, *whenever the composition $\circ$ is well-defined.*

The relation (4) shows that the Euclidean derivative is the appropriate limit of the hyperbolic derivative when $t \to +\infty$. (This limit will frequently occur in due course.)

Considerations in the calculus of primitives within the hyperbolic context lead to the following definition.

Definition 2.6. *An h-differentiable function $F\colon [a,b] \subset \mathbb{R}_t \to \mathbb{R}$ is called an h-primitive of f in $[a,b]$ if $F'_h(x) = f(x)$, for all $x \in (a,b)$.*

Theorem 2.1 ([10]). *Let f be an h-continuous function in $[a,b] \subset \mathbb{R}_t$, and let*

$$d\mu_t(u) = \left(1 - \frac{u^2}{t^2}\right)^{-1} du$$

be the hyperbolic measure on $\mathbb{R}_t$ (which is translation invariant under the operation $\oplus$). Then the function

$$F(x) = \int_a^x f(u)\, d\mu_t(u) \tag{5}$$

is an h-primitive of f. Moreover,

$$\int_a^b f(u)\, d\mu_t(u) = G(b) - G(a), \tag{6}$$

where G is any h-primitive of f.

Proposition 2.2 ([10]). *Let f and g be continuously h-differentiable functions defined in $\mathbb{R}_t$. The formula for integrating by parts is as follows:*

$$\int f'_h(u)g(u)\, d\mu_t(u) = f(u)g(u) - \int f(u)g'_h(u)\, d\mu_t(u). \tag{7}$$

2.2 Quaternion signals in hyperbolic space

We will use standard notation for the skew field of (real) quaternions [23]

$$\mathbb{H} := \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \colon q_i \in \mathbb{R}\}.$$

We will identify $\mathbb{H}$ with the real vector space $\mathbb{R}^4$; the binary operations of addition of two quaternions and multiplication of a quaternion by a scalar coincide with the usual operations on vectors in $\mathbb{R}^4$. The multiplication in $\mathbb{H}$ is given in terms of the canonical basis $(1, \mathbf{i}, \mathbf{j}, \mathbf{k})$ by the formulas

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Using the customary notation, one writes $q = \mathrm{Sc}(q) + \mathrm{Vec}(q)$, where $\mathrm{Sc}(q) = q_0$ and $\mathrm{Vec}(q) = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ are the scalar and vector parts of q. The conjugate of a quaternion q is defined by $\overline{q} = \mathrm{Sc}(q) - \mathrm{Vec}(q)$, with the property that the conjugate of a product is the product of the conjugates in the reverse order, i.e., $\overline{qr} = \overline{r}\,\overline{q}$,

$\forall q, r \in \mathbb{H}$. The (algebraic) norm of q is defined by $|q|^2 = q\overline{q} = \overline{q}q = \sum_{i=0}^{3} q_i^2$. It further follows that $|qr| = |q|\,|r|$, $\forall q, r \in \mathbb{H}$. A unit quaternion q is a quaternion with $|q| = 1$.

Throughout this paper, we will adopt the following notation:

$$\mathbb{R}^2_{t_1,t_2} = \{(x_1, x_2) \in \mathbb{R}^2 \colon |x_1| < t_1,\ |x_2| < t_2,\ t_1, t_2 \in \mathbb{R}^+\}$$

is an open rectangle centered at the origin in the two-dimensional Euclidean space $\mathbb{R}^2$. We define the operations of hyperbolic addition and hyperbolic scalar multiplication in $\mathbb{R}^2_{t_1,t_2}$ as

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus y_1, x_2 \oplus y_2)$$

and

$$\lambda \otimes (x_1, x_2) = (\lambda \otimes x_1, \lambda \otimes x_2)$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2_{t_1,t_2}$ and all $\lambda \in \mathbb{R}$.

We have found it convenient to introduce a special symbol to denote the extensions of the variables $(x_1, x_2) \in \mathbb{R}^2_{t_1,t_2}$ to the whole of the space $\mathbb{R}^2$ by putting

$$\underline{x} := (\underline{x}_1, \underline{x}_2) = \big(t_1 \tanh^{-1}\big(x_1/t_1\big),\ t_2 \tanh^{-1}\big(x_2/t_2\big)\big) \in \mathbb{R}^2. \tag{8}$$

We will consider functions $f \colon \mathbb{R}^2_{t_1,t_2} \to \mathbb{H}$ of the form

$$f(x) = f_0(x) + \mathbf{i} f_1(x) + \mathbf{j} f_2(x) + \mathbf{k} f_3(x), \tag{9}$$

where $x := (x_1, x_2) \in \mathbb{R}^2_{t_1,t_2}$ and $f_i \colon \mathbb{R}^2_{t_1,t_2} \to \mathbb{R}$. Properties (such as h-continuity, h-differentiability, or h-integrability) ascribed to f are defined coordinatewise.

We introduce the following quaternionic spaces, which will be of use in further discussion.

Definition 2.7. *Let $1 \le p < \infty$. The left-linear modules $L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ are defined to be the classes of all measurable $\mathbb{H}$-valued functions f defined on $\mathbb{R}^2_{t_1,t_2}$ such that $|f|^p \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$, i.e.,*

$$L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H}) = \Big\{ f \colon \mathbb{R}^2_{t_1,t_2} \to \mathbb{H} : \Big(\int_{\mathbb{R}^2_{t_1,t_2}} |f(x)|^p \, d\mu_{t_1,t_2}(x) \Big)^{1/p} < \infty \Big\},$$

where $d\mu_{t_1,t_2}(x) = d\mu_{t_1}(x_1) d\mu_{t_2}(x_2)$ denotes the hyperbolic measure on $\mathbb{R}^2_{t_1,t_2}$:

$$d\mu_{t_1,t_2}(x) = \frac{dx_1\, dx_2}{\Big(1 - \frac{x_1^2}{t_1^2}\Big)\Big(1 - \frac{x_2^2}{t_2^2}\Big)}. \tag{10}$$

For p = ∞, the space $L^\infty(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ contains essentially the bounded measurable functions f with norm $\|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^2_{t_1,t_2}} |f(x)|$.

The primary space $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ is endowed with the usual definition of the left-quaternionic inner product

$$\langle f, g \rangle_{L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})} := \int_{\mathbb{R}^2_{t_1,t_2}} f(x)\, \overline{g(x)}\, d\mu_{t_1,t_2}(x) \tag{11}$$

for $f, g \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. (It is a (left) quaternionic Hilbert space with the associated norm $\|f\|_{L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})} := \langle f, f \rangle^{1/2}_{L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})}$.)

Definition 2.8. *Let $C^\infty(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ be the space of smooth quaternion-valued functions defined in $\mathbb{R}^2_{t_1,t_2}$. We further denote by $\mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ the Schwartz space of all rapidly decreasing $\mathbb{H}$-valued smooth functions on $\mathbb{R}^2_{t_1,t_2}$:*

$$\mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H}) = \left\{ f \in C^\infty(\mathbb{R}^2_{t_1,t_2}, \mathbb{H}) : \sup_{x \in \mathbb{R}^2_{t_1,t_2}} |\underline{x}^\alpha\, \partial_h^\beta f(x)| < \infty \right\},$$

where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ are multi-indices of non-negative integers such that $\underline{x}^\alpha = \underline{x}_1^{\alpha_1} \underline{x}_2^{\alpha_2}$ and $\partial_h^\beta f := \frac{\partial_h^{\beta_1 + \beta_2}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} f$.

The hyperbolic partial derivatives $\partial_h f/\partial x_i$ $(i = 1, 2)$ in Definition 2.8 are constructed from (3) by

$$\frac{\partial_h f}{\partial x_1}(x_1, x_2) = \lim_{\epsilon\to 0} \frac{f(x_1 \oplus \epsilon, x_2) - f(x_1, x_2)}{\epsilon}$$

and

$$\frac{\partial_h f}{\partial x_2}(x_1, x_2) = \lim_{\epsilon\to 0} \frac{f(x_1, x_2 \oplus \epsilon) - f(x_1, x_2)}{\epsilon}.$$

For later purposes (see Section 4 below), a hyperbolic counterpart of Lebesgue's Dominated Convergence Theorem for quaternion-valued functions is given in the following theorem.

Theorem 2.2 (Lebesgue's Dominated Convergence Theorem)**.** *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of measurable $\mathbb{H}$-valued functions defined in $\mathbb{R}^2_{t_1,t_2}$. Suppose that*

1. *the pointwise limits $f(x) = \lim_{n\to\infty} f_n(x)$ exist a.e. on $x \in \mathbb{R}^2_{t_1,t_2}$,*
2. *there exists a nonnegative h-integrable function $g\colon \mathbb{R}^2_{t_1,t_2} \to [0,\infty)$ such that $|f_n(x)| \le g(x)$ a.e. on $x \in \mathbb{R}^2_{t_1,t_2}$ and for all $n \in \mathbb{N}$.*

Then f is in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, as is f_n for each n, and

$$\lim_{n\to\infty} \int_{\mathbb{R}^2_{t_1,t_2}} f_n(x)\, d\mu_{t_1,t_2}(x) = \int_{\mathbb{R}^2_{t_1,t_2}} f(x)\, d\mu_{t_1,t_2}(x).$$

Proof. This is proved similarly to its complex version. Since $|f_n(x)| \le g(x)$ a.e. on $x \in \mathbb{R}^2_{t_1,t_2}$, for all $n \in \mathbb{N}$, and g is h-integrable, it follows that

$$\int_{\mathbb{R}^2_{t_1,t_2}} |f_n(x)|\, d\mu_{t_1,t_2}(x) \le \int_{\mathbb{R}^2_{t_1,t_2}} g(x)\, d\mu_{t_1,t_2}(x) < \infty.$$

Hence, f_n is h-integrable for each n. Similarly, $|f(x)| = \lim_{n\to\infty} |f_n(x)| \le g(x)$ implies that f is h-integrable as well.

Now, since $|f_n(x)| \le g(x)$ a.e. on $\mathbb{R}^2_{t_1,t_2}$, for all $n \in \mathbb{N}$, then it follows that each component of f_n, say $f_{n,i}$ $(i \in \{0,\ldots,3\})$, satisfies $|f_{n,i}(x)| \le g(x)$ and $\lim_{n\to\infty} f_{n,i}(x) = f_i(x)$ for each $i \in \{0,\ldots,3\}$ a.e. on $\mathbb{R}^2_{t_1,t_2}$ and for all $n \in \mathbb{N}$. The rest of the proof follows from the Lebesgue Dominated Convergence Theorem for real-valued functions $f_{n,i} \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$, and so

$$\lim_{n\to\infty} \int_{\mathbb{R}^2_{t_1,t_2}} f_{n,i}(x)\, d\mu_{t_1,t_2}(x) = \int_{\mathbb{R}^2_{t_1,t_2}} f_i(x)\, d\mu_{t_1,t_2}(x),$$

which establishes the statement. □

When $t_1, t_2 \to +\infty$, the above theorem reduces to the corresponding result for quaternion-valued functions defined in the whole space of $\mathbb{R}^2$ (see [21]).

3 The partial QHLCTs and their properties

In this section, we introduce the definition of the right-sided partial QHLCTs and give their properties for functions in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. We further establish the Riemann-Lebesgue Lemma for the partial QHLCTs, which prescribes the asymptotic behavior of these transforms, and the Plancherel and Parseval Theorems. Besides, a result will be proven that gives an inversion formula for the partial QHLCTs. The treatment given here is a generalization of that considered by Kou et al. in [17] and [18].

3.1 Definition

Partially motivated by the results of [10], we will now introduce a four-parameter family of quaternionic hyperbolic plane waves on $\mathbb{R}_t$. (Only three of those parameters are free since there is a constraint involving the four parameters.)

Definition 3.1. *Let $t_1, t_2 \in \mathbb{R}^+$, and let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ for $k = 1, 2$. For $(\omega_1, \omega_2) \in \mathbb{R}^2$ and $(x_1, x_2) \in \mathbb{R}^2_{t_1,t_2}$, the four-parameter families $\{K^{\mathbf{i}}_{A_1}(x_1, \omega_1; t_1), K^{\mathbf{j}}_{A_2}(x_2, \omega_2; t_2)\}$ of 1D quaternionic hyperbolic plane waves are defined by*

$$K^{\mathbf{i}}_{A_1}(x_1, \omega_1; t_1) = \begin{cases} \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}}\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\, e^{\mathbf{i}\frac{d_1}{2b_1}\omega_1^2} \left(\frac{1+\frac{x_1}{t_1}}{1-\frac{x_1}{t_1}}\right)^{-\frac{\mathbf{i}\omega_1 t_1}{2b_1}}, & b_1 \neq 0, \\ \sqrt{d_1}\, e^{\mathbf{i}\frac{c_1 d_1}{2}\omega_1^2}\, \delta(\underline{x}_1 - d_1\,\omega_1), & b_1 = 0, \end{cases} \tag{12}$$

and

$$K^{\mathbf{j}}_{A_2}(x_2, \omega_2; t_2) = \begin{cases} \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}}\, e^{\mathbf{j}\frac{a_2}{2b_2}\underline{x}_2^2}\, e^{\mathbf{j}\frac{d_2}{2b_2}\omega_2^2} \left(\frac{1+\frac{x_2}{t_2}}{1-\frac{x_2}{t_2}}\right)^{-\frac{\mathbf{j}\omega_2 t_2}{2b_2}}, & b_2 \neq 0, \\ \sqrt{d_2}\, e^{\mathbf{j}\frac{c_2 d_2}{2}\omega_2^2}\, \delta(\underline{x}_2 - d_2\,\omega_2), & b_2 = 0, \end{cases} \tag{13}$$

where δ is the Dirac delta function. (Note that $1/\sqrt{\mathbf{i}} = e^{-\mathbf{i}\frac{\pi}{4}}$, $1/\sqrt{\mathbf{j}} = e^{-\mathbf{j}\frac{\pi}{4}}$.)

When $b_k \neq 0$ for $k = 1, 2$, it turns out that we can write (12) and (13) as

$$\begin{aligned} K^q_{A_k}(x_k, \omega_k; t_k) &= \frac{1}{\sqrt{2\pi b_k\, q}}\, e^{\frac{q}{2b_k}\left(a_k \underline{x}_k^2 + d_k \omega_k^2\right)} e^{-\frac{q\omega_k t_k}{2b_k}\ln\left(\frac{1+\frac{x_k}{t_k}}{1-\frac{x_k}{t_k}}\right)} \\ &= \frac{1}{\sqrt{2\pi b_k\, q}}\, e^{q\left(\frac{a_k}{2b_k}\underline{x}_k^2 - \frac{1}{b_k}\underline{x}_k\omega_k + \frac{d_k}{2b_k}\omega_k^2\right)}, \end{aligned} \tag{14}$$

where $e^{\frac{q}{2b_k}\left(a_k \underline{x}_k^2 + d_k \omega_k^2\right)}$ can be interpreted as quadratic modulations in both the hyperbolic spatial and frequency domains. Here $q = \mathbf{i}$ for $k = 1$ and $q = \mathbf{j}$ for $k = 2$. (We will often adopt this notation throughout the paper.)

In the limiting cases $b_k \to 0$, we use the well-known representation $\delta_{p_k}(\underline{x}_k) = 1/(\sqrt{\pi}|p_k|)e^{-(\underline{x}_k/p_k)^2}$ with $p_k = (2b_k q/a_k)^{1/2}$, combined with the fact that $1/a_k \to d_k$ as $b_k \to 0$, to show that

$$\begin{aligned} K^q_{A_k}(x_k, \omega_k; t_k) &= \frac{1}{\sqrt{2\pi b_k\, q}}\, e^{q\frac{a_k}{2b_k}\left(\underline{x}_k - \frac{\omega_k}{a_k}\right)^2} e^{q\frac{c_k}{2a_k}\omega_k^2} \\ &= \frac{1}{\sqrt{a_k}}\, \delta_{p_k}\left(\underline{x}_k - \frac{\omega_k}{a_k}\right) e^{q\frac{c_k}{2a_k}\omega_k^2} \\ &\to \sqrt{d_k}\, e^{q\frac{c_k d_k}{2}\omega_k^2}\, \delta(\underline{x}_k - d_k\,\omega_k) \ \text{ as } b_k \to 0. \end{aligned}$$

When $(a_k, b_k; c_k, d_k) = (0, 1; -1, 0)$, (14) reduces to the ordinary quaternionic hyperbolic plane waves (with regard to this special case, see [10]). From now on, we will omit the dependence of (14) on the parameters t_k and write $K^q_{A_k}(x_k, \omega_k)$ instead of $K^q_{A_k}(x_k, \omega_k; t_k)$.

When $b_k \neq 0$ for $k = 1, 2$, these kernels satisfy the following differential relations

$$\underline{x}_k K^q_{A_k}(x_k, \omega_k) = (d_k\omega_k + q b_k \partial_{\omega_k}) K^q_{A_k}(x_k, \omega_k) \tag{15}$$

$$\frac{\partial_h}{\partial x_k} K^q_{A_k}(x_k, \omega_k) = (c_k \omega_k q - a_k \partial_{\omega_k}) K^q_{A_k}(x_k, \omega_k) \tag{16}$$

which are the hyperbolic analogs of the relationships verified in the Euclidean setting. Thus, it is possible to relate the QHLCTs we will develop in this paper with quadratic hyperbolic quantum systems (see, e.g., [32, Sec. 1.2.2] for more details).

The following proposition summarizes some useful properties of (14). (Many of these properties are similar, more or less analogous, to the properties of the corresponding 1D quaternionic Euclidean plane waves in the QLCTs. For this reason, we shall be brief and omit details unless essential differences in statement or proof make the contrary necessary.)

Proposition 3.1. *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ with $b_k \neq 0$ for $k = 1, 2$, and let $t_k \in \mathbb{R}^+$ for $k = 1, 2$. For $\omega_k, \xi_k \in \mathbb{R}$ and $x_k, y_k \in \mathbb{R}_{t_k}$, we have*

(i) $K^q_{A_k}(x_k, \omega_k) = K^q_{B_k}(\omega_k, x_k)$, *where $B_k = (d_k, b_k; c_k, a_k)$,*

(ii) $K^q_{A_k}(\ominus x_k, \omega_k) = K^q_{A_k}(x_k, -\omega_k)$,

(iii) $K^q_{A_k}(\ominus x_k, -\omega_k) = K^q_{A_k}(x_k, \omega_k)$,

(iv) $\overline{K^q_{A_k}(x_k,\omega_k)} = K^q_{C_k}(x_k,\omega_k)$, *where* $C_k = (a_k, -b_k; -c_k, d_k)$,

(v) $K^q_{A_k}(x_k, \omega_k + \xi_k) = \frac{\sqrt{2\pi b_k}}{\sqrt{-q}}\, K^q_{A_k}(x_k,\omega_k) K^q_{A_k}(x_k,\xi_k)\, e^{q\left(\frac{d_k}{b_k}\omega_k\xi_k - \frac{a_k}{2b_k}\underline{x}_k^2\right)}$,

(vi) $K^q_{A_k}(x_k \oplus y_k, \omega_k) = \frac{\sqrt{2\pi b_k}}{\sqrt{-q}}\, K^q_{A_k}(x_k,\omega_k) K^q_{A_k}(y_k,\omega_k)\, e^{q\left(\frac{a_k}{b_k}\underline{x}_k\,\underline{y}_k - \frac{d_k}{2b_k}\omega_k^2\right)}$,

(vii) $\lim_{t_k\to+\infty} K^q_{A_k}(x_k,\omega_k) = \frac{1}{\sqrt{2\pi b_k\, q}}\, e^{q\left(\frac{a_k}{2b_k}x_k^2 - \frac{1}{b_k}x_k\omega_k + \frac{d_k}{2b_k}\omega_k^2\right)}$,

(viii) $\int_{\mathbb{R}_{t_k}} K^q_{A_k}(x_k,\underline{y}_k) K^q_{D_k}(\underline{y}_k,\omega_k)\, d\mu_{t_k}(y_k) = K^q_{D_kA_k}(x_k,\omega_k)$, *where* D_kA_k *corresponds to matrix multiplication.*

Proof. The first five properties can be directly verified using Definition 3.2. To prove Property (vi), we consider $x_k/t_k = \tanh(\alpha_k) \in (-1,1)$ and $y_k/t_k = \tanh(\beta_k) \in (-1,1)$. On account of the addition formula

$$\frac{\tanh(\alpha_k)+\tanh(\beta_k)}{1+\tanh(\alpha_k)\tanh(\beta_k)} = \tanh(\alpha_k+\beta_k),$$

and the fact that

$$(1/\sqrt{q})^{-1} = 1/\sqrt{-q}, \tag{17}$$

we readily get

$$\begin{aligned}
&K^q_{A_k}(x_k\oplus y_k,\omega_k)\\
&= \frac{1}{\sqrt{2\pi b_k\, q}}\, e^{q\left(\frac{a_k}{2b_k}t_k^2(\alpha_k+\beta_k)^2 - \frac{1}{b_k}t_k(\alpha_k+\beta_k)\omega_k + \frac{d_k}{2b_k}\omega_k^2\right)}\\
&= \frac{1}{\sqrt{2\pi b_k\, q}}\, e^{q\left(\frac{a_k}{2b_k}\underline{x}_k^2 - \frac{1}{b_k}\underline{x}_k\omega_k + \frac{d_k}{2b_k}\omega_k^2\right)}\, \frac{\sqrt{2\pi b_k}}{\sqrt{-q}}\, \frac{1}{\sqrt{2\pi b_k\, q}}\, e^{q\left(\frac{a_k}{2b_k}\underline{y}_k^2 - \frac{1}{b_k}\underline{y}_k\omega_k + \frac{a_k}{b_k}\underline{x}_k\,\underline{y}_k\right)},
\end{aligned}$$

from which the statement follows.

Property (vii) is based on the following limits computed using L'Hôpital's rule:

$$\lim_{t_k\to+\infty} t_k \tanh^{-1}(x_k/t_k) = \lim_{t_k\to+\infty} \frac{x_k}{1-\frac{x_k^2}{t_k^2}} = x_k.$$

To prove the reproducing property (viii), we first extend the definition (14) of the hyperbolic plane waves $K^q_{A_k}(x_k,\omega_k)$ from $(x_k,\omega_k) \in \mathbb{R}_{t_k}\times\mathbb{R}$ to the whole space of $\mathbb{R}^2$. The rest of the proof follows the same reasoning as in the Euclidean case. □

Due to the non-commutativity of the quaternions, different formulations are possible for the partial QHLCTs of quaternion-valued signals. We introduce the following definition, which gives the hyperbolic relativistic counterpart of the right-sided Euclidean partial QLCTs introduced in [17]. (The derivation of the left-sided partial QHLCTs is also possible, but we do not dwell further here on this structure.)

Definition 3.2 (The right-sided partial QHLCTs)**.** *Let* $A_k = (a_k, b_k; c_k, d_k)$ *be* 2×2 *matrices of real parameters satisfying* $a_kd_k - b_kc_k = 1$ *for* $k = 1,2$, *and let* $(\omega_1,\omega_2) \in \mathbb{R}^2$. *The right-sided partial QHLCTs of* $f \in L^1(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$ *are defined as*

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1,x_2) = \int_{\mathbb{R}_{t_1}} f(x_1,x_2) K^{\mathbf{i}}_{A_1}(x_1,\omega_1)\, d\mu_{t_1}(x_1) \tag{18}$$

and

$$\mathcal{L}^{\mathbf{j}}_{A_2}(f)(x_1,\omega_2) = \int_{\mathbb{R}_{t_2}} f(x_1,x_2) K^{\mathbf{j}}_{A_2}(x_2,\omega_2)\, d\mu_{t_2}(x_2). \tag{19}$$

Here $K^q_{A_k}(x_k,\omega_k)$ *for* $k = 1,2$ *have the same meaning as in* (14). *We refer to* (x_1,x_2) *as hyperbolic space-variables and* (ω_1,ω_2) *as angular-frequency variables.*

Notice that when $b_k = 0$, the partial QHLCTs reduce, respectively, to

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1,x_2) = f(d_1\,\omega_1,x_2)\sqrt{d_1}\, e^{\mathbf{i}\frac{c_1d_1}{2}\omega_1^2}$$

and

$$\mathcal{L}^{\mathbf{j}}_{A_2}(f)(x_1,\omega_2) = f(x_1, d_2\,\omega_2)\sqrt{d_2}\, e^{\mathbf{j}\frac{c_2 d_2}{2}\omega_2^2},$$

which are essentially chirp multiplications and are of no particular interest for the objective of this work. Therefore, from now on, we shall confine our attention to the cases $b_k \neq 0$.

According to Definition 3.2, since

$$|f(x_1,x_2)K^q_{A_k}(x_k,\omega_k)| = \frac{1}{\sqrt{2\pi|b_k|}}\,|f(x_1,x_2)| \tag{20}$$

for all $(x_1,x_2) \in \mathbb{R}^2_{t_1,t_2}$ and $(\omega_1,\omega_2) \in \mathbb{R}^2$, it follows that if $f \in L^1(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$, then the partial QHLCTs are well-defined, and the integrals (18) and (19) converge absolutely.

We shall observe that the factors in (18) and (19) must be written in a fixed order since the kernels $K^{\mathbf{i}}_{A_1}$ and $K^{\mathbf{j}}_{A_2}$ do not commute with every element of the quaternion algebra.

The particular cases when $A_k = (0,1;-1,0)$ yield the following result involving the basic properties of the partial Quaternion Hyperbolic Fourier Transforms (hereafter referred to as the partial QHFTs). (Although the proofs of the statements can be found in [10], we will include them for completeness.) Applications of these properties will be given in Section 4.

Proposition 3.2. *Let $(\omega_1,\omega_2) \in \mathbb{R}^2$, and let the right-sided partial QHFTs of $f \in L^1(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$ be defined as*

$$\mathcal{F}^{\mathbf{i}}(f)(\omega_1,x_2) = \int_{\mathbb{R}_{t_1}} f(x_1,x_2)e^{-\mathbf{i}\omega_1\underline{x}_1}\, d\mu_{t_1}(x_1),\;\; x_2 \in \mathbb{R}_{t_2}, \tag{21}$$
$$\mathcal{F}^{\mathbf{j}}(f)(x_1,\omega_2) = \int_{\mathbb{R}_{t_2}} f(x_1,x_2)e^{-\mathbf{j}\omega_2\underline{x}_2}\, d\mu_{t_2}(x_2),\;\; x_1 \in \mathbb{R}_{t_1}.$$

The following properties hold:

1. *(Partial h-derivatives) Let $m,n \in \mathbb{N}$. If $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$, then*

$$\mathcal{F}^{\mathbf{i}}\Big(\frac{\partial_h^m}{\partial x_1^m}f(x_1,x_2)\Big)(\omega_1,x_2) = \mathcal{F}^{\mathbf{i}}(f)(\omega_1,x_2)\,(\mathbf{i}\omega_1)^m,$$
$$\mathcal{F}^{\mathbf{j}}\Big(\frac{\partial_h^n}{\partial x_2^n}f(x_1,x_2)\Big)(x_1,\omega_2) = \mathcal{F}^{\mathbf{j}}(f)(x_1,\omega_2)\,(\mathbf{j}\omega_2)^n.$$

2. *(Riemann-Lebesgue Lemma)*

$$\mathcal{F}^{\mathbf{i}}(f)(\omega_1,x_2) \to 0 \;\; as \;\; |\omega_1| \to \infty$$

for all $x_2 \in \mathbb{R}_{t_2}$, and

$$\mathcal{F}^{\mathbf{j}}(f)(x_1,\omega_2) \to 0 \;\; as \;\; |\omega_2| \to \infty$$

for all $x_1 \in \mathbb{R}_{t_1}$.

3. *(Inversion) For a.e. on $(x_1,x_2) \in \mathbb{R}^2_{t_1,t_2}$,*

$$f(x_1,x_2) = (\mathcal{F}^{\mathbf{i}})^{-1}[\mathcal{F}^{\mathbf{i}}(f)](x_1,x_2) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}^{\mathbf{i}}(f)(\omega_1,x_2)e^{\mathbf{i}\omega_1\underline{x}_1}\, d\omega_1, \tag{22}$$

whenever the integral exists. A sufficient condition for the integral to exist is that $\mathcal{F}^{\mathbf{i}}(f) \in L^1(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})$. (A similar statement holds for the transform $\mathcal{F}^{\mathbf{j}}(f)$.)

Proof. Bearing in mind that

$$\frac{\partial_h}{\partial x_1}\, e^{-\mathbf{i}\omega_1\underline{x}_1} = -\mathbf{i}\omega_1\, e^{-\mathbf{i}\omega_1\underline{x}_1} \tag{23}$$

then, by induction, for $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$ the integration by parts formula (7) yields

$$\int_{\mathbb{R}_{t_1}} \Big(\frac{\partial_h^m}{\partial x_1^m}\, f(x_1,x_2)\Big)\, e^{-\mathbf{i}\omega_1\underline{x}_1}\, d\mu_{t_1}(x_1) = (-1)^m \int_{\mathbb{R}_{t_1}} f(x_1,x_2)\,\Big(\frac{\partial_h^m}{\partial x_1^m}e^{-\mathbf{i}\omega_1\underline{x}_1}\Big)\, d\mu_{t_1}(x_1)$$
$$= \mathcal{F}^{\mathbf{i}}(f)(\omega_1,x_2)\,(\mathbf{i}\omega_1)^m.$$

The second equality can be established analogously.

We now prove Property 2. Using the change of variable

$$\underline{x}_1 + \underline{\widetilde{\pi/\omega_1}} = \underline{y}_1 \Leftrightarrow y_1 = x_1 \oplus \widetilde{\pi/\omega_1},$$

where

$$\widetilde{\pi/\omega_1} = t_1 \tanh\left(\frac{\pi/\omega_1}{t_1}\right),$$

direct computation shows that

$$\begin{aligned}\mathcal{F}^{\mathbf{i}}(f)(\omega_1, x_2) &= -\int_{\mathbb{R}_{t_1}} f(x_1, x_2) e^{-\mathbf{i}\omega_1\left(\underline{x}_1 + \frac{\pi}{\omega_1}\right)} \, d\mu_{t_1}(x_1) \\ &= -\int_{\mathbb{R}_{t_1}} f\left(y_1 \ominus \widetilde{\pi/\omega_1}, x_2\right) e^{-\mathbf{i}\omega_1 \underline{y}_1} \, d\mu_{t_1}(y_1).\end{aligned}$$

Therefore,

$$\begin{aligned}2\left|\mathcal{F}^{\mathbf{i}}(f)(\omega_1, x_2)\right| &= \left|\int_{\mathbb{R}_{t_1}} \left(f(x_1, x_2) - f\left(x_1 \ominus \widetilde{\pi/\omega_1}, x_2\right)\right) e^{-\mathbf{i}\omega_1 \underline{x}_1} \, d\mu_{t_1}(x_1)\right| \\ &\leq \int_{\mathbb{R}_{t_1}} \left|f(x_1, x_2) - f\left(x_1 \ominus \widetilde{\pi/\omega_1}, x_2\right)\right| \, d\mu_{t_1}(x_1).\end{aligned}$$

Since $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, by dominated convergence theorem, it follows that

$$\lim_{|\omega_1| \to \infty} \mathcal{F}^{\mathbf{i}}(f)(\omega_1, x_2) \to 0$$

for all $x_2 \in \mathbb{R}_{t_2}$, which establishes the statement.

By inserting (21) into (22) and using the relation $\delta(\underline{x}_1 - \underline{y}_1) = \delta(x_1 - y_1)$, it follows that

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}^{\mathbf{i}}(f)(\omega_1, x_2) e^{\mathbf{i}\omega_1 \underline{x}_1} \, d\omega_1 &= \int_{\mathbb{R}_{t_1}} f(y_1, x_2) \left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathbf{i}\omega_1(\underline{x}_1 - \underline{y}_1)} d\omega_1\right) d\mu_{t_1}(y_1) \\ &= \int_{\mathbb{R}_{t_1}} f(y_1, x_2)\delta(\underline{x}_1 - \underline{y}_1) \, d\mu_{t_1}(y_1) \\ &= f(x_1, x_2),\end{aligned}$$

which proves Property 3.

This completes the proof of the theorem. □

Remark 3.1. *It shall be noted that Property 1 remains valid whenever $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ and the m-th partial h-derivative of f with respect to the variable x_1 (resp., the n-th partial h-derivative of f with respect to the variable x_2) exists and is in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$.*

We are thus led to the following remarkable relationships between the partial QHLCTs and the partial QHFTs.

Lemma 3.1. *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ with $b_k \neq 0$ for $k = 1, 2$. If $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then*

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) = \mathcal{F}^{\mathbf{i}}\left(f(x_1, x_2) \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\right)\left(\frac{\omega_1}{b_1}, x_2\right) e^{\mathbf{i}\frac{d_1}{2b_1}\omega_1^2} \tag{24}$$

and

$$\mathcal{L}^{\mathbf{j}}_{A_2}(f)(x_1, \omega_2) = \mathcal{F}^{\mathbf{j}}\left(f(x_1, x_2) \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}} e^{\mathbf{j}\frac{a_2}{2b_2}\underline{x}_2^2}\right)\left(x_1, \frac{\omega_2}{b_2}\right) e^{\mathbf{j}\frac{d_2}{2b_2}\omega_2^2}. \tag{25}$$

Proof. By the definition (21) of the partial QHFT of $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, we have, as is easily seen,

$$\begin{aligned}\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) &= \left(\int_{\mathbb{R}_{t_1}} \left(f(x_1, x_2) \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\right) e^{-\mathbf{i}\frac{\omega_1}{b_1}\underline{x}_1} \, d\mu_{t_1}(x_1)\right) e^{\mathbf{i}\frac{d_1}{2b_1}\omega_1^2} \\ &= \mathcal{F}^{\mathbf{i}}\left(f(x_1, x_2) \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\right)\left(\frac{\omega_1}{b_1}, x_2\right) e^{\mathbf{i}\frac{d_1}{2b_1}\omega_1^2}.\end{aligned}$$

An argument similar to that used leads to the statement for $\mathcal{L}^{\mathbf{j}}_{A_2}(f)$. □

3.2 Properties, Plancherel's Theorem, and inversion formula

In this subsection, we study the elementary operational properties of the proposed partial QHLCTs, which will be used to establish the properties of the right-sided QHLCT in Section 4. In the following considerations, we again assume that $b_k \neq 0$.

Proposition 3.3. *Let $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, and let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2_{t_1,t_2}$, $(\omega_1, \omega_2), (\theta_1, \theta_2) \in \mathbb{R}^2$, and $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$. Then*

1. *(Hyperbolic translation)*

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f(x_1 \oplus y_1, x_2))(\omega_1, x_2) = \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1 + a_1\underline{y}_1, x_2)e^{-\mathbf{i}c_1\left(\omega_1\underline{y}_1 + \frac{a_1}{2}\underline{y}_1^2\right)},$$
$$\mathcal{L}^{\mathbf{j}}_{A_2}(f(x_1, x_2 \oplus y_2))(x_1, \omega_2) = \mathcal{L}^{\mathbf{j}}_{A_2}(f)(x_1, \omega_2 + a_2\underline{y}_2)e^{-\mathbf{j}c_2\left(\omega_2\underline{y}_2 + \frac{a_2}{2}\underline{y}_2^2\right)}.$$

2. *(Modulation)*

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f(x_1, x_2)\, e^{\mathbf{i}\underline{x}_1\theta_1})(\omega_1, x_2) = \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1 - b_1\theta_1, x_2)\, e^{-\mathbf{i}\frac{d_1 b_1}{2}\theta_1^2}\, e^{\mathbf{i}d_1\theta_1\omega_1},$$
$$\mathcal{L}^{\mathbf{j}}_{A_2}(f(x_1, x_2)\, e^{\mathbf{j}\underline{x}_2\theta_2})(x_1, \omega_2) = \mathcal{L}^{\mathbf{j}}_{A_2}(f)(x_1, \omega_2 - b_2\theta_2)\, e^{-\mathbf{j}\frac{d_2 b_2}{2}\theta_2^2}\, e^{\mathbf{j}d_2\theta_2\omega_2}.$$

3. *(Hyperbolic dilation/scaling)*

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f(\lambda_1 \otimes x_1, x_2))(\omega_1, x_2) = \frac{1}{|\lambda_1|}\, \mathcal{L}^{\mathbf{i}}_{B_1}(f)\Big(\frac{\omega_1}{\lambda_1}, x_2\Big),$$
$$\mathcal{L}^{\mathbf{j}}_{A_2}(f(x_1, \lambda_2 \otimes x_2))(x_1, \omega_2) = \frac{1}{|\lambda_2|}\, \mathcal{L}^{\mathbf{j}}_{B_2}(f)\Big(x_1, \frac{\omega_2}{\lambda_2}\Big),$$

where $B_k = (a_k/\lambda_k^2, b_k; c_k, d_k\lambda_k^2)$.

4. *(Symmetry)*

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f(\ominus x_1, x_2))(\omega_1, x_2) = \mathcal{L}^{\mathbf{i}}_{A_1}(f)(-\omega_1, x_2),$$
$$\mathcal{L}^{\mathbf{j}}_{A_2}(f(x_1, \ominus x_2))(x_1, \omega_2) = \mathcal{L}^{\mathbf{j}}_{A_2}(f)(x_1, -\omega_2).$$

5. *(Additivity)*

$$\mathcal{L}^{\mathbf{i}}_{B_1}(\mathcal{L}^{\mathbf{i}}_{A_1})(f) = \mathcal{L}^{\mathbf{i}}_{B_1 A_1}(f),$$
$$\mathcal{L}^{\mathbf{j}}_{B_2}(\mathcal{L}^{\mathbf{j}}_{A_2})(f) = \mathcal{L}^{\mathbf{j}}_{B_2 A_2}(f).$$

6. *(Reversibility)*

$$\mathcal{L}^{\mathbf{i}}_{A_1}(\mathcal{L}^{\mathbf{i}}_{A_1^{-1}})(f) = \mathcal{L}^{\mathbf{i}}_{A_1^{-1}}(\mathcal{L}^{\mathbf{i}}_{A_1})(f) = f,$$
$$\mathcal{L}^{\mathbf{j}}_{A_2}(\mathcal{L}^{\mathbf{j}}_{A_2^{-1}})(f) = \mathcal{L}^{\mathbf{j}}_{A_2^{-1}}(\mathcal{L}^{\mathbf{j}}_{A_2})(f) = f.$$

7. *(Partial h-derivatives) Let $m, n \in \mathbb{N}$. If $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then*

$$\omega_1^m \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) = \mathcal{F}^{\mathbf{i}}\Big(\frac{\partial_h^m}{\partial x_1^m}\Big(f(x_1, x_2)\,\frac{1}{\sqrt{2\pi b_1 \mathbf{i}}}\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\Big)\Big)\Big(\frac{\omega_1}{b_1}, x_2\Big)(-b_1\mathbf{i})^m e^{\mathbf{i}\frac{d_1}{2b_1}\omega_1^2},$$
$$\omega_2^n \mathcal{L}^{\mathbf{j}}_{A_2}(f)(x_1, \omega_2) = \mathcal{F}^{\mathbf{j}}\Big(\frac{\partial_h^n}{\partial x_2^n}\Big(f(x_1, x_2)\,\frac{1}{\sqrt{2\pi b_2 \mathbf{j}}}\, e^{\mathbf{j}\frac{a_2}{2b_2}\underline{x}_2^2}\Big)\Big)\Big(x_1, \frac{\omega_2}{b_2}\Big)(-b_2\mathbf{j})^n e^{\mathbf{j}\frac{d_2}{2b_2}\omega_2^2}.$$

8. *(Riemann-Lebesgue Lemma)*

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) \to 0 \quad as \quad |\omega_1| \to \infty$$

for all $x_2 \in \mathbb{R}_{t_2}$, and

$$\mathcal{L}^{\mathbf{j}}_{A_2}(f)(x_1, \omega_2) \to 0 \quad as \quad |\omega_2| \to \infty$$

for all $x_1 \in \mathbb{R}_{t_1}$.

Proof. The first property follows by using the change of variables $x_k \oplus y_k = z_k$ for $k = 1, 2$ (which are equivalent to $x_k = z_k \ominus y_k$) together with the hyperbolic translation invariance property of the hyperbolic measure (10) and Property (vi) in Proposition 3.1.

Property 2 follows directly from the definition. To prove Property 3, we make the change of variables $\lambda_k \otimes x_k = y_k$ for $k = 1, 2$ (which are equivalent to $x_k = (1/\lambda_k) \otimes y_k$.) Since

$$\frac{|\lambda_k|}{\cosh^2\left(\lambda_k \tanh^{-1}(y_k/t_k)\right)}\left(1 - \frac{y_k^2}{t_k^2}\right)^{-1}$$

gives the Jacobian of the change of variables, then by direct computations, we obtain

$$d\mu_{t_k}\left(\frac{1}{\lambda_k} \otimes y_i\right) = \frac{1}{|\lambda_k|}\, d\mu_{t_k}(y_k),$$

which establishes the statement.

The property of symmetry follows from the previous one.

Now, we prove Property 5. We can naturally extend Definition 3.2 of partial QHLCTs to all functions in $L^1(\mathbb{R}^2, \mathbb{H})$ so that a composition of two transforms can be made. With regard to this fact, we combine Fubini's Theorem and Property (viii) of Proposition 3.1 to obtain

$$\begin{aligned}
\mathcal{L}^{\mathbf{i}}_{B_1}\left(\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2)\right)(\omega_1, x_2) &= \int_{\mathbb{R}_{t_1}} \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\underline{y}_1, x_2)\, K^{\mathbf{i}}_{B_1}(\underline{y}_1, \omega_1)\, d\mu_{t_1}(y_1) \\
&= \int_{\mathbb{R}_{t_1}} f(x_1, x_2)\Big(\int_{\mathbb{R}_{t_1}} K^{\mathbf{i}}_{A_1}(x_1, \underline{y}_1) K^{\mathbf{i}}_{B_1}(\underline{y}_1, \omega_1)\, d\mu_{t_1}(y_1)\Big) d\mu_{t_1}(x_1) \\
&= \mathcal{L}^{\mathbf{i}}_{B_1 A_1}(f)(\omega_1, x_2).
\end{aligned}$$

A similar computation can likewise be used to prove the Property of additivity for the other partial QHLCT.

The Property of reversibility is an immediate consequence of the Property of additivity.

We now prove Property 7. By (23), for $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, the integration by parts formula (7) yields

$$\begin{aligned}
\mathcal{F}^{\mathbf{i}}\Big(\frac{\partial_h^m}{\partial x_1^m}\Big(f(x_1, x_2)\, \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}}\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\Big)\Big)(\omega_1, x_2) &= \int_{\mathbb{R}_{t_1}} \Big(\frac{\partial_h^m}{\partial x_1^m}\Big(f(x_1, x_2)\, \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}}\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\Big)\Big)\, e^{-\mathbf{i}\,\underline{x}_1 \omega_1}\, d\mu_{t_1}(x_1) \\
&= (-1)^m \int_{\mathbb{R}_{t_1}} f(x_1, x_2)\, \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}}\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\, \Big(\frac{\partial_h^m}{\partial x_1^m}\, e^{-\mathbf{i}\,\underline{x}_1 \omega_1}\Big) d\mu_{t_1}(x_1) \\
&= \mathcal{F}^{\mathbf{i}}\Big(f(x_1, x_2)\, \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}}\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\Big)(\omega_1, x_2)\, (\mathbf{i}\omega_1)^m. \qquad (26)
\end{aligned}$$

By combining (24) and (26), the statement follows.

For the proof of the Riemann-Lebesgue Lemma, we use a density argument as in the classical case and assume that both f and $(\partial_h/\partial x_1)f$ are h-continuous with compact support. Obviously, such functions form a dense subspace in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$.

By assumption $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then

$$f(x_1, x_2)\, \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}}\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2} \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H}).$$

Combining Lemma 3.1 and Proposition 3.2, direct calculation leads readily to

$$|\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2)| = \Big|\mathcal{F}^{\mathbf{i}}\Big(f(x_1, x_2)\, \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}}\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}\Big)\Big(\frac{\omega_1}{b_1}, x_2\Big)\Big| \to 0$$

as $|\omega_1| \to \infty$. Hence,

$$\lim_{|\omega_1| \to \infty} \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) = 0$$

for all $x_2 \in \mathbb{R}_{t_2}$.

Now, for any given $\epsilon > 0$, we approximate $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ by a compactly supported and smooth function f_ϵ. We choose such an f_ϵ so that $\|f - f_\epsilon\|_{L^1(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} \leq \epsilon$. By (20), we find

$$\begin{aligned}\lim \sup_{|\omega_1|\to\infty} |\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2)| &\leq \lim_{|\omega_1|\to\infty} \Big| \int_{\mathbb{R}_{t_1}} (f(x_1, x_2) - f_\epsilon(x_1, x_2))\, K^{\mathbf{i}}_{A_1}(x_1, \omega_1)\, d\mu_{t_1}(x_1) \Big| \\ &\quad + \lim_{|\omega_1|\to\infty} \Big| \int_{\mathbb{R}_{t_1}} f_\epsilon(x_1, x_2) K^{\mathbf{i}}_{A_1}(x_1, \omega_1)\, d\mu_{t_1}(x_1) \Big| \\ &\leq \frac{1}{\sqrt{2\pi|b_1|}}\, \epsilon\end{aligned}$$

for all $x_2 \in \mathbb{R}_{t_2}$.

Since this holds for any arbitrary $\epsilon > 0$, the statement follows. (A similar statement can be proved for $\mathcal{L}^{\mathbf{j}}_{A_2}(f)$.) □

Now we establish a Plancherel's formula for the partial QHLCTs, which shows that $\mathcal{L}^{\mathbf{i}}_{A_1}(f)$ and $\mathcal{L}^{\mathbf{j}}_{A_2}(f)$ are unitary operators on $L^2(\mathbb{R} \times \mathbb{R}_{t_2}, \mathbb{H})$ and $L^2(\mathbb{R}_{t_1} \times \mathbb{R}, \mathbb{H})$, respectively. (We shall return to this result in Section 4, where it will be used to establish hyperbolic counterparts of the Plancherel and Parseval Theorems for the right-sided QHLCT.)

Theorem 3.1 (Plancherel Theorem for the partial QHLCTs)**.** *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ with $b_k \neq 0$ for $k = 1, 2$. If $f, g \in L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then*

$$\langle f, g\rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} = \langle \mathcal{L}^{\mathbf{i}}_{A_1}(f), \mathcal{L}^{\mathbf{i}}_{A_1}(g)\rangle_{L^2(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})} \tag{27}$$

and

$$\langle f, g\rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} = \langle \mathcal{L}^{\mathbf{j}}_{A_2}(f), \mathcal{L}^{\mathbf{j}}_{A_2}(g)\rangle_{L^2(\mathbb{R}_{t_1}\times\mathbb{R},\mathbb{H})}. \tag{28}$$

In particular, if $f = g$, Parseval's identities read as

$$\|f\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} = \|\mathcal{L}^{\mathbf{i}}_{A_1}(f)\|_{L^2(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})} \tag{29}$$

and

$$\|f\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} = \|\mathcal{L}^{\mathbf{j}}_{A_2}(f)\|_{L^2(\mathbb{R}_{t_1}\times\mathbb{R},\mathbb{H})}. \tag{30}$$

Further, the map $f \mapsto \mathcal{L}^{\mathbf{i}}_{A_1}(f)$ (resp., $f \mapsto \mathcal{L}^{\mathbf{j}}_{A_2}(f)$) has a unique extension to a continuous linear map from $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ into $L^2(\mathbb{R}\times\mathbb{R}_{t_2}, \mathbb{H})$ (resp., $L^2(\mathbb{R}_{t_1}\times\mathbb{R}, \mathbb{H})$) and (29) *(resp.,* (30)*) holds, whenever $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$.*

Proof. Combining Fubini's Theorem with the definition (11) of the quaternionic inner product, we obtain

$$\begin{aligned}&\langle \mathcal{L}^{\mathbf{i}}_{A_1}(f), \mathcal{L}^{\mathbf{i}}_{A_1}(g)\rangle_{L^2(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})} \\ &= \int_{\mathbb{R}_{t_2}} \int_{\mathbb{R}} \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2)\, \overline{\mathcal{L}^{\mathbf{i}}_{A_1}(g)(\omega_1, x_2)}\, d\omega_1 d\mu_{t_2}(x_2) \\ &= \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}_{t_1}} f(x_1, x_2) \Big(\int_{\mathbb{R}} K^{\mathbf{i}}_{A_1}(x_1, \omega_1)\, \overline{K^{\mathbf{i}}_{A_1}(y_1, \omega_1)}\, d\omega_1 \Big)\, \overline{g(y_1, x_2)}\, d\mu_{t_1}(y_1) d\mu_{t_1,t_2}(x_1, x_2) \\ &= \frac{1}{2\pi b_1} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}_{t_1}} f(x_1, x_2)\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_1^2 - \underline{y}_1^2)} \Big(\int_{\mathbb{R}} e^{-\mathbf{i}\frac{\omega_1}{b_1}(\underline{y}_1 - \underline{x}_1)}\, d\omega_1 \Big)\, \overline{g(y_1, x_2)}\, d\mu_{t_1}(y_1) d\mu_{t_1,t_2}(x_1, x_2) \\ &= \int_{\mathbb{R}^2_{t_1,t_2}} \Big(\int_{\mathbb{R}_{t_1}} f(x_1, x_2)\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_1^2 - \underline{y}_1^2)}\, \delta(\underline{y}_1 - \underline{x}_1)\, d\mu_{t_1}(x_1) \Big) \overline{g(y_1, x_2)}\, d\mu_{t_1,t_2}(y_1, x_2).\end{aligned} \tag{31}$$

Bearing in mind that

$$\delta(\underline{y}_1 - \underline{x}_1) = \delta(y_1 - x_1), \tag{32}$$

it follows that

$$\int_{\mathbb{R}_{t_1}} f(x_1, x_2)\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_1^2 - \underline{y}_1^2)}\, \delta(\underline{y}_1 - \underline{x}_1)\, d\mu_{t_1}(x_1) = f(y_1, x_2). \tag{33}$$

Now, putting all these facts together, we can simplify (31) to

$$\begin{aligned}\langle \mathcal{L}^{\mathbf{i}}_{A_1}(f), \mathcal{L}^{\mathbf{i}}_{A_1}(g)\rangle_{L^2(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})} &= \int_{\mathbb{R}^2_{t_1,t_2}} f(y_1, x_2)\, \overline{g(y_1, x_2)}\, d\mu_{t_1,t_2}(y_1, x_2) \\ &= \langle f, g\rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}.\end{aligned}$$

For the second part of the statement, we shall first notice that the left-linearity of the map $f \mapsto \mathcal{L}^{\mathbf{i}}_{A_1}(f)$ on the whole of $L^1(\mathbb{R} \times \mathbb{R}_{t_2}, \mathbb{H})$ follows from the linearity of integration. Now, we prove that the map is one-to-one. Suppose that $f, g \in L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ are such that $\mathcal{L}^{\mathbf{i}}_{A_1}(f) = \mathcal{L}^{\mathbf{i}}_{A_1}(g)$. Then $\mathcal{L}^{\mathbf{i}}_{A_1}(f - g) = 0$ by linearity, so $\|\mathcal{L}^{\mathbf{i}}_{A_1}(f - g)\|_{L^2(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})} = 0$. Hence, by Parseval's identity, $\|f - g\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} = 0$, and so $f = g$ a.e. on $\mathbb{R}^2_{t_1,t_2}$.

Now, let f be in $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ but not in $L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. Since $L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ is dense in $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$, there exists a sequence $(f)_j \in L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ such that $\|(f)_j - f\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{R})} \to 0$. By Parseval's identity,

$$\|(\mathcal{L}^{\mathbf{i}}_{A_1}(f))_j - (\mathcal{L}^{\mathbf{i}}_{A_1}(f))_m\|_{L^2(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})} = \|(f)_j - (f)_m\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{R})},$$

and hence $(\mathcal{L}^{\mathbf{i}}_{A_1}(f))_j$ is a Cauchy sequence in $L^2(\mathbb{R} \times \mathbb{R}_{t_2}, \mathbb{H})$ that converges to some function in $L^2(\mathbb{R} \times \mathbb{R}_{t_2}, \mathbb{H})$, which we still denote by $\mathcal{L}^{\mathbf{i}}_{A_1}(f)$. Then we have

$$\|\mathcal{L}^{\mathbf{i}}_{A_1}(f)\|_{L^2(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})} = \lim_{j\to\infty} \|(\mathcal{L}^{\mathbf{i}}_{A_1}(f))_j\|_{L^2(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})} = \lim_{j\to\infty} \|(f)_j\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{R})} = \|f\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{R})}.$$

Finally, let $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. The previous argument can be extended to every component f_i of f in (9) and, hence, $\mathcal{L}^{\mathbf{i}}_{A_1} \in L^2(\mathbb{R} \times \mathbb{R}_{t_2}, \mathbb{H})$. The statement is thus established. □

The following preliminary result will be useful to prove the inversion formula stated in Theorem 3.2. (Such an approach to the inversion differs from that used in [10].)

Lemma 3.2. *Let b_k $(k = 1, 2)$ be positive constants. Then*

$$\lim_{\alpha\to\infty} \int_{\mathbb{R}_{t_k}} e^{q\frac{a_k}{2b_k}(\underline{x}_k - \underline{y}_k)^2} \frac{\sin(\frac{\alpha}{b_k}\underline{y}_k)}{\underline{y}_k} \, d\mu_{t_k}(y_k) = \pi \, e^{q\frac{a_k}{2b_k}\underline{x}_k^2}.$$

Proof. A first observation shows that the QHFT of the function $f(x_1) = e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}$ exists in view of the fact that

$$\int_{\mathbb{R}_{t_1}} e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2} \, e^{-\mathbf{i}\omega_1\underline{x}_1} \, d\mu_{t_1}(x_1) = \int_{\mathbb{R}} e^{\mathbf{i}\frac{a_1}{2b_1}z_1^2} \, e^{-\mathbf{i}\omega_1 z_1} \, dz_1 = \Big(\frac{2\pi b_1 \mathbf{i}}{a_1}\Big)^{1/2} e^{-\mathbf{i}\frac{b_1}{2a_1}\omega_1^2}.$$

But since

$$\int_{\mathbb{R}_{t_1}} e^{-\mathbf{i}\omega_1\underline{x}_1} \frac{\sin(\frac{\alpha}{b_1}\underline{y}_1)}{\frac{\alpha}{b_1}\underline{y}_1} \, d\mu_{t_1}(y_1) = \frac{b_1}{\alpha} \int_{\mathbb{R}} e^{-\mathbf{i}\frac{b_1}{\alpha}\omega_1 u_1} \operatorname{sinc}(u_1) \, du_1 = \frac{b_1 \pi}{\alpha} \operatorname{rect}\Big(\frac{b_1}{\alpha}\omega_1\Big),$$

where rect is the rectangular function given by

$$\operatorname{rect}\Big(\frac{b_1}{\alpha}\omega_1\Big) = \begin{cases} 1, & \text{if} \quad |\omega_1| < \alpha/b_1, \\ 0, & \text{if} \quad |\omega_1| > \alpha/b_1, \end{cases}$$

we have, after applying the convolution theorem,

$$\begin{aligned} \lim_{\alpha\to\infty} \int_{\mathbb{R}_{t_1}} e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_1 - \underline{y}_1)^2} \frac{\sin(\frac{\alpha}{b_1}\underline{y}_1)}{\underline{y}_1} \, d\mu_{t_1}(y_1) &= \lim_{\alpha\to\infty} (\mathcal{F}^{\mathbf{i}})^{-1}[\mathcal{F}^{\mathbf{i}}(e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2})(\omega_1) \, \mathcal{F}^{\mathbf{i}}(\operatorname{sinc}(\frac{\alpha}{b_1}\underline{y}_1))(\omega_1)](\underline{x}_1) \, \frac{\alpha}{b_1} \\ &= \pi (\mathcal{F}^{\mathbf{i}})^{-1}[\mathcal{F}(e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2})(\omega_1)](\underline{x}_1) \\ &= \pi \, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2}. \end{aligned}$$

This establishes the lemma for $k = 1$. An argument similar to that used leads to the statement for $k = 2$. □

We now derive an inversion formula for the right-sided partial QHLCTs. The result shows that the kernels of the inverse transforms are obtained from the kernels of the forward transforms by replacing the matrices $A_k = (a_k, b_k; c_k, d_k)$ with $A_1^{-1} = (d_k, -b_k; -c_k, a_k)$. (In the large limits of t_1 and t_2, i.e., $t_1, t_2 \to +\infty$, the inversion formula reduces to the corresponding result for the Euclidean right-sided partial QLCTs; see [17].)

Theorem 3.2 (Inversion theorem for the right-sided partial QHLCT)**.** *Let $A_1 = (a_1, b_1; c_1, d_1)$ be a 2×2 matrix of real parameters satisfying $a_1 d_1 - b_1 c_1 = 1$ with $b_1 \neq 0$, and let $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. Suppose that f is h-continuous except for a finite number of finite jumps in any finite interval and has the self-averaging property*

$f(x,y) = (1/2)(f(x^+,y) + f(x^-,y))$ for all $(x,y) \in \mathbb{R}^2_{t_1,t_2}$. The inversion formula for the right-sided partial QHLCT is

$$f(x_0,y) = \lim_{\alpha\to\infty} \int_{-\alpha}^{\alpha} \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1,y) K^{\mathbf{i}}_{A_1^{-1}}(\omega_1,x_0)\, d\omega_1 \tag{34}$$

for every x_0 and y where f has finite left and right partial h-derivatives, whenever the integral exists. In particular, if f is piecewise smooth (i.e., h-continuous and with piecewise first-order h-continuous derivatives), then the formula holds for all $x_0 \in \mathbb{R}_{t_1}$ and uniformly in y. Further, if $\mathcal{L}^{\mathbf{i}}_A(f) \in L^1(\mathbb{R}\times\mathbb{R}_{t_2},\mathbb{H})$, the inversion formula (34) takes the form of the absolutely convergent integral

$$f(x_0,y) = \int_{\mathbb{R}} \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1,y) K^{\mathbf{i}}_{A_1^{-1}}(\omega_1,x_0)\, d\omega_1.$$

(A similar statement holds for the transform $\mathcal{L}^{\mathbf{j}}_{A_2}(f)$.)

Proof. Let $x_0 \in \mathbb{R}_{t_1}$ be a fixed number and assume without loss of generality that $b_1 > 0$. By (18) and Fubini's Theorem, direct computations show that

$$\begin{aligned}
I(x_0,y;\alpha) &= \int_{-\alpha}^{\alpha} \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1,y) K^{\mathbf{i}}_{A_1^{-1}}(\omega_1,x_0)\, d\omega_1 \\
&= \int_{-\alpha}^{\alpha} \Big(\int_{\mathbb{R}_{t_1}} f(z_1,y) K^{\mathbf{i}}_{A_1}(z_1,\omega_1)\, d\mu_{t_1}(z_1) \Big) K^{\mathbf{i}}_{A_1^{-1}}(\omega_1,x_0)\, d\omega_1 \\
&= \int_{\mathbb{R}_{t_1}} f(z_1,y) \Big(\int_{-\alpha}^{\alpha} K^{\mathbf{i}}_{A_1}(z_1,\omega_1) K^{\mathbf{i}}_{A_1^{-1}}(\omega_1,x_0)\, d\omega_1 \Big) d\mu_{t_1}(z_1) \\
&= \int_{\mathbb{R}_{t_1}} f(z_1,y) \Big(\frac{1}{2\pi b_1}\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{z}_1^2 - \underline{x}_0^2)} \int_{-\alpha}^{\alpha} e^{\mathbf{i}\frac{\omega_1}{b_1}(\underline{x}_0 - \underline{z}_1)} d\omega_1 \Big) d\mu_{t_1}(z_1) \\
&= \frac{1}{\pi} \Big(\int_{\mathbb{R}_{t_1}} f(z_1,y)\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{z}_1^2}\, \frac{\sin\left(\frac{\alpha}{b_1}(\underline{x}_0 - \underline{z}_1)\right)}{\underline{x}_0 - \underline{z}_1}\, d\mu_{t_1}(z_1) \Big) e^{-\mathbf{i}\frac{a_1}{2b_1}\underline{x}_0^2} \\
&= \frac{1}{\pi} \Big(\int_{\mathbb{R}_{t_1}} f(x_0 \ominus u_1,y)\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0 - \underline{u}_1)^2}\, \frac{\sin(\frac{\alpha}{b_1}\underline{u}_1)}{\underline{u}_1}\, d\mu_{t_1}(u_1) \Big) e^{-\mathbf{i}\frac{a_1}{2b_1}\underline{x}_0^2}.
\end{aligned}$$

From Lemma 3.2, it follows that

$$\begin{aligned}
&\frac{2}{\pi} \Big(\int_0^{t_1} f(x_0 \ominus u_1,y)\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0 - \underline{u}_1)^2}\, \frac{\sin(\frac{\alpha}{b_1}\underline{u}_1)}{\underline{u}_1}\, d\mu_{t_1}(u_1) \Big) e^{-\mathbf{i}\frac{a_1}{2b_1}\underline{x}_0^2} - f(x_0^-,y) \\
&= \frac{2}{\pi} \Big(\int_0^{t_1} \big(f(x_0 \ominus u_1,y) - f(x_0^-,y) \big)\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0 - \underline{u}_1)^2}\, \frac{\sin(\frac{\alpha}{b_1}\underline{u}_1)}{\underline{u}_1}\, d\mu_{t_1}(u_1) \Big) e^{-\mathbf{i}\frac{a_1}{2b_1}\underline{x}_0^2}
\end{aligned} \tag{35}$$

as $\alpha \to \infty$.

We proceed by splitting the integral on the right-hand side of Eq. (35) into three terms:

$$\begin{aligned}
&\frac{2}{\pi} \Big(\int_0^{\beta_1} \Big(\frac{f(x_0 \ominus u_1,y) - f(x_0^-,y)}{\underline{u}_1} \Big) e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0 - \underline{u}_1)^2} \sin\left(\frac{\alpha}{b_1}\underline{u}_1\right) d\mu_{t_1}(u_1) \Big) e^{-\mathbf{i}\frac{a_1}{2b_1}\underline{x}_0^2} \\
&+ \frac{2}{\pi} \Big(\int_{\beta_1}^{t_1} f(x_0 \ominus u_1,y)\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0 - \underline{u}_1)^2}\, \frac{\sin(\frac{\alpha}{b_1}\underline{u}_1)}{\underline{u}_1}\, d\mu_{t_1}(u_1) \Big) e^{-\mathbf{i}\frac{a_1}{2b_1}\underline{x}_0^2} \\
&- \frac{1}{2\pi} f(x_0^-,y) \Big(\int_{\beta_1}^{t_1} e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0 - \underline{u}_1)^2}\, \frac{\sin(\frac{\alpha}{b_1}\underline{u}_1)}{\underline{u}_1}\, d\mu_{t_1}(u_1) \Big) e^{-\mathbf{i}\frac{a_1}{2b_1}\underline{x}_0^2} \\
&= I_1 + I_2 - I_3.
\end{aligned}$$

Taking account of

$$\int_{\beta_1}^{t_1} e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0 - \underline{u}_1)^2}\, \frac{\sin(\frac{\alpha}{b_1}\underline{u}_1)}{\underline{u}_1}\, d\mu_{t_1}(u_1) = \int_{\frac{\alpha}{b_1}\otimes\beta_1}^{t_1} e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0 - \frac{b_1}{\alpha}\underline{v}_1)^2}\, \frac{\sin \underline{v}_1}{\underline{v}_1}\, d\mu_{t_1}(v_1) \to 0 \tag{36}$$

as $\alpha \to \infty$, we find that the term I_3 tends to zero as $b_1 > 0$ and $\alpha \to \infty$.

Now let $\epsilon > 0$ be given. Since $f \in L^1(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$, there exists a number $\beta_1 \in \mathbb{R}_{t_1}$ such that

$$\frac{2}{\pi} \int_{\beta_1}^{t_1} |f(x_0 \ominus u_1,y)|\, d\mu_{t_1}(u_1) \le \epsilon.$$

Thus, it follows that the term I_2 is bounded:

$$\begin{aligned}|I_2| &= \Big|\frac{2}{\pi}\Big(\int_{\beta_1}^{t_1} f(x_0 \ominus u_1, y)\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0-\underline{u}_1)^2}\,\frac{\sin(\frac{\alpha}{b_1}\underline{u}_1)}{\underline{u}_1}\, d\mu_{t_1}(u_1)\Big)e^{-\mathbf{i}\frac{a_1}{2b_1}\underline{x}_0^2}\Big| \\ &\leq \frac{2\alpha}{\pi b_1}\int_{\beta_1}^{t_1} |f(x_0 \ominus u_1, y)|\, d\mu_{t_1}(u_1).\end{aligned}$$

Now, we see that the function in the term I_1,

$$g(u_1, y) = \frac{f(x_0 \ominus u_1, y) - f(x_0^-, y)}{\underline{u}_1}$$

is h-continuous except for jumps in the interval $(0, \beta_1) \times \mathbb{R}_{t_2}$, and it has the finite limit

$$g(0^+, y) = \lim_{u_1 \to 0^+} \frac{f(x_0 \ominus u_1, y) - f(x_0^-, y)}{\underline{u}_1} = \frac{\partial^h f}{\partial x_0^+}(x_0, y).$$

This means that g is uniformly bounded in y and thus is h-integrable on the interval. By Property 8 of Proposition 3.3, we conclude that $I_1 \to 0$ as $\beta_1 \to t_1$.

All this together gives, since ϵ can be taken as small as we wish,

$$\frac{2}{\pi}\Big(\int_0^{t_1} f(x_0 \ominus u_1, y)\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{x}_0-\underline{u}_1)^2}\,\frac{\sin(\frac{\alpha}{b_1}\underline{u}_1)}{\underline{u}_1}\, d\mu_{t_1}(u_1)\Big)e^{-\mathbf{i}\frac{a_1}{2b_1}\underline{x}_0^2} \to f(x_0^-, y)$$

as $\alpha \to \infty$.

A similar argument also implies that the corresponding integral over the interval $(-t_1, 0)$ tends to $f(x_0^+, y)$ uniformly in y.

By taking the mean value of the two results, the statement of the theorem is completed. □

4 The right-sided QHLCT

In this section, we introduce the definition of the right-sided QHLCT associated with two-dimensional quaternion-valued signals and study some of its elementary properties, such as the Riemann-Lebesgue Lemma, the Plancherel and Parseval Theorems, and an inversion formula.

4.1 Definition and properties

The right-sided QHLCT, which gives the hyperbolic counterpart of the corresponding Euclidean right-sided QLCT [17], is based on two distinct families of 1D quaternionic hyperbolic plane waves of the form (14), each depending on a different quaternion basis unit (an essential feature to maintain the separation between the two dimensions).

Definition 4.1 (The right-sided QHLCT)**.** *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ for $k = 1, 2$, and let $(\omega_1, \omega_2) \in \mathbb{R}^2$. The right-sided QHLCT of quaternion signals f are defined as*

$$\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) = \begin{cases} \int_{\mathbb{R}^2_{t_1,t_2}} f(x_1,x_2)K^{\mathbf{i}}_{A_1}(x_1,\omega_1)K^{\mathbf{j}}_{A_2}(x_2,\omega_2)\, d\mu_{t_1,t_2}(x_1,x_2), & b_1, b_2 \neq 0, \\ \sqrt{d_1}\int_{\mathbb{R}_{t_2}} f(d_1\omega_1, x_2)\, e^{\mathbf{i}\frac{c_1 d_1}{2}\omega_1^2} K^{\mathbf{j}}_{A_2}(x_2,\omega_2)\, d\mu_{t_2}(x_2), & b_1 = 0, b_2 \neq 0, \\ \sqrt{d_2}\int_{\mathbb{R}_{t_1}} f(x_1, d_2\omega_2)K^{\mathbf{i}}_{A_1}(x_1,\omega_1)\, e^{\mathbf{j}\frac{c_2 d_2}{2}\omega_2^2}\, d\mu_{t_1}(x_1), & b_1 \neq 0, b_2 = 0, \\ \sqrt{d_1 d_2}\, f(d_1\omega_1, d_2\omega_2)\, e^{\mathbf{i}\frac{c_1 d_1}{2}\omega_1^2} e^{\mathbf{j}\frac{c_2 d_2}{2}\omega_2^2}, & b_1 = b_2 = 0. \end{cases} \tag{37}$$

Here $K^q_{A_k}(x_k, \omega_k)$ for $k = 1, 2$ have the same meaning as in (14)*. A sufficient condition for the integrals to exist is that $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$.*

The following lemma shows that the right-sided QHLCT separates a real signal f into four quaternionic components, i.e., the even-even, odd-even, even-odd, and odd-odd components of f.

Lemma 4.1. *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_kd_k - b_kc_k = 1$ with $b_k \neq 0$ for $k = 1, 2$. The following closed-form representation of the right-sided QHLCT of a function $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ holds:*

$$\begin{aligned}
&\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1, \omega_2)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2_{t_1,t_2}} f(x_1, x_2)\,\frac{1}{\sqrt{b_1\mathbf{i}}}\cos\left(\frac{a_1}{2b_1}\underline{x}_1^2 - \frac{1}{b_1}\underline{x}_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2\right)\cos\left(\frac{a_2}{2b_2}\underline{x}_2^2 - \frac{1}{b_2}\underline{x}_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2\right)\frac{1}{\sqrt{b_2\mathbf{j}}}\,d\mu_{t_1,t_2}(x_1, x_2)\\
&+ \frac{1}{2\pi}\int_{\mathbb{R}^2_{t_1,t_2}} f(x_1, x_2)\,\frac{\mathbf{i}}{\sqrt{b_1\mathbf{i}}}\sin\left(\frac{a_1}{2b_1}\underline{x}_1^2 - \frac{1}{b_1}\underline{x}_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2\right)\cos\left(\frac{a_2}{2b_2}\underline{x}_2^2 - \frac{1}{b_2}\underline{x}_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2\right)\frac{1}{\sqrt{b_2\mathbf{j}}}\,d\mu_{t_1,t_2}(x_1, x_2)\\
&+ \frac{1}{2\pi}\int_{\mathbb{R}^2_{t_1,t_2}} f(x_1, x_2)\,\frac{\mathbf{j}}{\sqrt{b_1\mathbf{i}}}\cos\left(\frac{a_1}{2b_1}\underline{x}_1^2 - \frac{1}{b_1}\underline{x}_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2\right)\sin\left(\frac{a_2}{2b_2}\underline{x}_2^2 - \frac{1}{b_2}\underline{x}_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2\right)\frac{1}{\sqrt{b_2\mathbf{j}}}\,d\mu_{t_1,t_2}(x_1, x_2)\\
&+ \frac{1}{2\pi}\int_{\mathbb{R}^2_{t_1,t_2}} f(x_1, x_2)\,\frac{\mathbf{k}}{\sqrt{b_1\mathbf{i}}}\sin\left(\frac{a_1}{2b_1}\underline{x}_1^2 - \frac{1}{b_1}\underline{x}_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2\right)\sin\left(\frac{a_2}{2b_2}\underline{x}_2^2 - \frac{1}{b_2}\underline{x}_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2\right)\frac{1}{\sqrt{b_2\mathbf{j}}}\,d\mu_{t_1,t_2}(x_1, x_2).
\end{aligned} \tag{38}$$

Figure 1 shows some of the basis functions of the QHLCT in the hyperbolic spatial domain. The frequency parameter is modified from image to image.

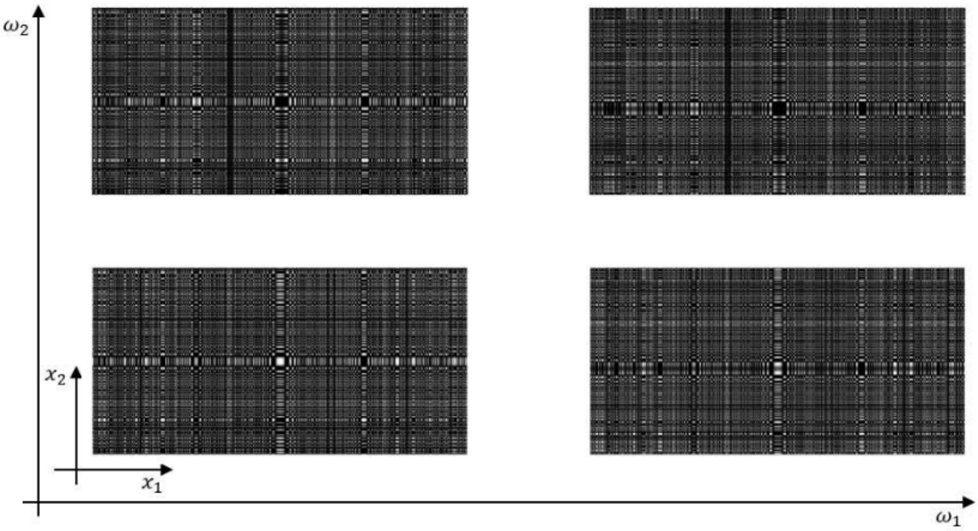

Figure 1: The small images are intensity images of the basis function of the first component in (38), up to the dilation and phase factors $1/\sqrt{b_1\mathbf{i}}$ and $1/\sqrt{b_2\mathbf{j}}$, with parameters $t_1 = 20$ and $t_2 = 10$, $A = (5, 3; 3, 2)$, $\omega_1 \in \{\pi/2, 2\pi\}$, and $\omega_2 \in \{\pi/4, 8\pi\}$.

A relationship between the right-sided QHLCT and the right-sided partial QHLCTs follows.

Lemma 4.2. *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_kd_k - b_kc_k = 1$ with $b_k \neq 0$ for $k = 1, 2$, and let $(\omega_1, \omega_2) \in \mathbb{R}^2$. The right-sided QHLCT of a function $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ is obtained from the composition of two right-sided partial QHLCTs of f as follows:*

$$\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1, \omega_2) = \mathcal{L}^{\mathbf{j}}_{A_2}\left(\mathcal{L}^{\mathbf{i}}_{A_1}(f)\right)(\omega_1, \omega_2). \tag{39}$$

Proof. According to Definitions 3.2 and 4.1, direct computation yields to

$$\begin{aligned}
\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1, \omega_2) &= \int_{\mathbb{R}_{t_2}}\Big(\int_{\mathbb{R}_{t_1}} f(x_1, x_2)K^{\mathbf{i}}_{A_1}(x_1, \omega_1)\,d\mu_{t_1}(x_1)\Big)K^{\mathbf{j}}_{A_2}(x_2, \omega_2)\,d\mu_{t_2}(x_2)\\
&= \int_{\mathbb{R}_{t_2}} \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2)K^{\mathbf{j}}_{A_2}(x_2, \omega_2)\,d\mu_{t_2}(x_2)\\
&= \mathcal{L}^{\mathbf{j}}_{A_2}\left(\mathcal{L}^{\mathbf{i}}_{A_1}(f)\right)(\omega_1, \omega_2).
\end{aligned}$$

□

The following proposition describes the fundamental mapping properties of the right-sided QHLCT. (We assume here that $b_k \neq 0$ for $k = 1, 2$ as before.)

Proposition 4.1. *If $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then*

1. $\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}$ *is bounded and uniformly h-continuous (and hence a measurable function).*

2. *(Riemann Lebesgue Lemma)*
$$\lim_{|\omega_1|\to\infty} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) = 0$$
for all $\omega_2 \in \mathbb{R}$, and
$$\lim_{|\omega_2|\to\infty} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) = 0$$
for all $\omega_1 \in \mathbb{R}$.

Proof. In view of Definition 4.1 and (20), it follows that

$$|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2)| \leq \frac{1}{2\pi\sqrt{|b_1 b_2|}} \|f\|_{L^1(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}, \tag{40}$$

which shows that the transform is bounded.

We pass now to the uniform h-continuity of $\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)$. Since h-continuous functions are uniformly h-continuous in compact sets, it suffices to show that $\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)$ is h-continuous at every point (ω_1,ω_2).

Direct computation shows that

$$\begin{aligned}
&\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(\omega_1+\rho_1,\omega_2+\rho_2) - \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(\omega_1,\omega_2) \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} f(x_1,x_2)\left(K^{\mathbf{i}}_{A_1}(x_1,\omega_1+\rho_1)K^{\mathbf{j}}_{A_2}(x_2,\omega_2+\rho_2) - K^{\mathbf{i}}_{A_1}(x_1,\omega_1)K^{\mathbf{j}}_{A_2}(x_2,\omega_2+\rho_2)\right) d\mu_{t_1,t_2}(x_1,x_2).
\end{aligned}$$

For any $\rho_1,\rho_2 > 0$, the integrand is dominated by a constant multiple of $|f(x_1,x_2)|$. Now, since the factor inside the parentheses tends to zero, by Theorem 2.2, we find

$$\lim_{\rho_1,\rho_2\to 0}\left(\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(\omega_1+\rho_1,\omega_2+\rho_2) - \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(\omega_1,\omega_2)\right) = 0.$$

This establishes the statement of uniform h-continuity.

For the proof of the Riemann-Lebesgue Lemma, we use a density argument similar to Property 7 of Proposition 3.3 and assume that both f and $(\partial_h/\partial x_1)f$ are h-continuous with compact support. Such functions form a dense subspace in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. Combining Lemmas 3.1 and 4.2, we find the relation

$$\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) = \mathcal{F}^{\mathbf{j}}\left(\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1,x_2)\frac{1}{\sqrt{2\pi b_2 \mathbf{j}}}\, e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2}\right)\left(\omega_1,\frac{\omega_2}{b_2}\right)e^{\mathbf{j}\frac{d_2}{2b_2}\omega_2^2}. \tag{41}$$

Since $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1,x_2)\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}\, e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2} \in L^1(\mathbb{R}\times\mathbb{R}_{t_2}, \mathbb{H})$$

for $b_2 \neq 0$.

Now, according to Property 2 of Proposition 3.2, it follows that

$$|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2)| = \left|\mathcal{F}^{\mathbf{j}}\left(\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1,x_2)\frac{1}{\sqrt{2\pi b_2\mathbf{j}}}\, e^{\mathbf{j}\frac{a_2}{2b_2}x_2^2}\right)\left(\omega_1,\frac{\omega_2}{b_2}\right)\right| \to 0$$

as $|\omega_1| \to \infty$. Thus,

$$\lim_{|\omega_1|\to\infty} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) = 0$$

for all $\omega_2 \in \mathbb{R}$. Similarly, we can prove that

$$\lim_{|\omega_2|\to\infty} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) = 0$$

for all $\omega_1 \in \mathbb{R}$.

For any given $\epsilon > 0$, there exists a function f_ϵ in the above-mentioned dense class, such that

$$\|f - f_\epsilon\|_{L^1(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} \leq \epsilon.$$

Thus, under these conditions, by (40), we obtain

$$\begin{aligned}
|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2)| &\leq |\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f_\epsilon)(\omega_1,\omega_2)| + \frac{1}{2\pi\sqrt{|b_1 b_2|}}\, \|f - f_\varepsilon\|_{L^1(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} \\
&\leq |\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f_\epsilon)(\omega_1,\omega_2)| + \frac{1}{2\pi\sqrt{|b_1 b_2|}}\, \epsilon.
\end{aligned}$$

On account of the result just proved for the density class, it follows that

$$\lim_{\omega_1\to\pm\infty} |\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2)| \leq \frac{1}{2\pi\sqrt{|b_1 b_2|}}\, \epsilon$$

for all $\omega_2 \in \mathbb{R}$. Since ϵ is arbitrary, we have

$$\lim_{\omega_1\to\pm\infty} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) = 0$$

for all $\omega_2 \in \mathbb{R}$. Similarly,

$$\lim_{\omega_2\to\pm\infty} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) = 0$$

for all $\omega_1 \in \mathbb{R}$.

This furnishes the complete proof. □

We will now derive the Plancherel's Theorem for the right-sided QHLCT. It states that the quaternionic inner product of two quaternion signals is independent of the domain of description (hyperbolic spatial or frequency). The primary tool of the proof is Theorem 3.1.

Theorem 4.1 (Plancherel's Theorem for the right-sided QHLCT)**.** *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ with $b_k \neq 0$ for $k = 1, 2$. If $f, g \in L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then*

$$\langle f, g\rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} = \langle \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f), \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(g)\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{42}$$

In particular, if $f = g$, Parseval's identity reads as

$$\|f\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} = \|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)\|_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{43}$$

Further, the map $f \mapsto \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)$ has a unique extension to a continuous linear map from $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ into $L^2(\mathbb{R}^2, \mathbb{H})$ and (43) holds, whenever $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$.

Proof. Let $f, g \in L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. According to the definition (11) of the quaternionic inner product and using the identities (32) and (33), Fubini's Theorem, and Theorem 3.1, we obtain

$$\begin{aligned}
&\langle \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f), \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(g)\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2_{t_1,t_2}}\int_{\mathbb{R}^2_{t_1,t_2}} f(x_1,x_2) K^{\mathbf{i}}_{A_1}(x_1,\omega_1)\Big(K^{\mathbf{j}}_{A_2}(x_2,\omega_2)\,\overline{K^{\mathbf{j}}_{A_2}(y_2,\omega_2)}\Big) \\
&\qquad \times \overline{g(y_1,y_2)K^{\mathbf{i}}_{A_1}(y_1,\omega_1)}\, d\mu_{t_1,t_2}(x_1,x_2)d\mu_{t_1,t_2}(y_1,y_2)d\omega_1 d\omega_2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^2_{t_1,t_2}}\int_{\mathbb{R}^2_{t_1,t_2}} f(x_1,x_2) K^{\mathbf{i}}_{A_1}(x_1,\omega_1)\Big(\frac{1}{2\pi b_2}\, e^{\mathbf{j}\frac{a_2}{2b_2}(\underline{x}_2^2-\underline{y}_2^2)}\int_{\mathbb{R}} e^{-\mathbf{j}\frac{\omega_2}{b_2}(\underline{x}_2-\underline{y}_2)}d\omega_2\Big) \\
&\qquad \times \overline{g(y_1,y_2)K^{\mathbf{i}}_{A_1}(y_1,\omega_1)}\, d\mu_{t_1,t_2}(x_1,x_2)d\mu_{t_1,t_2}(y_1,y_2)d\omega_1 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^2_{t_1,t_2}}\int_{\mathbb{R}_{t_1}}\Big(\int_{\mathbb{R}_{t_2}} f(x_1,x_2) K^{\mathbf{i}}_{A_1}(x_1,\omega_1) e^{\mathbf{j}\frac{a_2}{2b_2}(\underline{x}_2^2-\underline{y}_2^2)}\delta(\underline{y}_2-\underline{x}_2)\, d\mu_{t_2}(x_2)\Big) \\
&\qquad \times \overline{g(y_1,y_2)K^{\mathbf{i}}_{A_1}(y_1,\omega_1)}\, d\mu_{t_1}(x_1)d\mu_{t_1,t_2}(y_1,y_2)d\omega_1 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}_{t_2}}\Big(\int_{\mathbb{R}_{t_1}} f(x_1,y_2)K^{\mathbf{i}}_{A_1}(x_1,\omega_1)\, d\mu_{t_1}(x_1)\Big) \\
&\qquad \times \Big(\int_{\mathbb{R}_{t_1}} \overline{g(y_1,y_2)K^{\mathbf{i}}_{A_1}(y_1,\omega_1)}\, d\mu_{t_1}(y_1)\Big)\, d\mu_{t_2}(y_2)d\omega_1 \\
&= \langle \mathcal{L}^{\mathbf{i}}_{A_1}(f), \mathcal{L}^{\mathbf{i}}_{A_1}(g)\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} \\
&= \langle f, g\rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}.
\end{aligned}$$

From this point, the statement that the right-sided QHLCT can be extended uniquely to the whole of $L^2(\mathbb{R}^2, \mathbb{H})$ follows the same way as in the proof of Theorem 3.1. This completes the proof. □

Parseval's identity (43) shows that the total energy of a signal $f(x_1, x_2)$ can be obtained by calculating the energy in the hyperbolic spatial domain or in the frequency domain. As the signal f and the spectrum $\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)$ contain the same amount of energy, the law of conservation of energy applies to the right-sided QHLCT.

4.2 Inversion formula

We now derive an inversion formula for the right-sided QHLCT, which includes the corresponding result of [10] as a particular case. (This is proved in a similar manner as in [20], but it is necessary to employ the definitions (18) and (37).)

Theorem 4.2 (Inversion theorem for the right-sided QHLCT)**.** *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ with $b_k \neq 0$ for $k = 1, 2$, and let $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. The inversion formula for the right-sided QHLCT given by (37) is*

$$f(x_1, x_2) = \int_{\mathbb{R}^2} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1, \omega_2) K^{\mathbf{j}}_{A_2^{-1}}(\omega_2, x_2) K^{\mathbf{i}}_{A_1^{-1}}(\omega_1, x_1)\, d\omega_1 d\omega_2 \tag{44}$$

for a.e. $(x_1, x_2) \in \mathbb{R}^2_{t_1,t_2}$, whenever the integral exists. A sufficient condition for the integral to exist is that $\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f) \in L^1(\mathbb{R}^2, \mathbb{H})$.

Proof. Since by assumption $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then $\mathcal{L}^{\mathbf{i}}_{A_1}(f) \in L^1(\mathbb{R}_{t_2}, \mathbb{H})$ (of the variable $x_2 \in \mathbb{R}_{t_2}$.) This implies that

$$\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}}\, e^{\mathbf{j}\frac{a_2}{2b_2}\underline{x}_2^2} \in L^1(\mathbb{R}^2, \mathbb{H}).$$

Now, since $\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f) \in L^1(\mathbb{R}^2, \mathbb{H})$, by identity (41), we find

$$\mathcal{F}^{\mathbf{j}}\Big(\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}}\, e^{\mathbf{j}\frac{a_2}{2b_2}\underline{x}_2^2}\Big)\Big(\omega_1, \frac{\omega_2}{b_2}\Big) e^{\mathbf{j}\frac{d_2}{2b_2}\omega_2^2} \in L^1(\mathbb{R}^2, \mathbb{H}).$$

Moreover, by the inversion of the right-sided partial QHFT (22) and using (41) again, it follows that

$$\begin{aligned}
\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}}\, e^{\mathbf{j}\frac{a_2}{2b_2}\underline{x}_2^2} &= \frac{1}{2\pi b_2} \int_{\mathbb{R}} \mathcal{F}^{\mathbf{j}}\Big(\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) \frac{1}{\sqrt{2\pi b_2 \mathbf{j}}}\, e^{\mathbf{j}\frac{a_2}{2b_2}\underline{x}_2^2}\Big)\Big(\omega_1, \frac{\omega_2}{b_2}\Big) e^{\mathbf{j}\frac{\omega_2}{b_2}\underline{x}_2}\, d\omega_2 \\
&= \frac{1}{2\pi b_2} \int_{\mathbb{R}} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1, \omega_2) e^{-\mathbf{j}\left(\frac{d_2}{2b_2}\omega_2^2 - \frac{1}{b_2}\omega_2\underline{x}_2\right)}\, d\omega_2
\end{aligned}$$

for almost every x_2.

With these computations at hand, together with (17), we further obtain

$$\begin{aligned}
\mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) &= \int_{\mathbb{R}} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1, \omega_2) e^{-\mathbf{j}\left(\frac{d_2}{2b_2}\omega_2^2 - \frac{1}{b_2}\omega_2\underline{x}_2 + \frac{a_2}{2b_2}\underline{x}_2^2\right)} \frac{1}{\sqrt{2\pi(-b_2)\mathbf{j}}}\, d\omega_2 \\
&= \int_{\mathbb{R}} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1, \omega_2) K^{\mathbf{j}}_{A_2^{-1}}(\omega_2, x_2)\, d\omega_2
\end{aligned} \tag{45}$$

for almost every x_2.

Now an argument similar to that used above, in combination with (24) and the fact that $\mathcal{L}^{\mathbf{i}}_{A_1}(f) \in L^1(\mathbb{R}, \mathbb{H})$ (in the variable ω_1), give after some simplification

$$f(x_1, x_2) \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}}\, e^{\mathbf{j}\frac{d_1}{2b_1}\underline{x}_1^2} = \frac{1}{2\pi b_1} \int_{\mathbb{R}} \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) e^{-\mathbf{i}\left(\frac{d_1}{2b_1}\omega_1^2 - \frac{1}{b_1}\omega_1\underline{x}_1\right)}\, d\omega_1.$$

As a consequence of the last equality, from (45) and Fubini's Theorem we obtain

$$\begin{aligned}
f(x_1, x_2) &= \int_{\mathbb{R}} \mathcal{L}^{\mathbf{i}}_{A_1}(f)(\omega_1, x_2) K^{\mathbf{i}}_{A_1^{-1}}(\omega_1, x_1)\, d\omega_1 \\
&= \int_{\mathbb{R}^2} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1, \omega_2) K^{\mathbf{j}}_{A_2^{-1}}(\omega_2, x_2) K^{\mathbf{i}}_{A_1^{-1}}(\omega_1, x_1)\, d\omega_1 d\omega_2
\end{aligned}$$

for a.e. $(x_1, x_2) \in \mathbb{R}^2_{t_1,t_2}$, which is the inversion formula (44) for the right-sided QHLCT. (Here the interchange of the order of integration is permitted since the integrals converge absolutely.)

This establishes the statement. ∎

In the large limits of t_1 and t_2, $t_1, t_2 \to +\infty$, the inversion formula (44) reduces to the corresponding result for the standard Euclidean right-sided QLCT; see [20].

4.3 Hyperbolic differentiation properties

In this subsection, we derive some equalities between integrals in spatial and frequency domains involving the hyperbolic spatial differentiation property of a quaternion signal f and the right-sided QHLCT of f, which will be used to establish the hyperbolic counterparts of the Heisenberg uncertainty principle for the proposed transform. (For the proof, we use the techniques of Theorem 4.1.)

Theorem 4.3. *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_kd_k - b_kc_k = 1$ with $b_k \neq 0$ for $k = 1, 2$, and let $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. If $(\partial_h/\partial x_k)f$ exists for $k = 1, 2$ and is in $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then*

$$\int_{\mathbb{R}^2} \omega_1^2 \left|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}\left(e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2} f(x_1,x_2)\right)(\omega_1,\omega_2)\right|^2 d\omega_1 d\omega_2 = b_1^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left|\frac{\partial_h}{\partial x_1} f(x_1,x_2) + \frac{a_1}{b_1}\,\underline{x}_1 \mathbf{i} f(x_1,x_2)\right|^2 d\mu_{t_1,t_2}(x_1,x_2) \tag{46}$$

and

$$\int_{\mathbb{R}^2} \omega_2^2 \left|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}\left(e^{\mathbf{j}\frac{a_2}{2b_2}\underline{x}_2^2} f(x_1,x_2)\right)(\omega_1,\omega_2)\right|^2 d\omega_1 d\omega_2 = b_2^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left|\frac{\partial_h}{\partial x_2} f(x_1,x_2) + \frac{a_2}{b_2}\,\underline{x}_2 \mathbf{j} f(x_1,x_2)\right|^2 d\mu_{t_1,t_2}(x_1,x_2). \tag{47}$$

Proof. We only prove (46). (No other modifications in the argument are necessary to establish (47).)

From the definition (11) of the quaternionic inner product and Fubini's Theorem, combined with (32) and (33), we find

$$\begin{aligned}
&\int_{\mathbb{R}^2} \omega_1^2 \left|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}\left(e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2} f(x_1,x_2)\right)(\omega_1,\omega_2)\right|^2 d\omega_1 d\omega_2\\
&= \int_{\mathbb{R}^2} \omega_1^2 \Big(\int_{\mathbb{R}^2_{t_1,t_2}} e^{\mathbf{i}\frac{a_1}{2b_1}\underline{y}_1^2} f(y_1,y_2) K^{\mathbf{i}}_{A_1}(y_1,\omega_1) K^{\mathbf{j}}_{A_2}(y_2,\omega_2)\, d\mu_{t_1,t_2}(y_1,y_2)\Big)\\
&\quad \times \Big(\overline{\int_{\mathbb{R}^2_{t_1,t_2}} e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2} f(x_1,x_2) K^{\mathbf{i}}_{A_1}(x_1,\omega_1) K^{\mathbf{j}}_{A_2}(x_2,\omega_2)\, d\mu_{t_1,t_2}(x_1,x_2)}\Big)\, d\omega_1 d\omega_2\\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} \omega_1^2\, e^{\mathbf{i}\frac{a_1}{2b_1}\underline{y}_1^2} f(y_1,y_2) K^{\mathbf{i}}_{A_1}(y_1,\omega_1) \left(\int_{\mathbb{R}} K^{\mathbf{j}}_{A_2}(y_2,\omega_2) \overline{K^{\mathbf{j}}_{A_2}(x_2,\omega_2)}\, d\omega_2\right)\\
&\quad \times \overline{K^{\mathbf{i}}_{A_1}(x_1,\omega_1)}\; \overline{e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2} f(x_1,x_2)}\, d\mu_{t_1,t_2}(y_1,y_2) d\mu_{t_1,t_2}(x_1,x_2) d\omega_1\\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}_{t_1}} \omega_1^2 \Big(\int_{\mathbb{R}_{t_2}} e^{\mathbf{i}\frac{a_1}{2b_1}\underline{y}_1^2} f(y_1,y_2) K^{\mathbf{i}}_{A_1}(y_1,\omega_1) e^{\mathbf{j}\frac{a_2}{2b_2}(\underline{y}_2^2-\underline{x}_2^2)} \delta(\underline{x}_2 - \underline{y}_2)\, d\mu_{t_2}(y_2)\Big)\\
&\quad \times \overline{K^{\mathbf{i}}_{A_1}(x_1,\omega_1)}\; \overline{e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2} f(x_1,x_2)}\, d\mu_{t_1}(y_1) d\mu_{t_1,t_2}(x_1,x_2) d\omega_1\\
&= \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}_{t_1}} e^{\mathbf{i}\frac{a_1}{2b_1}\underline{y}_1^2} f(y_1,y_2) \Big(\int_{\mathbb{R}} \omega_1^2\, K^{\mathbf{i}}_{A_1}(y_1,\omega_1) \overline{K^{\mathbf{i}}_{A_1}(x_1,\omega_1)}\, d\omega_1\Big)\\
&\quad \times \overline{e^{\mathbf{i}\frac{a_1}{2b_1}\underline{x}_1^2} f(x_1,x_2)}\, d\mu_{t_1}(y_1) d\mu_{t_1,t_2}(x_1,x_2).
\end{aligned} \tag{48}$$

Now, we evaluate the integral inside the parentheses. According to Property 1 of Proposition 3.2, we have, after some simplification

$$\int_{\mathbb{R}} \omega_1^2\, K^{\mathbf{i}}_{A_1}(y_1,\omega_1) \overline{K^{\mathbf{i}}_{A_1}(x_1,\omega_1)}\, d\omega_1 = b_1^2\, e^{\mathbf{i}\frac{a_1}{2b_1}(\underline{y}_1^2-\underline{x}_1^2)} \frac{1}{2\pi} \int_{\mathbb{R}} \omega_1^2\, e^{\mathbf{i}\omega_1(\underline{x}_1-\underline{y}_1)}\, d\omega_1 = -\frac{\partial_h^2}{\partial x_1^2}\, \delta(x_1 - y_1).$$

With these computations at hand, we can simplify (48) to

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \omega_1^2 \left| \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)} \left(e^{\mathbf{i} \frac{a_1}{2b_1} \underline{x}_1^2} f(x_1,x_2) \right)(\omega_1,\omega_2) \right|^2 d\omega_1 d\omega_2 \\
&= b_1^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left(\int_{\mathbb{R}_{t_1}} e^{\mathbf{i} \frac{a_1}{2b_1} \underline{y}_1^2} f(y_1,x_2) e^{\mathbf{i} \frac{a_1}{2b_1} (\underline{y}_1^2 - \underline{x}_1^2)} \frac{\partial_h^2}{\partial x_1^2} \delta(y_1 - x_1)\, d\mu_{t_1}(y_1) \right) \overline{e^{\mathbf{i} \frac{a_1}{2b_1} \underline{x}_1^2} f(x_1,x_2)}\, d\mu_{t_1,t_2}(x_1,x_2) \\
&= b_1^2 \int_{\mathbb{R}^2_{t_1,t_2}} e^{\mathbf{i} \frac{a_1}{2b_1} \underline{x}_1^2} f(x_1,x_2) \frac{\partial_h^2}{\partial x_1^2} \overline{e^{\mathbf{i} \frac{a_1}{2b_1} \underline{x}_1^2} f(x_1,x_2)}\, d\mu_{t_1,t_2}(x_1,x_2) \\
&= b_1^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left| \frac{\partial_h}{\partial x_1} \left(e^{\mathbf{i} \frac{a_1}{2b_1} \underline{x}_1^2} f(x_1,x_2) \right) \right|^2 d\mu_{t_1,t_2}(x_1,x_2) \\
&= b_1^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left| e^{\mathbf{i} \frac{a_1}{2b_1} \underline{x}_1^2} \left(\frac{\partial_h}{\partial x_1} f(x_1,x_2) + \frac{a_1}{b_1} \underline{x}_1 \mathbf{i} f(x_1,x_2) \right) \right|^2 d\mu_{t_1,t_2}(x_1,x_2) \\
&= b_1^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left| \frac{\partial_h}{\partial x_1} f(x_1,x_2) + \frac{a_1}{b_1} \underline{x}_1 \mathbf{i} f(x_1,x_2) \right|^2 d\mu_{t_1,t_2}(x_1,x_2),
\end{aligned}
$$

which is the same as (46). This establishes the statement. □

By similar arguments used in the proof of the previous theorem, we also obtain the following result.

Theorem 4.4. *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ with $b_k \neq 0$ for $k = 1, 2$, and let $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. If $(\partial_h / \partial x_k) f$ exists for $k = 1, 2$ and is in $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then*

$$
\int_{\mathbb{R}^2} \omega_k^2 \, |\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2 \; = \; b_k^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left| \frac{\partial_h}{\partial x_k} f(x_1,x_2) \right|^2 d\mu_{t_1,t_2}(x_1,x_2). \tag{49}
$$

Equality (49) is the generalization of [17, Lemma 17], which corresponds to the case of the Euclidean right-sided QLCT.

Properties of the right-sided QHLCT are summarized in Table 1.

Table 1: Properties of the right-sided QHLCT.

Left-linearity	$\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(\alpha f(x_1,x_2) + \beta g(x_1,x_2)) =$	$\alpha \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) + \beta \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(g)(\omega_1,\omega_2)$, $\alpha, \beta \in \mathbb{H}$
Inversion	$f(x_1,x_2) =$	$\int_{\mathbb{R}^2} \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2) K^{\mathbf{j}}_{A_2^{-1}}(\omega_2,x_2) K^{\mathbf{i}}_{A_1^{-1}}(\omega_1,x_1) d\omega_1 d\omega_2$
Plancherel	$\langle f, g \rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} =$	$\langle \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f), \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(g) \rangle_{L^2(\mathbb{R}^2,\mathbb{H})}$
Parseval	$\\|f\\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} =$	$\\|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)\\|_{L^2(\mathbb{R}^2,\mathbb{H})}$
h-Derivatives	$\int_{\mathbb{R}^2} \omega_k^2 \vert\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f)(\omega_1,\omega_2)\vert^2 d\omega_1 d\omega_2 =$	$b_k^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left\vert \frac{\partial_h}{\partial x_k} f(x_1,x_2) \right\vert^2 d\mu_{t_1,t_2}(x_1,x_2)$, $k = 1, 2$
	$\int_{\mathbb{R}^2} \omega_k^2 \vert\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(e^{q \frac{a_k}{2b_k} \underline{x}_k^2} f(x_1,x_2))(\omega_1,\omega_2)\vert^2 d\omega_1 d\omega_2 =$	$b_k^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left\vert \frac{\partial_h}{\partial x_k} f(x_1,x_2) + \frac{a_k}{b_k} \underline{x}_k q f(x_1,x_2) \right\vert^2 d\mu_{t_1,t_2}(x_1,x_2)$, $k = 1, 2$

5 Uncertainty principles for the right-sided QHLCT

In this section, we derive two quaternionic analogues of the Heisenberg uncertainty principle for the right-sided QHLCT, which generalize the uncertainty principle due to Kou et al. in [17] within the hyperbolic context. These results assert that a quaternion signal and its right-sided QHLCT cannot be well-concentrated around their respective means: narrowing one broadens necessarily the other. More precisely, we give a lower bound on the product of the effective widths of quaternion signals in the hyperbolic spatial and frequency domains. It is shown that only hyperbolic Gaussian quaternion functions minimize the uncertainty relations. Many variations and related information about this result can be found in [19]. Other versions of the Heisenberg uncertainty principle were given for the right-sided QFT by Bahri et al. [1], for the two-sided QFT by Hitzer [15] and Chen et al. [5], and the QHFT by Ferreira et al. [10].

We have the following definition.

Definition 5.1. *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ with $b_k \neq 0$ for $k = 1, 2$, and let $g, \underline{x}_k g \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, and $\omega_k \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(g) \in L^2(\mathbb{R}^2, \mathbb{H})$. We define the normalized* *effective hyperbolic spatial width (or spatial uncertainty) of g as*

$$\triangle x_k := \frac{\|\underline{x}_k g\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}}{\|g\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}},$$

and the normalized effective spectral width (or frequency uncertainty) of $\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(g)$ *as*

$$\triangle \omega_k := \frac{\|\omega_k \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(g)\|_{L^2(\mathbb{R}^2,\mathbb{H})}}{\|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(g)\|_{L^2(\mathbb{R}^2,\mathbb{H})}}.$$

Now we are able to prove our main results of this section. Theorem 5.1 gives the uncertainty associated with the right-sided QHLCT for quaternion-valued signals of the form $e^{q\frac{a_k}{2b_k}\underline{x}_k^2} f(x_1,x_2)$, with $f \in L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$, while Theorem 5.2 gives the classical uncertainty for signals $f(x_1,x_2)$.

Theorem 5.1 (Heisenberg-type Uncertainty Principle I)**.** *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ with $b_k \neq 0$ for $k = 1,2$. If $f, \underline{x}_k f \in L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$, $(\partial_h/\partial x_k) f$ exists for $k=1,2$ and is in $L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$, and $\omega_k \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(e^{q\frac{a_k}{2b_k}\underline{x}_k^2} f) \in L^2(\mathbb{R}^2,\mathbb{H})$, then the following inequality holds:*

$$\triangle x_1 \, \triangle x_2 \, \triangle \omega_1 \, \triangle \omega_2 \geq \frac{b_1 b_2}{4}. \tag{50}$$

Further, equality in (50) *holds when f is a hyperbolic Gaussian quaternion function of the form*

$$f(x_1,x_2) = Ce^{-\frac{1}{2}\left(\left(t_1 \tanh^{-1}\left(\sqrt{\frac{1}{\beta_1}+\frac{a_1}{b_1}\mathbf{i}} \otimes \frac{x_1}{t_1}\right)\right)^2 + \left(t_2 \tanh^{-1}\left(\sqrt{\frac{1}{\beta_2}+\frac{a_2}{b_2}\mathbf{j}} \otimes \frac{x_2}{t_2}\right)\right)^2\right)}, \tag{51}$$

where β_1, β_2 are real positive constants and $C = \|f\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} (\beta_1\beta_2\pi^2)^{-1/4}$.

Proof. Because the proof is relatively technical, we will break it into two parts to make it easier for the reader to follow. In part one, we show that

$$\triangle x_k \, \triangle \omega_k \geq \frac{b_k}{2} \tag{52}$$

for $k = 1,2$. In part two, we prove that (50) becomes an equality when f is of the form (51).

Part One: Let $f \in L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})$. Applying Definition 5.1 to $g(x_1,x_2) = e^{q\frac{a_k}{2b_k}\underline{x}_k^2} f(x_1,x_2)$, combined with Parseval's identity (43) and Proposition 4.3, yields

$$\begin{aligned}(\triangle x_k \, \triangle \omega_k)^2 &= \frac{\left(\int_{\mathbb{R}^2_{t_1,t_2}} \underline{x}_k^2 \, |e^{q\frac{a_k}{2b_k}\underline{x}_k^2} f(x_1,x_2)|^2 \, d\mu_{t_1,t_2}(x_1,x_2)\right)}{\|e^{q\frac{a_k}{2b_k}\underline{x}_k^2} f\|^2_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}} \frac{\left(\int_{\mathbb{R}^2} \omega_k^2 \, |\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(e^{q\frac{a_k}{2b_k}\underline{x}_k^2} f)(\omega_1,\omega_2)|^2 \, d\omega_1 d\omega_2\right)}{\|\mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(e^{q\frac{a_k}{2b_k}\underline{x}_k^2} f)\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}} \\ &= \frac{\int_{\mathbb{R}^2_{t_1,t_2}} \underline{x}_k^2 \, |f(x_1,x_2)|^2 \, d\mu_{t_1,t_2}(x_1,x_2)}{\|f\|^4_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}} b_k^2 \int_{\mathbb{R}^2_{t_1,t_2}} \left|\frac{\partial_h}{\partial x_k} f(x_1,x_2) + \frac{a_k}{b_k} q \underline{x}_k f(x_1,x_2)\right|^2 d\mu_{t_1,t_2}(x_1,x_2).\end{aligned} \tag{53}$$

Now, using the Cauchy-Bunyakovsky-Schwarz inequality,

$$|\langle h_1, h_2\rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}| \leq \|h_1\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} \, \|h_2\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}, \tag{54}$$

where

$$h_1(x_1,x_2) = \underline{x}_k \, f(x_1,x_2) \tag{55}$$

and

$$h_2(x_1,x_2) = \frac{\partial_h}{\partial x_k} f(x_1,x_2) + \frac{a_k}{b_k} \, \underline{x}_k q f(x_1,x_2), \tag{56}$$

combined with Property 2 of Proposition 2.1 and (53), we have after some simplification

$$(\triangle x_k \triangle \omega_k)^2 \geq \frac{b_k^2}{4} \frac{\left(\int_{\mathbb{R}^2_{t_1,t_2}} \underline{x}_k \, \frac{\partial_h}{\partial x_k}(|f(x_1,x_2)|^2) \, d\mu_{t_1,t_2}(x_1,x_2)\right)^2}{\|f\|^4_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}}. \tag{57}$$

This reduces the statement to the discussion of the integral in (57).

Integration by parts (see Proposition 2.2 above) yields

$$\int_{\mathbb{R}^2_{t_1,t_2}} \underline{x}_k \frac{\partial_h}{\partial x_k}(|f(x_1,x_2)|^2)\, d\mu_{t_1,t_2}(x_1,x_2) = \int_{\mathbb{R}_{t_m}} \Big(\underline{x}_k\, |f(x_1,x_2)|^2 \big|_{x_k=-t_k}^{x_k=t_k} - \int_{\mathbb{R}_{t_k}} |f(x_1,x_2)|^2\, d\mu_{t_k}(x_k) \Big)\, d\mu_{t_m}(x_m)$$
$$= -\|f\|^2_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})},$$

where $m \in \{1,2\}$, $m \neq k$. This proves (52) and, hence, (50).

Part Two: We show that the equality in (50) is satisfied when f is of the form (51).

First, we observe that (54) becomes an equality when the two functions, h_1 and h_2, defined respectively by (55) and (56), are proportional, i.e.,

$$\underline{x}_k\, f(x_1,x_2) = \beta_k u\Big(\frac{\partial_h}{\partial x_1}\, f(x_1,x_2) + \frac{a_k}{b_k}\, \underline{x}_k q f(x_1,x_2) \Big), \tag{58}$$

where β_k are real positive constants and u is a unit quaternion.

Moreover, since

$$|\mathrm{Sc}(\langle h_1,h_2\rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})})| \leq |\langle h_1,h_2\rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}|,$$

the sign of equality in (54) also implies that

$$-\,\mathrm{Sc}(h_1(x_1,x_2)\, \overline{h_2(x_1,x_2)}) = |h_1(x_1,x_2)\, \overline{h_2(x_1,x_2)}|. \tag{59}$$

From (59), we obtain

$$-\, \underline{x}_k\, f(x_1,x_2)\, \overline{\Big(\frac{\partial_h}{\partial x_k} f(x_1,x_2) + \frac{a_k}{b_k}\, \underline{x}_k q f(x_1,x_2) \Big)} \geq 0. \tag{60}$$

Now, multiplying both sides of (58) from the right by

$$\overline{\frac{\partial_h}{\partial x_k} f(x_1,x_2) + \frac{a_k}{b_k}\, \underline{x}_k q f(x_1,x_2)}$$

it easily follows that

$$-\underline{x}_k\, f(x_1,x_2) \overline{\Big(\frac{\partial_h}{\partial x_k} f(x_1,x_2) + \frac{a_k}{b_k}\, \underline{x}_k q f(x_1,x_2) \Big)} \;=\; -\beta_k u \left| \frac{\partial_h}{\partial x_1}\, f(x_1,x_2) + \frac{a_k}{b_k}\, \underline{x}_k q f(x_1,x_2) \right|^2.$$

Applying (60), we find that $u = -1$. Combining this with our previous results gives

$$\frac{\partial_h}{\partial x_k} f(x_1,x_2) = -\Big(\frac{1}{\beta_k} + \frac{a_k}{b_k} q \Big)\, \underline{x}_k\, f(x_1,x_2). \tag{61}$$

Finally, solving (61), we find that f must be a hyperbolic Gaussian-type quaternion signal of the form

$$f(x_1,x_2) = C\, e^{-\frac{1}{2}\left(\left(\frac{1}{\beta_1} + \frac{a_1}{b_1}\mathbf{i}\right)\underline{x}_1^2 + \left(\frac{1}{\beta_2} + \frac{a_2}{b_2}\mathbf{j}\right)\underline{x}_2^2 \right)}, \tag{62}$$

where the value of $C \in \mathbb{R}$ is found to ensure that the function f is of unit norm.

Since $1/\beta_k + (a_k/b_k)q \in \mathbb{R} \oplus q\mathbb{R}$, we extend the definition of the operation (2) such that $\sqrt{1/\beta_k + a_k/b_k q} \otimes (x_k/t_k)$ represent relativistic multiplications in $(-1,1)$,

$$\sqrt{\frac{1}{\beta_k} + \frac{a_k}{b_k} q} \otimes \frac{x_k}{t_k} = \tanh\left(\sqrt{\frac{1}{\beta_k} + \frac{a_k}{b_k} q}\, \tanh^{-1}\Big(\frac{x_k}{t_k}\Big) \right) \in (-1,1) \oplus q(-1,1).$$

In this way, the hyperbolic Gaussian-type quaternion signal defined by (62) can be written as

$$f(x_1,x_2) = Ce^{-\frac{1}{2}\left(\left(t_1 \tanh^{-1}\left(\sqrt{\frac{1}{\beta_1} + \frac{a_1}{b_1}\mathbf{i}} \otimes \frac{x_1}{t_1}\right)\right)^2 + \left(t_2 \tanh^{-1}\left(\sqrt{\frac{1}{\beta_2} + \frac{a_2}{b_2}\mathbf{j}} \otimes \frac{x_2}{t_2}\right)\right)^2 \right)}.$$

This establishes the statement. □

Finally, we obtain the hyperbolic analogue of the Heisenberg-type uncertainty principle associated with the Euclidean right-sided QLCT; see [17]. The proof is similar to that of Theorem 5.1; of course, (63) also follows directly from the formula (53) combined with (49).

Theorem 5.2 (Heisenberg-type Uncertainty Principle II). *Let $A_k = (a_k, b_k; c_k, d_k)$ be 2×2 matrices of real parameters satisfying $a_k d_k - b_k c_k = 1$ with $b_k \neq 0$ for $k = 1, 2$. If $f, \underline{x}_k f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, $(\partial_h/\partial x_k) f$ exists for $k = 1, 2$ and is in $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, and $\omega_k \mathcal{L}^{\mathbf{i},\mathbf{j}}_{(A_1,A_2)}(f) \in L^2(\mathbb{R}^2, \mathbb{H})$, then the following inequality holds:*

$$\triangle x_1 \, \triangle x_2 \, \triangle \omega_1 \, \triangle \omega_2 \geq \frac{b_1 b_2}{4}. \tag{63}$$

Further, equality in (50) *holds when f is a hyperbolic Gaussian function of the form*

$$f(x_1, x_2) = C e^{-\frac{1}{2}\left(\left(t_1 \tanh^{-1}\left(\frac{1}{\sqrt{\beta_1}} \otimes \frac{x_1}{t_1}\right)\right)^2 + \left(t_2 \tanh^{-1}\left(\frac{1}{\sqrt{\beta_2}} \otimes \frac{x_2}{t_2}\right)\right)^2\right)}, \tag{64}$$

where β_1, β_2 are real positive constants and $C = \|f\|_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})} (\beta_1 \beta_2 \pi^2)^{-1/4}$.

In the large limits of t_1 and t_2, i.e., $t_1, t_2 \to +\infty$, the particular case when $A_1 = A_2 = (0, 1; -1, 0)$ boils down to the classical Heisenberg uncertainty principle for the standard Euclidean right-sided QFT investigated by Bahri et al. in [1].

6 Conclusion

We introduced the right-sided QHLCT associated with two-dimensional quaternion signals defined in an open rectangle of the Euclidean plane endowed with a hyperbolic measure. The new transform uses a four-parameter family of two-dimensional quaternionic hyperbolic plane waves and contains the QLCTs as particular cases. Although we confined the analysis to the right-sided QHLCT, there are two other types of QHLCTs due to the lack of commutativity of quaternions: the left-sided QHLCT and the two-sided QHLCT. With a slight modification, the same techniques can be used to study the left-sided QHLCT employing the definition of a right-quaternionic inner product. Further investigations on the two-sided QHLCT are now under research and will be reported in a forthcoming paper.

Various properties of the right-sided QHLCT were discussed, such as the Riemann-Lebesgue Lemma and Plancherel and Parseval Theorems, and an inversion formula was obtained. The algebraic approach requires the introduction of novel hyperbolic differentiation and integration concepts. The results were applied to establish two quaternionic versions of the Heisenberg uncertainty principle for the right-sided QHLCT. These uncertainty principles give a lower bound on the product of the effective hyperbolic spatial width and the effective spectral width of a hyperbolic quaternion signal. It was shown that only hyperbolic Gaussian quaternion signals minimize the uncertainty relations.

All the results presented in this paper have a Euclidean counterpart in the large limit of t_1 and t_2, $t_1, t_2 \to \infty$.

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References

[1] M. Bahri, E. Hitzer, A. Hayashi, and R. Ashino, (2008), *An uncertainty principle for quaternion Fourier transform.* Comput. Math. with Appl., **56**, 2398–2410.

[2] H. De. Bie, N. De. Schepper, T. A. Ell, K. Rubrecht, and S. J. Sangwine, (2015), *Connecting spatial and frequency domains for the quaternion Fourier transform,* Appl. Math. Comput., **271**, 581–593.

[3] D. Cardona, R. Duduchava, A. Hendrickx, and M. Ruzhansky, (2022), *Global pseudo-differential operators on the Lie group $G = (-1, 1)^n$*, arXiv:2209.09751.

[4] S. A. Collins, (1970), *Lens-system Diffraction Integral Written in Terms of Matrix Optics*, J. Opt. Soc. Am., **60**, 1168–1177.

[5] L. P. Chen, K. I. Kou, and M. S. Liu, (2015), *Pitt's inequality and the uncertainty principle associated with the quaternion Fourier transform*, J. Math. Anal. Appl., **423**(1), 681–700.

[6] R. Duduchava, (2022), *Convolution equations on the Lie group* $(-1, 1)$, arXiv:2208.08765.

[7] T. A. Ell, (1993), *Quaternion-Fourier transforms for analysis of two-dimensional linear time-invariant partial differential systems*, in: Proceeding of the 32nd Conference on Decision and Control, San Antonio, Texas, 1830–1841.

[8] M. Ferreira, (2014), *Harmonic analysis on the Einstein gyrogroup*, J. Geom. Symm. Phys., **35**, 21–60.

[9] M. Ferreira, (2015), *Harmonic analysis on the Möbius gyrogroup*, J. Fourier Anal. Appl., **21**(2), 281–317.

[10] M. Ferreira and J. Morais, *Quaternion hyperbolic Fourier transforms and uncertainty principles.* Submitted.

[11] N. Goel, K. Singh, R. Saxena, and A. K. Singh, (2016), *Multiplicative filtering in the linear canonical transform domain,* IET Signal Process., **10**(2), 173–181.

[12] K. Gröchenig, (2000), *Foundations of time-frequency analysis.* Birkhäuser.

[13] J. J. Healy, M. A. Kutay, H. M. Ozaktas, and J. T. Sheridan, (2015), *Linear Canonical Transforms: Theory and Applications,* Vol. 198. Springer, New York.

[14] E. Hitzer, (2007), *Quaternion Fourier transform on quaternion fields and generalizations,* Adv. Appl. Clifford Algebras, **17**(3), 497–517.

[15] E. Hitzer, (2010), *Directional uncertainty principle for quaternion Fourier transform,* Adv. Appl. Clifford Algebras, **20**, 271–284.

[16] K. I. Kou, J. Morais, and Y. Zhang, (2013), *Generalized prolate spheroidal wave functions for offset linear canonical transform in Clifford analysis,* Math. Methods Appl. Sci., **36**(9), 1028–1041.

[17] K. I. Kou, J. Ou, and J. Morais, (2013), *On uncertainty principle for quaternionic linear canonical transform.* Abstr. Appl. Anal., Article ID725952.

[18] K. I. Kou and J. Morais, (2014), *Asymptotic behaviour of the quaternion linear canonical transform and the Bochner-Minlos theorem.* Appl. Math. Comput., **247**(15), 675–688.

[19] K. I. Kou, J. Ou, and J. Morais, (2016), *Uncertainty principles associated with quaternionic linear canonical transform.* Math. Meth. Appl. Sci., **39**, 2722–2736.

[20] X. X. Hu and K. I. Kou, (2017),*Quaternion Fourier and linear canonical inversion theorems,* Math. Meth. Appl. Sci., **40**(7), 2421–2440.

[21] K. I. Kou, M. S. Liu, J. Morais, and C. Zou, (2017), *Envelope detection using generalized analytic signal in 2D QLCT domains.* Multidimens. Syst. Signal Process., **28**, 1343–1366.

[22] Y. Liu, K. Kou, and I. Ho, (2010), *New sampling formulae for non-bandlimited signals associated with linear canonical transform and nonlinear Fourier atoms.* Signal Processing, **90**(3), 933–945.

[23] J. Morais, S. Georgiev, and W. Sprössig, (2014), *Real quaternionic calculus handbook.* Birkhäuser, Basel.

[24] M. Moshinsky and C. Quesne, (1971), *Linear canonical transforms and their unitary representations.* J. Math. Phys., **12**, 1772–1783.

[25] V. E. Petrov, (2005), *Integral transform on a segment,* Problemy Matem. Analiza **31**, 67–95; English transl. J. Math. Sci. **132**(4), 451–481 (2006).

[26] V. E. Petrov, (2006), *The generalized singular Tricomi equation as a convolution equation,* Dokl. Ros. Akad. Nauk **411**(2), 1–5; English transl. Dokl. Math. **74**(3), 901–905.

[27] K. K. Sharma and S. D. Joshi, (2008), *Uncertainty principles for real signals in linear canonical transform domains*. IEEE Trans. Signal Processing, **56**(7), 2677–2683.

[28] A. Stern, (2007), *Sampling of compact signals in offset linear canonical transform domains*, Signal, Image Video Process., **1**(4), 259–367.

[29] A. Stern, (2008), *Uncertainty principles in linear canonical transform domains and some of their implications in optics*, J. Opt. Soc. AM. A, **25**(3), 647–652.

[30] A. Ungar, (1988), *Thomas rotation and the parametrization of the Lorentz transformation group. Found. Phys. Lett.,* **1**(1), 57–89.

[31] A. Ungar, (2005), *Analytic hyperbolic geometry. Mathematical foundations and applications*, Hackensack, NJ: World Scientific.

[32] K. B. Wolf, (2016), *Development of linear canonical transforms: a historical sketch*, In Linear canonical transforms. Theory and applications, Healy, John J. (ed.) et al., Springer Series in Optical Sciences 198, 3–28, New York, NY: Springer.

[33] G. L. Xu, X. T. Wang, and X. G. Xu, (2009), *Three uncertainty relations for real signals associated with linear canonical transform*, IET Signal Process., **3**(1), 85–92.