Spherical continuous wavelet transforms arising from sections of the Lorentz group[†]

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Abstract

We consider the conformal group of the unit sphere S^{n-1}, the so-called proper Lorentz group $\mathrm{Spin}^+(1,n)$, for the study of spherical continuous wavelet transforms. Our approach is based on the method for construction of general coherent states associated to square integrable group representations over homogeneous spaces. The underlying homogeneous space is an extension to the whole of the group $\mathrm{Spin}^+(1,n)$ of the factorization of the gyrogroup of the unit ball by an appropriate gyro-subgroup. Sections on it are constituted by rotations of the subgroup $\mathrm{Spin}(n)$ and Möbius transformations of the type $\varphi_a(x) = (x-a)(1+ax)^{-1}$, where a belongs to a given section on a quotient space of the unit ball. This extends in a natural way the work of Antoine and Vandergheynst to anisotropic conformal dilations on the unit sphere.

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1 Introduction

In the last decade there has been a growing interest in wavelet theory on the unit sphere for an efficient analysis of spherical data. Several wavelet transforms on the sphere have been proposed in recent years and their performance depends on the application. In 1995, Schröder and Sweldens [26] describe an orthogonal wavelet transform on the sphere based on the Haar wavelet function but this construction is limited to the Haar function. Holschneider [22] was the first to build a genuine spherical continuous wavelet transform (CWT), but it contains a parameter that although interpreted as a dilation parameter it is difficult to compute. A satisfactory approach was obtained by Antoine and Vandergheynst [7, 8] building a rigorous spherical CWT based on a group theoretical approach. Using appropriate combinations of spherical harmonics, Freeden et al. [17] defined approximation kernels and introduced a discrete wavelet transform through a kind of multiresolution on S^2 on the Fourier domain. As demonstrated in Antoine et al. [6], the approximation scheme of Freeden et al. [17] can be realized

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in the general continuous wavelet theory, based on group-theoretical considerations. Many other papers [9, 19] describe applications of spherical wavelets for detection of non-Gaussianity in Cosmic Microwave Background (CMB). These works have been extended to directional wavelet transforms [6, 20, 21, 23]. All these new continuous wavelet decompositions are interesting for data analysis. There is a growing interest in directional or anisotropic wavelets on the sphere. Recently, new multiscale transforms were introduced on the sphere based on curvelets and ridgelets [1] with applications to denoising and filtering methods. Also, the anisotropic dilation operator introduced in Hobson et al. [20] and used in Frossard et al. [18] allows the construction of anisotropic wavelets through dilations in two orthogonal directions. However, in this anisotropic setting, a bounded admissibility integral cannot be determined (even in the plane) and thus the synthesis of a signal from its coefficients cannot be performed. This is a direct consequence of the absence of a group structure and an easy evaluation of the proper measure in the reconstruction formula. However, in our approach we are able to derive anisotropic conformal dilations from Möbius transformations, which are natural transformations on the unit sphere and, hence, to construct anisotropic spherical CWT using sections on a homogeneous space.

The construction of integral transforms related to group representations on L^2-spaces on manifolds is a non-trivial problem because the square integrability property may fail to hold. For example, the Lorentz group $SO_o(1,n)$ has no square integrable unitary representations for $n > 3$ [27, 30]. A way to overcome this fact is to make the group smaller, i.e., to factor out a suitable closed subgroup H and to restrict the representation to a quotient $X = G/H$. This approach was successful realized in the case of the unit sphere [7, 8]. The authors used the conformal group of S^{n-1}, the connected component of the Lorentz group $SO_0(1,n)$ and its Iwasawa decomposition, or $KAN-$decomposition, where K is the maximal compact subgroup $SO(n)$, $A = SO(1,1) \cong (\mathbb{R}^+_*, \times)$ is the subgroup of Lorentz boosts in the x_n-direction and $N \cong \mathbb{R}^{n-1}$ is a nilpotent subgroup. The homogeneous space $X = SO_0(1,n)/N \cong SO(n) \times \mathbb{R}^+_*$ gives rise to the parameter space of the isotropic spherical CWT. This approach is restricted to pure dilations on the unit sphere, according to the inverse stereographic projection of dilations from the plane to the sphere [7, 10]. However, as we will show in this paper, the spherical CWT can be extended to anisotropic conformal dilations, while preserving the main properties of square-integrability and invertibility. Our approach allows us to connect the geometry of conformal transformations on the sphere with wavelet theory.

In our paper [10] we used rotations of the group $\mathrm{Spin}(n)$ and Möbius transformations of the form $\varphi_a(x) = (x-a)(1+ax)^{-1}$, $a \in B^n$, being B^n the open unit ball in $\mathbb{R}^n$, as the basic and natural transformations for building a CWT on the sphere. While in [10] we study dilations on the unit sphere by the action of Möbius transformations on spherical caps, in this paper we will make a rigorous construction of continuous wavelet transforms on the unit sphere arising from sections of the proper Lorentz group $\mathrm{Spin}^+(1,n)$, which is the symmetry group of conformal geometry on the unit sphere. The use of the group $\mathrm{Spin}^+(1,n)$ instead of $SO_0(1,n)$ gives us the possibility of working on spherical geometry in a very natural and elegant way. The benefits of this approach are the explicit construction of a wide variety of cross-sections associated to anisotropic conformal dilations on the sphere. The main result is the proof that admissible functions (wavelets) for the fundamental section are also admissible functions for any continuous global section constructed in our homogeneous space (Theorem 5.12).

The paper consists of two parts, a geometrical and algebraic one concerned with the definition of a suitable quotient space and associated cross-sections (Sections 2-4) and a second part applying these results to the construction of continuous wavelet transforms on the unit sphere and analysis of the wavelet systems obtained (Sections 5 and 6). The first part is predominantly of auxiliary nature and it describes the algebraic structures needed for the construction of our parameter space. In Section 2 we present some basic setting of Clifford algebra (more details can be found e.g. in Delanghe et al. [15] and Cnops [12]). In Section 3 we present two realizations of the group $\mathrm{Spin}^+(1,n)$, via Vahlen matrices and via the gyrosemidirect product of the gyrogroup of the unit ball and the group $\mathrm{Spin}(n)$ and we study properties of this group. In Section 4 we factorize the gyrogroup of the unit ball $(B^n,\oplus)$ by an appropriate gyro-subgroup $(D^{n-1}_{e_n},\oplus)$. The arising quotient space $Y = B^n/(D^{n-1}_{e_n},\sim_l)$ is not a homogeneous space and, hence, the equivalence classes are not cosets of a fixed subgroup. However, with the additional extension of the equivalence relation to $\mathrm{Spin}^+(1,n)$, by the inclusion of rotations, we generate a homogeneous space X which happens to be equipotent, up to a covering, to the homogeneous space constructed by Antoine and Vandergheynst via the Iwasawa decomposition of $SO_0(1,n)$ [8]. An isomorphism between both spaces is not easy to obtain since the construction of both spaces is very different and its algebraic properties are also very different. The second part of the paper contains the main results. First we establish an equivalence between our approach and the approach of Antoine and Vandergheynst showing that both are equivalent when we restrict ourselves to the fundamental section of our homogeneous space. Then we extend the results for an arbitrary global continuous section, replacing the isotropic dilation operator by an anisotropic dilation operator. With the results of our paper [10] we can characterize geometrically the anisotropic dilations introduced in this paper.

2 Preliminaries

Let $e_1,\ldots,e_n$ be an orthonormal basis of $\mathbb{R}^n$. The universal real Clifford algebra $\mathrm{Cl}_{0,n}$ is the free algebra over $\mathbb{R}^n$ generated modulo the relation $x^2 = -|x|^2$, for $x \in \mathbb{R}^n$. The non-commutative multiplication in $\mathrm{Cl}_{0,n}$ is governed by the rules

$$e_ie_j + e_je_i = -2\delta_{ij}, \quad \forall\, i,j \in \{1,\ldots,n\}.$$

For a set $A = \{i_1,\ldots,i_k\} \subset \{1,\ldots,n\}$, with $1 \leq i_1 < \ldots < i_k \leq n$, let $e_A = e_{i_1}e_{i_2}\cdots e_{i_k}$ and $e_\emptyset = 1$. Then $\{e_A : A \subset \{1,\ldots,n\}\}$ is a basis for $\mathrm{Cl}_{0,n}$. Thus, any $a \in \mathrm{Cl}_{0,n}$ may be written as $a = \sum_A a_Ae_A$, with $a_A \in \mathbb{R}$ or still as $a = \sum_{k=0}^n [a]_k$, where $[a]_k = \sum_{|A|=k} a_Ae_A$ is a so-called *k-vector* ($k = 0,1,\ldots,n$). We can identify a vector $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ with the one-vector $x = \sum_{j=1}^n x_je_j$. There are two linear anti-automorphisms * (reversion) and $^-$ (conjugation) and an automorphism $'$ of $\mathrm{Cl}_{0,n}$ defined on its basis elements $e_A = e_{i_1}e_{i_2}\cdots e_{i_k}$, by the rules:

$$(e_A)^* = (-1)^{\frac{k(k-1)}{2}}e_A, \qquad \overline{e_A} = (-1)^{\frac{k(k+1)}{2}}e_A, \qquad e'_A = (-1)^k e_A.$$

Thus, we have $(ab)^* = b^*a^*$, $\overline{ab} = \overline{b}\,\overline{a}$ and $(ab)' = a'b'$, for any $a,b \in \mathrm{Cl}_{0,n}$. In particular, for vectors $\overline{x} = x' = -x$ and $x^* = x$. The Clifford algebra decomposes as $\mathrm{Cl}_{0,n} = \mathrm{Cl}^+_{0,n} \oplus \mathrm{Cl}^-_{0,n}$, where $\mathrm{Cl}^+_{0,n} = \{a \in \mathrm{Cl}_{0,n} : a = a'\}$ denote the even subalgebra of $\mathrm{Cl}_{0,n}$. Each non-zero vector $x \in \mathbb{R}^n$ is invertible with $x^{-1} := \frac{\overline{x}}{|x|^2}$. Finite products of invertible vectors are invertible in $\mathrm{Cl}_{0,n}$ and form the Clifford group $\Gamma(n)$ [12]. The even Clifford group is defined by $\Gamma^+(n) = \Gamma(n) \cap \mathrm{Cl}^+_{0,n}$. Elements $a \in \Gamma(n)$ such that $a\overline{a} = \pm 1$ form the $\mathrm{Pin}(n)$ group - the double cover of the group of orthogonal transformations $O(n)$, and elements $a \in \Gamma(n)$ such that $a\overline{a} = 1$ form the $\mathrm{Spin}(n)$ group - the double cover of the group of special orthogonal transformations $SO(n)$. The $\mathrm{Spin}(n)$ group can also be viewed as $\mathrm{Spin}(n) = \mathrm{Pin}(n) \cap \mathrm{Cl}^+_{0,n}$. For each $s \in \mathrm{Spin}(n)$, the mapping $\chi_s : \mathbb{R}^n \to \mathbb{R}^n, \chi_s(x) = sx\overline{s}$ defines a rotation. As $\ker\chi = \{-1,1\}$ the group $\mathrm{Spin}(n)$ is a double covering group of $SO(n)$.

In Clifford algebra the group of Möbius transformations in $\mathbb{R}^n$ can be described by $2\times 2-$matrices with entries in $\Gamma(n) \cup \{0\}$, forming the group $\Gamma(1,n+1)$ [2, 12].

Theorem 2.1 *(Vahlen [12], 1904/ Alfhors [2], 1986) A matrix* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *represents a Möbius transformation in* $\mathbb{R}^n$ *if and only if*

1. $a, b, c, d \in \Gamma(n) \cup \{0\}$;
2. $ab^*, cd^*, c^*a, d^*b \in \mathbb{R}^n$;
3. *the pseudodeterminant of* A, $\lambda = ad^* - bc^*$, *is real and non zero.*

For a matrix A fulfilling the conditions of Theorem 2.1 we can define the respective fractional linear transformation of $\overline{\mathbb{R}}^n$ (here $\overline{\mathbb{R}}^n$ denotes the compactification of $\mathbb{R}^n$ by the point at infinity) given by $y = (ax+b)(cx+d)^{-1}$. Thus, we have the analogy with fractional-linear transformations of the complex plane $\mathbb{C}$. Due to the non-commutative character of Clifford algebras $xy^{-1} \neq y^{-1}x$ in general. From now on we identify $e_1, \ldots, e_n$ with the canonical basis in $\mathbb{R}^n$.

3 The proper Lorentz group Spin$^+(1, n)$

The group of all Lorentz transformations preserving both orientation and the direction of time is called the proper orthochronous Lorentz group and it is denoted by $SO_0(1, n)$. It is generated by spatial rotations of the maximal compact subgroup $K = SO(n)$ and hyperbolic rotations of the subgroup $A = SO(1, 1)$, accordingly to the Cartan decomposition KAK [27, 30]. The double covering group of $SO_0(1, n)$ is the group Spin$^+(1, n)$ and it is described in Clifford algebra by rotations of the group Spin(n) and hyperbolic rotations (or boosts) in a given direction $\omega \in S^{n-1}$. By the Fillmore-Springer construction the conformal group of the unit ball can de described by Vahlen matrices.

The isomorphisms Spin$^+(1, 3)/\mathbb{Z}_2 \cong SL(2, \mathbb{C})/\mathbb{Z}_2 \cong SO_0(1, 3)$ are well-known in complex analysis. In $\mathbb{R}^n$, Clifford algebra allows us to obtain analogous relations since we have $SO_0(1, n+1) \cong$ Spin$^+(1, n+1)/\mathbb{Z}_2 \cong SL(2, \Gamma^+(n) \cup \{0\})/\mathbb{Z}_2$ ([2, 12]), where $SL(2, \Gamma^+(n) \cup \{0\})/\mathbb{Z}_2$ is the group of Vahlen matrices with entries in the even Clifford group or zero, and pseudodeterminant 1. The conformal group of the unit disc in $\mathbb{C}$ is the group $SU(1, 1)$, which is a subgroup of $SL(2, \mathbb{C})$ and the isomorphisms $SU(1, 1)/\mathbb{Z}_2 \cong SO_0(1, 2) \cong$ Spin$^+(1, 2)/\mathbb{Z}_2$ hold. Thus, the special conformal group of the unit ball $\mathcal{M}^+(B^n)$ in $\mathbb{R}^n$ is the analogous of the group $SU(1, 1)$ in $\mathbb{C}$, and we have the isomorphisms $\mathcal{M}^+(B^n)/\mathbb{Z}_2 \cong$ Spin$^+(1, n)/\mathbb{Z}_2 \cong SO_0(1, n)$.

Theorem 3.1 *[2, 12, 25] The group* $\mathcal{M}^+(B^n)$, *of orientation preserving automorphisms of the unit ball* B^n *consists of Vahlen matrices of the form*

$$\begin{pmatrix} u & v' \\ v & u' \end{pmatrix}, \quad u, v \in \Gamma^+(n) \cup \{0\}, \quad uv^* \in \mathbb{R}^n, \quad |u|^2 - |v|^2 = 1. \tag{3.1}$$

Matrices of type (3.1) can be decomposed into the maximal compact subgroup Spin(n) and the set of Möbius transformations which map the closed unit ball $\overline{B^n}$ onto itself, as follows

$$\begin{pmatrix} u & v' \\ v & u' \end{pmatrix} = |u| \begin{pmatrix} \frac{u}{|u|} & 0 \\ 0 & \frac{u'}{|u|} \end{pmatrix} \begin{pmatrix} 1 & \frac{\overline{u}}{|u|^2} v' \\ \frac{u^*}{|u|^2} v & 1 \end{pmatrix} = \frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} s & 0 \\ 0 & s' \end{pmatrix} \begin{pmatrix} 1 & a' \\ a & 1 \end{pmatrix}, \tag{3.2}$$

where

$$s = \frac{u}{|u|} \in \text{Spin}(n), \quad a = \frac{u^*}{|u|^2} v \in B^n, \quad \sqrt{1 - |a|^2} = |u|^{-1}. \tag{3.3}$$

From (3.2) we will consider the action of rotations and Möbius transformations on the unit sphere S^{n-1} given by $sx\overline{s}$, $s \in \mathrm{Spin}(n)$, and $\varphi_a(x) := (x-a)(1+ax)^{-1}$, $a \in B^n$, respectively. Möbius transformations $\varphi_a(x)$ are the multi-dimensional analogous transformations of the well-known Möbius transformations of the complex plane $\varphi_u(z) = \frac{z-u}{1-\overline{u}z}, u \in D$, which maps the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ onto itself. They can be written as

$$\varphi_a(x) = \frac{(1-|a|^2)x - (1+|x|^2 - 2\langle a, x\rangle)a}{1 - 2\langle a, x\rangle + |a|^2|x|^2}. \tag{3.4}$$

We are interested in the study of Möbius transformations of type (3.4) since these transformations are associated to dilations on the unit sphere. First we will study the algebraic properties of the unit ball since it is the manifold associated to the transformations $\varphi_a(x)$. The composition of two Möbius transformations of type (3.4) is (up to a rotation) again a Möbius transformation of the same type:

$$(\varphi_a \circ \varphi_b)(x) = q\, \varphi_{(1-ab)^{-1}(a+b)}(x)\, \overline{q}, \quad q = \frac{1-ab}{|1-ab|} \in \mathrm{Spin}(n). \tag{3.5}$$

From (3.5) we can endow the unit ball B^n with a binary operation $\oplus$ defined by

$$b \oplus a := (1-ab)^{-1}(a+b) = (a+b)(1-ba)^{-1} = \varphi_{-b}(a). \tag{3.6}$$

The neutral element on $(B^n, \oplus)$ is zero while each element $a \in B^n$ has an inverse given by the symmetric element $-a$. This operation is non-associative since it satisfies

$$a \oplus (b \oplus c) = (a \oplus b) \oplus (qc\overline{q}), \quad \text{with} \quad q = \frac{1-ab}{|1-ab|} \in \mathrm{Spin}(n). \tag{3.7}$$

and is non-commutative since

$$b \oplus a = \overline{q}\ (a \oplus b)\ q, \quad q = \frac{1-ab}{|1-ab|} \in \mathrm{Spin}(n), \tag{3.8}$$

for all $a, b, c \in B^n$.

Associativity and commutativity only fails up to a rotation. The group structure is replaced by a new algebraic structure called gyrogroup. This algebraic structure was introduced by A. Ungar [28, 29] and it appears by the extension of the Einstein relativistic groupoid and the Thomas precession effect (related by the rotation induced by q). The gyrolanguage was adopted by Ungar since 1991. It rests on the unification of Euclidean and hyperbolic geometry in terms of the analogies shared [28]. The relation (3.7) is called the left gyroassociative law, the relation (3.8) is called the gyrocommutative law and the rotation induced by q is a gyroautomorphism of B^n and it obeys the left loop property

$$\frac{1-(a \oplus b)b}{|1-(a \oplus b)b|} = \frac{1-ab}{|1-ab|}. \tag{3.9}$$

Thus, $(B^n, \oplus)$ is a gyrogroup. This algebraic structure is important in order to factorize the unit ball and to obtain sections that allows us to define dilation operators on the unit sphere as we will see on Section 4. Although the operation $\oplus$ is not strictly associative it becomes strictly associative in some special cases as we can see in the next lemma.

Lemma 3.2 *If $a, b, c \in B^n$ such that $a // b$ or $a \perp c$ and $b \perp c$ then the operation $\oplus$ is associative, i.e.*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c.$$

This result is important in the construction of quotient spaces for the gyrogroup of the unit ball. For our constructions we need also the left and right cancelation laws:

Lemma 3.3 *For all $a, b \in B^n$ it holds*

$$(-b) \oplus (b \oplus a) = a \tag{3.10}$$

$$(a \oplus b) \oplus (q(-b)\overline{q}) = a, \quad \textit{with } q = \frac{1-ab}{|1-ab|}. \tag{3.11}$$

One of the most important results of the theory of gyrogroups is that the gyro-semidirect product of a gyrogroup $(G, \oplus)$ with a gyroautomorphism group $H \subset Aut(G, \oplus)$ is a group [28]. In our case the result is stated as

Proposition 3.4 *The gyro-semidirect product between $(B^n, \oplus)$ and $Spin(n)$ is the group of pairs (s, a) where $a \in B^n$ and $s \in \mathrm{Spin}(n)$, with operation $\times$ given by the gyro-semidirect product*

$$(s_1, a) \times (s_2, b) = (s_1 s_2 q, b \oplus (\overline{s_2} a s_2)), \quad \textit{with } q = \frac{1 - \overline{s_2} a s_2 b}{|1 - \overline{s_2} a s_2 b|}. \tag{3.12}$$

The group operation obtained from the decomposition (3.2) coincides with (3.12) and thus we have $\mathrm{Spin}^+(1,n) \cong \mathrm{Spin}(n) \times B^n$, with $\times$ being the gyro-semidirect product. This description gives us the possibility of working with the whole of the group in a very simple geometric way. The gyrosemidirect product is a generalization of the external semidirect product of groups (see [24]). Some properties between Möbius transformations and rotations are established next.

Lemma 3.5 *For $s \in \mathrm{Spin}(n)$ and $a \in B^n$ we have*

$$(i)\ \varphi_a(sx\overline{s}) = s\varphi_{\overline{s}as}(x)\overline{s} \quad \textit{and} \quad (ii)\ s\varphi_a(x)\overline{s} = \varphi_{sa\overline{s}}(sx\overline{s}). \tag{3.13}$$

These properties are easily transferred to the gyrogroup $(B^n, \oplus)$.

Corollary 3.6 *The following equalities hold*

$$(i)\ (sa\overline{s}) \oplus b = s(a \oplus (\overline{s}bs))\overline{s} \quad \textit{and} \quad (ii)\ s(a \oplus b)\overline{s} = (sa\overline{s}) \oplus (sb\overline{s}). \tag{3.14}$$

The relation $(sa\overline{s}) \oplus (sb\overline{s}) = s(a \oplus b)\overline{s}$ defines a homomorphism of $\mathrm{Spin}(n)$ onto the gyrogroup $(B^n, \oplus)$. From the decomposition (3.12) we can derive the Cartan or KAK decomposition of the group $\mathrm{Spin}^+(1,n)$, where $K = \mathrm{Spin}(n)$ and $A = \mathrm{Spin}(1,1)$ is the subgroup of Lorentz boosts in the e_n-direction.

Lemma 3.7 *Each $a \in B^n$ can be written as $a = sre_n\overline{s}$, where $r = |a| \in [0,1[$ and $s = s_1 \cdots s_{n-1} \in \mathrm{Spin}(n)$ with*

$$s_i = \cos\frac{\theta_i}{2} + e_{i+1}e_i \sin\frac{\theta_i}{2}, \quad i = 1, \ldots, n-1, \tag{3.15}$$

where $0 \leq \theta_1 < 2\pi$ $\ 0 \leq \theta_i < \pi$, $i = 2, \ldots, n-1$.

This follows from the description of $a \in B^n$ in spherical coordinates using the rotors (3.15). These rotors are very useful in Clifford algebra in the description of rotations in coordinate planes. For $s = \cos\left(\frac{\theta}{2}\right) + e_i e_j \sin\left(\frac{\theta}{2}\right)$, $i \neq j$ we have

$$s x \overline{s} = (\cos\theta\, x_i - \sin\theta\, x_j)e_i + (\cos\theta\, x_j + \sin\theta\, x_i)e_j + \sum_{k\neq i,j} x_k e_k\,,$$

which represents a rotation of angle θ in the e_ie_j-plane. In general we have $s_is_j \neq s_js_i$, $i \neq j$. The order of the rotors is important since different choices lead to different systems of coordinates. Due to the relevance of the rotor s_{n-1} for our work we shall denote $\theta_{n-1} := \phi$.

Lemma 3.8 *For $a \in B^n$ as described in Lemma 3.7 it follows*

$$\varphi_a(x) = \varphi_{sre_n\overline{s}}(x) = s\varphi_{re_n}(\overline{s}xs)\overline{s}.$$

Combining Lemma 3.8 with the rotation in decomposition (3.12) yields the Cartan decomposition or KAK decomposition of $\text{Spin}^+(1,n)$.

For the construction of a theory of sections we need to consider some special sub-structures contained in the gyrogroup.

Definition 3.9 *Let $(G,\oplus)$ be a gyrogroup and H a non-empty subset of G. H is a gyro-subgroup of $(G,\oplus)$ if it is a gyrogroup for the operation induced from G and $gyr[a,b] \in Aut(H)$ for all $a,b \in H$.*

For a fixed $\omega \in S^{n-1}$, we consider the hyperplane $H_\omega = \{x \in \mathbb{R}^n : \langle\omega,x\rangle = 0\}$, the hyperdisc $D^{n-1}_\omega = H_\omega \cap B^n$ and $L_\omega = \{x \in B^n : x = t\omega,\ -1 < t < 1\}$. Using (3.6) and (3.4) we derive the following gyro-subgroups.

Proposition 3.10 *The sets D^{n-1}_ω and L_ω endowed with the operation $\oplus$ are gyro-subgroups of $(B^n,\oplus)$. Moreover, $(L_\omega,\oplus)$ is a group.*

The last property comes directly from Lemma 3.2. The operation $\oplus$ in L_ω corresponds to the Einstein velocity addition of parallel velocities in the special theory of relativity. The particular group $(L_{e_n},\oplus)$ is isomorphic to the $\text{Spin}(1,1)$ group. In the next section we will see the importance of these sub-structures in order to factorize the gyrogroup of the unit ball.

4 Sections of the proper Lorentz group

We want to factorize the proper Lorentz group and to construct sections on the respective quotient space. First we will study the factorization of the gyrogroup $(B^n,\oplus)$ by the gyro-subgroup $(D^{n-1}_{e_n},\oplus)$, taking into consideration only left cosets.

Theorem 4.1 *[16] For each $c \in B^n$ there exist unique $a \in D^{n-1}_{e_n}$ and $b \in L_{e_n}$ such that $c = b \oplus a$.*

Proof: Let $c = (c_1,\ldots,c_n) \in B^n$ be an arbitrary point. By Lemma 3.7 we can write $c = s_*c_*\overline{s_*}$, with $s_* = s_1\ldots s_{n-2} \in \text{Spin}(n-1)$, (this rotation leaves the x_n-axis invariant) and $c_* = (0,\ldots,0,c^*_{n-1},c_n)$, where $c^*_{n-1} = r\sin\phi$ and $c_n = r\cos\phi$, with $r = |c| \in [0,1[$ and $\phi = \arccos(c_n) \in [0,\pi]$.

First we prove the existence of the decomposition for c_*. If $c^*_{n-1}=0$ then it suffices to consider $a=0$ and $b=c_*$. If $c^*_{n-1}\neq 0$ then we consider $a=\lambda e_{n-1}$ and $b=te_n$ such that

$$\lambda=\frac{|c_*|^2-1+\sqrt{((c_n+1)^2+c^{*^2}_{n-1})((c_n-1)^2+c^{*^2}_{n-1})}}{2c^*_{n-1}} \quad \text{and} \quad t=\frac{c_n}{\lambda c^*_{n-1}+1}. \tag{4.1}$$

We can see that $-1<\lambda,t<1$. Thus, $a\in D^{n-1}_{e_n}$ and $b\in L_{e_n}$. Taking into account that $a\perp b$, that is $\langle a,b\rangle=0$, we obtain

$$b\oplus a=\left(0,\ldots,0,\frac{\lambda(1-t^2)}{1+\lambda^2t^2},\frac{t(1+\lambda^2)}{1+\lambda^2t^2}\right)=c_*.$$

Now, we prove the existence result for the arbitrary point $c\in B^n$. Considering $a_*=s_*a\overline{s_*}$ then $a_*\in D^{n-1}_{e_n}$ since the rotation induced by s_* leaves the x_n-axis invariant. Thus, by (3.14) we have

$$b\oplus(s_*a\overline{s_*})=s_*((\overline{s_*}bs_*)\oplus a)\overline{s_*}=s_*(b\oplus a)\overline{s_*}=s_*c_*\overline{s_*}=c,$$

which shows that $c=b\oplus a_*$ and the existence of the decomposition is proved.

To prove the uniqueness we suppose that there exist $a,d\in D^{n-1}_{e_n}$ and $b,f\in L_{e_n}$ such that $c=b\oplus a=f\oplus d$. Then $a=(-b)\oplus(f\oplus d)$, by (3.10). As $b\perp d$ and $f\perp d$ we have that $a=((-b)\oplus f)\oplus d$, by Lemma 3.2. Since by hypothesis $a,d\in D^{n-1}_{e_n}$, $(-b)\oplus f$ must be an element of $D^{n-1}_{e_n}$. This is true if and only if $(-b)\oplus f=0$. This implies $b=f$ and $a=d\oplus 0=d$.

■

This unique decomposition theorem allow us to construct left cosets for a convenient factorization of the unit ball by the gyro-subgroup $D^{n-1}_{e_n}$. If we consider the subgroup $D^{n-1}_{e_n}$ defined by

$$\forall\, c,d\in B^n,\quad c\;R\;d\;\Leftrightarrow\;\exists\, a\in D^{n-1}_{e_n}:\quad c=d\oplus a,$$

then R is a reflexive relation but it is not symmetric nor transitive because the operation $\oplus$ is neither commutative nor associative. Thus R is not an equivalence relation. However, an equivalence relation can be built if we have a partition of the unit ball.

Proposition 4.2 *The family $\{S_b:b\in L_{e_n}\}$, with $S_b=\{b\oplus a: a\in D^{n-1}_{e_n}\}$ is a disjoint partition of B^n.*

Proof: We first prove that this family is indeed disjoint. Let $b=t_1e_n$ and $c=t_2e_n$ with $t_1\neq t_2$, and assume that $S_b\cap S_c\neq\emptyset$. Then there exists $f\in B^n$ such that $f=b\oplus a$ and $f=c\oplus d$ for some $a,d\in D^{n-1}_{e_n}$. By (3.10) and (3.7) we have $a=(-b)\oplus(c\oplus d)=((-b)\oplus c)\oplus(qd\overline{q})$, with $q=\frac{1+bc}{|1+bc|}$. But $q=\frac{1+bc}{|1+bc|}=\frac{1-t_1t_2}{|1-t_1t_2|}=1$, hence $a=((-b)\oplus c)\oplus d$. Then $(-b)\oplus c\in D^{n-1}_{e_n}$ because $a,d\in D^{n-1}_{e_n}$. Therefore, $(-b)\oplus c=0$ or either $b=c$. But this contradicts our assumption. Finally, by Theorem 4.1 we have that $\cup_{b\in L_{e_n}}S_b=B^n$.

■

This partition induces an equivalence relation on B^n whose equivalence classes are the surfaces S_b. Two points $c,d\in B^n$ are said to be in relation if and only if there exists $b\in L_{e_n}$ such that $c,d\in S_b$, which means

$$c\sim_l d\;\Leftrightarrow\;\exists b\in L_{e_n},\,\exists\, a,f\in D^{n-1}_{e_n}:\;c=b\oplus a\;\text{ and }\;d=b\oplus f. \tag{4.2}$$

This relation is equivalent to

$$c \sim_l d \;\Leftrightarrow\; \exists\, b \in L_{e_n},\; \exists\, a, f \in D^{n-1}_{e_n}:\; c \oplus (q_1(-a)\overline{q_1}) = d \oplus (q_2(-f)\overline{q_2}),$$

with $q_1 = \frac{1-ab}{|1-ab|}$ and $q_2 = \frac{1-fb}{|1-fb|}$.

Thus, we have proved the bijection $B^n/(D^{n-1}_{e_n}, \sim_l) \;\cong\; L_{e_n}$. More generally, we have the bijections $B^n/(D^{n-1}_{\omega}, \sim_l) \;\cong\; L_{\omega}$, for each $\omega \in S^{n-1}$.

Now, we will give a characterization of the surfaces S_b.

Proposition 4.3 *For each $b = te_n \in L_{e_n}$ the surface S_b is a revolution surface in turn of the x_n-axis obtained by the intersection of B^n with the sphere orthogonal to S^{n-1}, with center in the point $C = (0, \ldots, 0, \frac{1+t^2}{2t})$ and radius $\tau = \frac{1-t^2}{2|t|}$.*

Proof: Let $b = te_n \in L_{e_n}$, $c = \lambda e_{n-1} \in D^{n-1}_{e_n}$ and

$$P := b \oplus c = \left(0, \ldots, 0, \frac{\lambda(1-t^2)}{1+\lambda^2 t^2}, \frac{t(1+\lambda^2)}{1+\lambda^2 t^2}\right).$$

Let $C_b = \{b \oplus c : c = \lambda e_{n-1}, 0 \leq \lambda < 1\}$. Each $a \in D^{n-1}_{e_n}$ can be described as $a = s_* c\overline{s_*}$, for some $c = \lambda e_{n-1}$ and $s_* = s_1 \cdots s_{n-2} \in \mathrm{Spin}(n-1)$. Then, by (3.14) we have

$$b \oplus (s_* c \overline{s_*}) = s_*((\overline{s_*} b s_*) \oplus c)\overline{s_*} = s_*(b \oplus c)\overline{s_*}.$$

Thus, S_b is a surface of revolution obtained by revolution in turn of the x_n-axis of the arc C_b. The last coordinate of the surface S_b gives the orientation of its concavity.

For all $\lambda \in [0, 1[$, we have $||P - C||^2 = \tau^2$, with $C = (0, \ldots, 0, \frac{1+t^2}{2t})$ and $\tau = \frac{1-t^2}{2|t|}$, which means that the arc C_b lies on the sphere centered at C and radius τ. Moreover, as t tends to zero the radius of the sphere tends to infinity thus proving that the surface S_0 coincides with the hyperdisc $D^{n-1}_{e_n}$.

Each S_b is orthogonal to S^{n-1} because $||C||^2 = 1 + \tau^2$. We recall that two spheres, S_1 and S_2, with centers m_1 and m_2 and radii τ_1 and τ_2, respectively, intersect orthogonally if and only if $\langle m_1 - y, m_2 - y\rangle = 0$, for all $y \in S_1 \cap S_2$, or equivalently, if $||m_1 - m_2||^2 = \tau_1^2 + \tau_2^2$.

■

We can observe the projection of these orbits in the $x_{n-1}x_n-$plane:

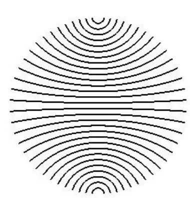

Figure 1: Cut of the surfaces S_b, in the $x_{n-1}x_n-$plane.

Let $Y = B^n/(D^{n-1}_{e_n}, \sim_l)$. Given $c \in B^n$ we denote by $[c]$ its equivalence class on Y. Thus, $[c] = b \oplus D_{e_n}$, for some $b \in L_{e_n}$ given by Theorem 4.1 and the mapping $\pi : Y \to L_{e_n}$ given by $\pi([c]) = b$ is bijective. For each $c \in B^n$ we define the mapping

$$\psi_c : Y \to Y$$
$$\psi_c(b \oplus D^{n-1}_{e_n}) := \pi([b \oplus c]) \oplus D^{n-1}_{e_n}. \tag{4.3}$$

Lemma 4.4 *For each $c \in B^n$ the mapping ψ_c is bijective.*

Proof: Suppose that $\pi([b \oplus c]) \oplus D_{e_n}^{n-1} = \pi([f \oplus c]) \oplus D_{e_n}^{n-1}$, for some $d, f \in L_{e_n}$ and $c \in B^n$. Thus, $[b \oplus c] = [f \oplus c]$ by Proposition 4.2 and since π is a bijection. Writing $c = b_* \oplus a$ for some $b_* \in L_{e_n}$ and $a \in D_{e_n}^{n-1}$ by Theorem 4.1, we obtain $[b \oplus (b_* \oplus a)] = [f \oplus (b_* \oplus a)]$. By Lemma 3.2 it follows $[(b \oplus b_*) \oplus a] = [(f \oplus b_*) \oplus a]$, which means that $b \oplus b_* = f \oplus b_*$. Since $(L_{e_n}, \oplus)$ is a group we have $b = f$ and therefore $b \oplus D_{e_n}^{n-1} = f \oplus D_{e_n}^{n-1}$. This proves the injectivity of ψ_c. As for each $b \in L_{e_n}$ there exists $f = b \oplus (-qc\overline{q})$, with $q = \frac{1-bc}{|1-bc|}$ such that $\pi([f \oplus c]) \oplus D_{e_n}^{n-1} = b \oplus D_{e_n}^{n-1}$, we conclude that the mapping ψ_c is bijective.

■

The mapping ψ_c defines a quasi-action of the gyrogroup $(B^n, \oplus)$ on Y in the sense that transitivity is preserved by the bijection of ψ_c, i.e. for any $x, y \in X$ there exists $c \in B^n$ such that $\psi_c(x) = y$. The space Y endowed with the mapping ψ_c will be called a quasi-homogeneous space. In this new setting we can introduce the concept of quasi-invariant measures in the same way as they are introduced for homogeneous spaces on groups, as we will see in the next Section.

We consider L_{e_n} the fundamental section σ_0, which corresponds to the Spin$(1,1)$ subgroup. From Proposition 4.3 an entire class of sections $\sigma : B^n/(D_{e_n}^{n-1}, \sim_l) \to B^n$ can be obtained from L_{e_n} by considering

$$\sigma(te_n) = te_n \oplus g(t)e_{n-1} = \left(0, \ldots, 0, \frac{g(t)(1-t^2)}{1+(tg(t))^2}, \frac{t(1+g(t)^2)}{1+(tg(t))^2}\right), \tag{4.4}$$

where $g :]-1,1[\to]-1,1[$ is called the generating function of the section. Depending on the properties of the function g we can have sections that are Borel maps and also smooth sections. If $g \in C^k(]-1,1[), k \in \mathbb{N}$, then the section generates a C^k-curve inside the unit ball. For example, for each $\lambda \in]-1,1[$ if we consider $g(t) = \lambda, (t \in]-1,1[)$ then we obtain the family of sections $\sigma_\lambda(t) = \left(0, \ldots, 0, \frac{\lambda(1-t^2)}{1+t^2\lambda^2}, \frac{t(1+\lambda^2)}{1+t^2\lambda^2}\right)$. Another example is the family of sections $\sigma_c(\phi)$ defined for each $c \in]-1,1[$ by $\sigma_c(\phi) = (0, \ldots, 0, c\sin\phi, -\cos\phi)$, $\phi \in]0, \pi[$. The intersection of each section σ_c with the orbits S_b gives us the generating function of σ_c. For $c \in]-1,1[\backslash\{0\}$ we obtain the generating function

$$g(t) = \begin{cases} \sqrt{\frac{t^2-1+\sqrt{(1-t^2)^2+4c^4t^2}}{2t^2c^2}}, & t \in]-1,1[\backslash\{0\} \\ c, & t = 0 \end{cases},$$

and the relation between the generating function and σ_c is given by

$$\cos\phi = \begin{cases} -\frac{t(1+g(t)^2)}{1+(tg(t))^2} = -\frac{t^2(2c^2+1)-1+\sqrt{(1-t^2)^2+4c^4t^2}}{t(2c^2+t^2-1+\sqrt{(1-t^2)^2+4c^4t^2})}, & t \in]-1,1[\backslash\{0\} \\ 0, & t = 0 \end{cases}.$$

For $\lambda = 0$ or $c = 0$ we obtain the fundamental section of the homogenous space Y.

Until now we only considered the factorization of the gyrogroup $(B^n, \oplus)$. In order to incorporate rotations in our scheme we now extend the equivalence relation (4.2) to the group Spin$^+(1,n)$ according to the group operation (3.12). It is easy to see that the direct product $\{1\} \times D_{e_n}^{n-1}$ is a gyrogroup, where 1 is the identity rotation in Spin(n). Our goal is to define an equivalence relation $\sim_l^*$ on Spin$(n) \times B^n$, which is an extension of the equivalence relation $\sim_l$ on B^n, such that the quotient space $X = (\text{Spin}(n) \times B^n)/(\{1\} \times D_{e_n}^{n-1}, \sim_l^*)$ is equipotent to Spin$(n) \times L_{e_n}$.

For $(s_1,c),(s_2,d)\in \mathrm{Spin}(n)\times B^n$ we define the equivalence relation $\sim_l^*$ by

$$
\begin{aligned}
&(s_1,c)\sim_l^*(s_2,d)\Leftrightarrow \exists\, s_3\in \mathrm{Spin}(n),\ \exists\, b\in L_{e_n},\ \exists\, a,f\in D_{e_n}^{n-1}:\\
&(s_1,c)=\left(s_3\frac{\overline{1-[(-a)\oplus(b\oplus a)]a}}{|1-[(-a)\oplus(b\oplus a)]a|},(-a)\oplus(b\oplus a)\right)\times(e,a)\\
&\text{and}\\
&(s_2,d)=\left(s_3\frac{\overline{1-[(-f)\oplus(b\oplus f)]f}}{|1-[(-f)\oplus(b\oplus f)]f|},(-f)\oplus(b\oplus f)\right)\times(e,f).
\end{aligned}
\tag{4.5}
$$

The equivalence relation (4.5) reduces to

$$
\begin{aligned}
&(s_1,c)\sim_l^*(s_2,d)\Leftrightarrow \exists\, s_3\in \mathrm{Spin}(n),\ \exists\, b\in L_{e_n},\ \exists\, a,f\in D_{e_n}^{n-1}:\\
&s_1=s_2\wedge(c=b\oplus a\ \wedge\ d=b\oplus f).
\end{aligned}
\tag{4.6}
$$

It is easy to see that the equivalence class associated to (s_1,c) is equal to $\{s_1\}\times[c]$, with $[c]=S_b\in B^n/(D_{e_n}^{n-1},\sim_l)$, and the quotient space is $X=\{\{s\}\times S_b : s\in \mathrm{Spin}(n), b\in L_{e_n}\}$, which is equipotent to $\mathrm{Spin}(n)\times L_{e_n}$. Thus, the space X is equipotent (up to covering) to the homogeneous space used by Antoine et. al. via the Iwasawa decomposition of $SO_0(1,n)$ [8]. The homogeneous space X will be the underlying space for the construction of spherical continuous wavelet transforms.

5 Continuous wavelet transforms on the unit sphere S^{n-1}

The theory of square integrability of group representations on homogeneous spaces was developed in a series of papers [3, 5] and applied for example in [13] for the case of the Weyl Heisenberg group and in [14] for Gabor analysis on the unit sphere. The theory depends only on the homogeneous space, on the choice of a section and a quasi-invariant measure on the homogeneous space, and on the representation of the group. Therefore, the general approach also works in our case since we can obtain these terms in our case.

For the construction of a spherical continuous wavelet transform we need to define motions (rotations and translations) and dilations on S^{n-1}. Translations correspond to rotations of the homogeneous space $\mathrm{Spin}(n)/\mathrm{Spin}(n-1)$ and rotations can be realized as rotations around a certain axis on the sphere. Thus, both translations and rotations can be associated with the action of $\mathrm{Spin}(n)$ on S^{n-1}. Dilations will arise from Möbius transformations of type $\varphi_a(x)$.

For $f\in L^2(S^{n-1},dS)$, where dS is the usual Lebesgue surface measure on S^{n-1}, we define the rotation and spherical dilation operators

$$R_sf(x)=f(\overline{s}xs),\qquad s\in \mathrm{Spin}(n) \tag{5.1}$$

and

$$D_af(x)=\left(\frac{1-|a|^2}{|1-ax|^2}\right)^{\frac{n-1}{2}}f(\varphi_{-a}(x)),\qquad a\in B^n, \tag{5.2}$$

where $\left(\frac{1-|a|^2}{|1-ax|^2}\right)^{n-1}$ is the Jacobian of $\varphi_{-a}(x)$ on S^{n-1}. From these operators we construct the representation U of $\mathrm{Spin}^+(1,n)$ in $L^2(S^{n-1},dS)$ defined by

$$U(s,a)f(x)=R_sD_af(x)=\left(\frac{1-|a|^2}{|1-a\overline{s}xs|^2}\right)^{\frac{n-1}{2}}f(\varphi_{-a}(\overline{s}xs)). \tag{5.3}$$

Proposition 5.1 *[16] $U(s,a)$ is a strongly continuous unitary representation of the group* $\mathrm{Spin}^+(1,n)$ *in $L^2(S^{n-1}, dS)$.*

This representation can be decomposed via the Cartan decomposition of $\mathrm{Spin}^+(1,n)$ giving rise to the well-known representations of the subgroups $K = \mathrm{Spin}(n)$ and $A = \mathrm{Spin}(1,1)$ [30]. It belongs to the principal series of $SO_0(1,n)$, being irreducible on the space $L^2(S^{n-1})$.

Following the general approach of [3] we will build a system of wavelets indexed by points of the space X. For each section σ on $Y = B^n/(D^{n-1}_{e_n}, \sim_l)$ we will consider the parameter space $W_\sigma = \{(s, \sigma(te_n)), s \in \mathrm{Spin}(n), t \in]-1,1[\}$. Thus, W_σ is a section on X and we will restrict our representation U to a given section W_σ. For the fundamental section W_{σ_0} the representation $U(s,\sigma_0)$ coincides with the representation used by Antoine and Vandergheynst in [8]. To see this, we only need to make the equivalence of the dilation operators since, up to a double covering, the actions of the rotation operators are the same. We recall that the dilation operator defined in [8] is

$$D^u\psi(\omega) = \lambda^{1/2}(u,\phi)\psi(\omega_{1/u}),$$

with $\omega = (\theta_1, \ldots, \theta_{n-2}, \phi) \in S^{n-1}$, $\lambda(u,\phi) = \left(\frac{4u^2}{[(u^2-1)\cos\phi + (u^2+1)]^2}\right)^{\frac{n-1}{2}}$ and $\omega_{1/u} = ((\theta_1)_u, \ldots, (\theta_{n-2})_u, \phi_u)$, with $(\theta_j)_u = \theta_j$, $j = 1\ldots, n-2$, and $\tan\frac{\phi_u}{2} = u\tan\frac{\phi}{2}$.

Proposition 5.2 *The dilation operators D^u and D_{te_n} are equivalent, in the sense that they produce the same action on S^{n-1}.*

Proof: The Möbius transformation φ_{te_n} corresponds to the $\mathrm{Spin}(1,1)$ action, which is the usual Euclidean dilation lifted on S^{n-1} by inverse stereographic projection from the South Pole [10]. It remains to show that the weights of the dilation operators D^u and D_{te_n} are equal. In fact, by the bijection between $]-1,1[$ and $]0,\infty[$ given by $t = \frac{u-1}{u+1}$, $t \in]-1,1[, u > 0$ we have

$$\left(\frac{1-t^2}{|1-te_nx|^2}\right)^{\frac{n-1}{2}} = \left(\frac{1-t^2}{1+2\langle te_n, x\rangle + t^2}\right)^{\frac{n-1}{2}} = \left(\frac{1-\left(\frac{u-1}{u+1}\right)^2}{1+2\frac{u-1}{u+1}x_n + \left(\frac{u-1}{u+1}\right)^2}\right)^{\frac{n-1}{2}} =$$

$$= \left(\frac{4u}{2(u^2+1)+2(u^2-1)x_n}\right)^{\frac{n-1}{2}} = \left(\frac{4u^2}{[(u^2-1)\cos\phi + (u^2+1)]^2}\right)^{\frac{n-1}{4}}.$$

■

Next, we construct a quasi-invariant measure on X. For that, we need first to construct a quasi-invariant measure on the space $Y = B^n/(D^{n-1}_{e_n}, \sim_l)$ and, afterwards, to extend it to the space X. First we will prove that the measure $d\mu(te_n) = \frac{2(1-t)^{n-2}}{(1+t)^n}dt$ is a quasi-invariant measure on Y using the mapping (4.3).

For all $a \in B^n$ and $b \in L_{e_n}$ such that $a = s_* a_* \overline{s_*}$, with $s_* = s_1 \ldots s_{n-2} \in \mathrm{Spin}(n-1)$ (c.f. Lemma 3.7) we have that $b \oplus a = b \oplus (s_*(a_*)\overline{s_*}) = s_*((\overline{s_*}bs_*) \oplus a_*)\overline{s_*} = s_*(b \oplus a_*)\overline{s_*}$. Thus, the equivalence classes $[b \oplus a]$ and $[b \oplus a_*]$ are equal. Therefore, to prove the quasi-invariance of our measure we only need to consider the quasi-action of $a = (0, \ldots, 0, a_{n-1}, a_n)$ on $b = te_n \in L_{e_n}$.

Let $d = b \oplus a = \varphi_{-b}(a) = \left(0, \ldots, 0, \frac{(1-t^2)a_{n-1}}{1+2a_nt+|a|^2t^2}, \frac{(1-t^2)a_n + (1+|a|^2+2ta_n)t}{1+2a_nt+|a|^2t^2}\right)$. Applying the projection formulas (4.1) we obtain the new equivalence class, $t^*e_n \oplus D^{n-1}_{e_n}$ where

$$t^* = \tau_a(t) = 2\frac{t^2a_n + (1+|a|^2)t + a_n}{\sqrt{C_a}(1-t^2) + (1+|a|^2)(1+t^2) + 4ta_n}$$

with $C_a := (a_{n-1}^2 + (1+a_n)^2)(a_{n-1}^2 + (1-a_n)^2)$. Putting a in spherical coordinates we can see that $t^* \in]-1,1[$, for each $t \in]-1,1[$. Differentiating τ_a with respect to t we obtain by straightforward computations

$$\tau_a'(t) = \frac{2\sqrt{C_a}}{\sqrt{C_a}(1-t^2) + (1+|a|^2)(1+t^2) + 4ta_n}.$$

Therefore, the Radon-Nikodym derivative of $d\mu([b \oplus a])$ with respect to $d\mu(b)$ is given by

$$\chi(a,b) = \frac{d\mu([b\oplus a])}{d\mu(b)} = \frac{(1-\tau_a(t))^{n-2}(1+t)^n}{(1+\tau_a(t))^n(1-t)^{n-2}}\tau_a'(t).$$

Since for each a we have that $\tau_a'(t) > 0$, for all $t \in]-1,1[$ we conclude that $\chi(a,b) \in \mathbb{R}^+$, for all $a \in B^n$ and $b \in L_{e_n}$, thus proving that the measure $d\mu(b)$ is quasi-invariant. This measure is equivalent to the measure $d\mu(u) = \frac{du}{u^n}$, by means of the bijection given by $t = \frac{u-1}{u+1}$ ($u \in \mathbb{R}^+$ and $t \in]-1,1[$). This bijection establishes a correspondence between the measure of our homogeneous space and the homogeneous space of Antoine and Vandergheynst [8]. Thus, a quasi-invariant measure for X is given by $d\mu(s, te_n) = \frac{2(1-t)^{n-2}}{(1+t)^n}\, dt\, d\mu(s)$, where $d\mu(s)$ is the invariant measure on $\mathrm{Spin}(n)$.

The measure $d\mu(s, \sigma(te_n)) = \chi(\sigma(te_n), te_n) d\mu(s, te_n)$ is the standard quasi-invariant measure for an arbitrary Borel section W_σ (see [3]). The quasi-invariant measure on a section is unique (up to equivalence), that is, if μ_1 and μ_2 are quasi-invariant measures on X then there is a Borel function $\beta : X \to \mathbb{R}^+$, such that $d\mu_1(x) = \beta(x) d\mu_2(x)$, for all $x \in X$. Thus, we will use the same measure for all Borel sections. To avoid questions of integrability we will restrict our attention only to continuous sections, that is sections $\sigma(te_n)$ such that the generating function g is of class $C(]-1,1[)$.

Let $N(n,l)$ be the number of all linearly independent homogeneous harmonic polynomials of degree l in n variables and $\{Y_l^M, M = 1, \ldots, N(n,l)\}_{l=0}^{\infty}$ be an orthonormal basis of spherical harmonics, i.e. $< Y_l^M, Y_{l'}^{M'} >_{L^2(S^{n-1})} = \delta_{ll'}\delta_{MM'}$. Thus, a function $f \in L^2(S^{n-1})$ has the Fourier expansion

$$f(x) = \sum_{l=0}^{\infty} \sum_{M=1}^{N(n,l)} \widehat{f}(l,M) Y_l^M(x), \tag{5.4}$$

where $\widehat{f}(l,M) := < Y_l^M, f >_{L^2(S^{n-1})}$ are the Fourier coefficients of f.

We will also make use of Wigner $D-$functions [8] which appear as the following transformation formula for spherical harmonics

$$(R(s)Y_l^M)(x) = Y_l^M(\overline{s}xs) = \sum_{M'=1}^{N(n,l)} D_{MM'}^l(s) Y_l^{M'}(x), \quad s \in \mathrm{Spin}(n)$$

and satisfy the orthogonality relations

$$\int_{\mathrm{Spin}(n)} \overline{D_{MN}^l(s)} D_{M'N'}^{l'}(s) d\mu(s) = \frac{1}{N(n,l)} \delta_{ll'}\delta_{MM'}\delta_{NN'}. \tag{5.5}$$

The next theorem is a consequence of a result of Antoine and Vandergheynst (Theorem 4.2 - [8]). This is due to the fact that the wavelet system obtained by W_{σ_0} coincides with the wavelet system previously obtained via the homogeneous space in [8], since the dilation operators are equivalent as proved in Proposition 5.2.

Theorem 5.3 *The representation U given in (5.3) is square integrable modulo $(\{1\} \times D^{n-1}_{e_n}, \sim^*_l)$ and the section W_σ if there exists a nonzero admissible vector $\psi \in L^2(S^{n-1}, dS)$ satisfying*

$$\frac{1}{N(n,l)} \sum_{M=1}^{N(n,l)} \int_{-1}^{1} |\widehat{\psi}_{\sigma(te_n)}(l,M)|^2 \, d\mu(te_n) < \infty, \tag{5.6}$$

uniformly in l, where $\widehat{\psi}_{\sigma(te_n)}(l,M) =< Y_l^M, D_{\sigma(te_n)}\psi >_{L^2(S^{n-1})}$.

In order to investigate the existence of admissible functions for a given section W_σ we will project the admissibility condition on the sphere to the tangent plane. The idea is to compare every global continuous section $\sigma(te_n)$ with the fundamental section σ_0, since by definition every section $\sigma(te_n) = te_n \oplus f(t)e_{n-1}$ is a Möbius deformation of the fundamental section.

We consider the tangent plane of the sphere S^{n-1} at the North Pole and we denote by Φ the stereographic projection map $\Phi : S^{n-1} \to \mathbb{R}^{n-1} \cup \{\infty\}$ given by

$$\Phi(x) = \left(\frac{2x_1}{1+x_n}, \ldots, \frac{2x_{n-1}}{1+x_n} \right) \quad x \in S^{n-1} \backslash \{(0,\ldots,0,-1)\},$$

and $\Phi(0,\ldots,0,-1) = \infty$. The inverse mapping $\Phi^{-1} : \mathbb{R}^{n-1} \cup \{\infty\} \to S^{n-1}$ is given by

$$\Phi^{-1}(y) = \left(\frac{4y_1}{4+|y|^2}, \ldots, \frac{4y_{n-1}}{4+|y|^2}, \frac{4-|y|^2}{4+|y|^2} \right), \quad y \in \mathbb{R}^{n-1}$$

and $\Phi^{-1}(\infty) = (0,\ldots,0,-1)$.

To perform the stereographic projection of $\varphi_a(x)$ we will use the Cayley transformation in $\mathbb{R}^n$. It is a conformal mapping from the unit ball B^n into the upper half space $H_n^+ = \{x \in \mathbb{R}^n : x_n > 0\}$ defined by $y = (x+e_n)(1+e_n x)^{-1}$. The stereographic projection Φ can be rewritten in terms of the inverse of the Cayley transformation that is

$$\Phi(x) = 2(x-e_n)(1-e_n x)^{-1} = \left(\frac{2x_1}{1+x_n}, \ldots, \frac{2x_{n-1}}{1+x_n} \right).$$

Theorem 5.4 *The intertwining relation $\Phi(\varphi_a(x)) = \widetilde{\varphi}(\Phi(x))$ holds, where $\widetilde{\varphi}(x)$ is the Möbius transformation in $\mathbb{R}^{n-1}$ defined by the Vahlen matrix*

$$\frac{1}{\sqrt{1-|a|^2}} \begin{pmatrix} 1+a_n & 2(-a+a_n e_n) \\ \frac{a-a_n e_n}{2} & 1-a_n \end{pmatrix}. \tag{5.7}$$

Proof: Solving the system of equations in order to $u, v, w, z \in \Gamma(n)$ we obtain:

$$\begin{pmatrix} 2 & -2e_n \\ -e_n & 1 \end{pmatrix} \begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} = \begin{pmatrix} u & v \\ w & z \end{pmatrix} \begin{pmatrix} 2 & -2e_n \\ -e_n & 1 \end{pmatrix}$$

$$\Leftrightarrow \left\{ \begin{array}{c} 2u - ve_n = 2 - 2e_n a \\ -2ue_n + v = -2a - 2e_n \\ 2w - ze_n = -e_n + a \\ -2we_n + z = 1 + e_n a \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} u = \frac{2-2e_n a + ve_n}{2} \\ v = -2a - 2e_n + 2ue_n \\ w = \frac{-e_n + a + ze_n}{2} \\ z = 1 + e_n a + 2we_n \end{array} \right. \Leftrightarrow \left\{ \begin{array}{c} u = 1 + a_n \\ v = -2a + 2a_n e_n \\ w = \frac{a - a_n e_n}{2} \\ z = 1 - a_n \end{array} \right. .$$

In the resolution of the system we used the identities $e_n a e_n = -2a_n e_n + a$ and $e_n a + a e_n = -2a_n$. After normalization we obtain the matrix (5.7), which is in the Clifford group $\Gamma(1, n+1)$, by Theorem 2.1, and thus, it represents a Möbius transformation in $\mathbb{R}^{n-1}$.

$\blacksquare$

Now, we will consider the unitary stereographic projection operator Θ from $L^2(S^{n-1}, dS)$ into $L^2(\mathbb{R}^{n-1}, r^{n-2}dr\, dS^{n-2})$ and we will find the stereographic mapping in $\mathbb{R}^{n-1}$ that intertwines with the dilation operator D_a. To distinguish functions from different spaces we will use the convention $f \in L^2(S^{n-1})$ and $F \in L^2(\mathbb{R}^{n-1})$.

Lemma 5.5 *The map $\Theta : L^2(S^{n-1}, dS) \to L^2(\mathbb{R}^{n-1}, r^{n-2}dr\, dS^{n-2})$ defined by*

$$f(\theta_1, \ldots, \theta_{n-2}, \phi) \mapsto F(\theta_1, \ldots, \theta_{n-2}, r) = \left(\frac{4}{4+r^2}\right)^{\frac{n-1}{2}} f(\theta_1, \ldots, \theta_{n-2}, 2\arctan(r/2))$$

is a unitary map. In cartesian coordinates it reads as

$$f(x) \mapsto F(y) = \left(\frac{4}{4+|y|^2}\right)^{\frac{n-1}{2}} f(\Phi_1^{-1}(y)). \tag{5.8}$$

Proof: By the change of variables $\phi = 2\arctan(r/2)$, that is, $r = 2\tan(\phi/2) = 2\sqrt{\frac{1-\cos\phi}{1+\cos\phi}}$ we have

$$\begin{aligned} ||\Theta f||^2 &= \int_{S^{n-2}} \int_0^\infty \left| \left(\frac{4}{4+r^2}\right)^{\frac{n-1}{2}} f(\theta_1, \ldots, \theta_{n-2}, 2\arctan(r/2)) \right|^2 r^{n-2} dr\, dS^{n-2} \\ &= \int_{S^{n-2}} \int_0^\pi |f(\theta_1, \ldots, \theta_{n-2}, \phi)|^2 \sin^{n-2}\phi\, d\phi\, dS^{n-2} \\ &= ||f||^2_{L^2(S^{n-1}, dS)}. \end{aligned}$$

$\blacksquare$

Theorem 5.6 *Let $y \in \mathbb{R}^{n-1}$, and $\widetilde{\varphi}(y) := \frac{c_1 y + c_2}{c_3 y + c_4}$ be the Möbius transformation obtained from the matrix (5.7), with $c_1 = \frac{1+a_n}{\sqrt{1-|a|^2}}$, $c_2 = \frac{2(-a+a_n e_n)}{\sqrt{1-|a|^2}}$, $c_3 = \frac{a-a_n e_n}{2\sqrt{1-|a|^2}}$ and $c_4 = \frac{1-a_n}{\sqrt{1-|a|^2}}$. Then we have the intertwining relation*

$$\Theta D_a \psi = M \Theta \psi, \tag{5.9}$$

where $MF(y) = \left(\frac{4(1-|a|^2)}{|-(a-a_n e_n)y + 2(1+a_n)|^2}\right)^{\frac{n-1}{2}} F(\widetilde{\varphi}^{-1}(y))$ is the unitary operator associated with $\widetilde{\varphi}^{-1}(y) = \frac{c_4 y - c_2}{-c_3 y + c_1}$.

Proof: By definition we have

$$\Theta D_a \psi(y) = \left(\frac{4}{4+|y|^2}\right)^{\frac{n-1}{2}} \left(\frac{1-|a|^2}{|1-a\Phi^{-1}(y)|^2}\right)^{\frac{n-1}{2}} \psi\left(\varphi_{-a}(\Phi_1^{-1}(y))\right) \tag{5.10}$$

and

$$M\Theta\psi(y) = \left(\frac{4(1-|a|^2)}{|-(a-a_n e_n)y + 2(1+a_n)|^2}\right)^{\frac{n-1}{2}} \left(\frac{4}{4+|\widetilde{\varphi}^{-1}(y)|^2}\right)^{\frac{n-1}{2}} \psi\left(\Phi_1^{-1}(\widetilde{\varphi}^{-1}(y))\right). \tag{5.11}$$

First we observe that

$$\varphi_{-a}(\Phi^{-1}(y)) = \Phi^{-1}(\widetilde{\varphi}^{-1}(y)), \quad \forall y \in \overline{\mathbb{R}^{n-1}}. \tag{5.12}$$

If $\varphi_{-a}(\Phi^{-1}(y)) = x \in S^{n-1}$ then $y = \Phi(\varphi_a(x))$. Moreover, if $\Phi^{-1}(\widetilde{\varphi}^{-1}(y)) = x$, then $y = \widetilde{\varphi}(\Phi(x))$. Since the relation $\Phi(\varphi_a(x)) = \widetilde{\varphi}(\Phi(x))$ holds (Theorem 5.4) we conclude that the relation (5.12) is true. It remains to prove that the weights in (5.10) and (5.11) are equal. On the one hand,

$$\begin{aligned}\frac{4}{4+|y|^2}\frac{1-|a|^2}{|1-a\Phi^{-1}(y)|^2} &= \frac{4}{4+|y|^2}\frac{1-|a|^2}{1+2\langle a,\Phi^{-1}(y)\rangle+|a|^2} \\ &= \frac{4(1-|a|^2)}{(1+|a|^2)(4+|y|^2)+2a_n(4-|y|^2)+8(\langle a,y\rangle-a_ny_n)}.\end{aligned}$$

On the other hand, since

$$|\widetilde{\varphi}^{-1}(y)|^2 = \frac{|c_4y-c_2|^2}{|-c_3y+c_1|^2} = 4\frac{(1-a_n)^2|y|^2+4(1-a_n)(\langle a,y\rangle-a_ny_n)+4(|a|^2-a_n^2)}{(|a|^2-a_n^2)|y|^2+4(1+a_n)(\langle a,y\rangle-a_ny_n)+4(1+a_n)^2}$$

and

$$|-(a-a_ne_n)y+2(1+a_n)|^2 = (|a|^2-a_n^2)|y|^2+4(1+a_n)(\langle a,y\rangle-a_ny_n)+4(1+a_n)^2$$

we obtain, after some computations

$$\frac{4(1-|a|^2)}{|-(a-a_ne_n)y+2(1+a_n)|^2}\,\frac{4}{4+|\widetilde{\varphi}^{-1}(y)|^2} =$$
$$\frac{4(1-|a|^2)}{(1+|a|^2)(4+|y|^2)+2a_n(4-|y|^2)+8(\langle a,y\rangle-a_ny_n)}.$$

Thus, the result follows.

■

Vahlen matrices with pseudodeterminant equals to 1 constitute the group $SL_2(\Gamma(n)\cup\{0\})$ and admit an Iwasawa decomposition similar to the decomposition of the group $SL_2(\mathbb{C})$. The Iwasawa decomposition of a generic element of $SL_2(\Gamma(n)\cup\{0\})$ is

$$\begin{pmatrix} u & v \\ w & z \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}\begin{pmatrix} \delta^{-1/2} & 0 \\ 0 & \delta^{1/2} \end{pmatrix}\begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \tag{5.13}$$

where $\alpha,\beta,\xi \in \Gamma(n)\cup\{0\}$ and $\delta\in\mathbb{R}^+$, $uz^*-vw^*=1$ and

$$\delta = (|u|^2+|w|^2)^{-1}, \qquad \alpha = u\delta^{1/2}, \qquad \beta = -\overline{w}\delta^{1/2},$$

$$\xi = u^{-1}(v+\overline{w}\delta),\ \text{if } u\neq 0 \qquad \text{or} \qquad \xi = w^{-1}(z-\overline{u}\delta),\ \text{if } w\neq 0.$$

The Iwasawa decomposition of the matrix 5.7 yields the parameters

$$\begin{aligned}\alpha &= \frac{2(1+a_n)}{\sqrt{4(1+a_n)^2+|a|^2-a_n^2}}, \qquad \beta = \frac{a-a_ne_n}{\sqrt{4(1+a_n)^2+|a|^2-a_n^2}}, \\ \delta &= \frac{4(1-|a|^2)}{4(1+a_n)^2+|a|^2-a_n^2}, \qquad \xi = \frac{2(-a+a_ne_n)(5+3a_n)}{4(1+a_n)^2+|a|^2-a_n^2}.\end{aligned} \tag{5.14}$$

Departing from the Iwasawa decomposition with parameters (5.14) we define the following unitary operators on $L^2(\mathbb{R}^{n-1})$:

$$R^{\alpha,\beta}F(y) = \left(\frac{1}{|-\beta\overline{y}+\alpha|^2}\right)^{\frac{n-1}{2}} F\left(\frac{\alpha y+\beta}{-\overline{\beta}y+\overline{\alpha}}\right);$$

$$D^{\delta}F(y) = \delta^{-\frac{n-1}{2}} F\left(\frac{y}{\delta}\right); \qquad T^{\xi}F(y) = F(y+\xi).$$

As these mappings are isometries whose ranges include the whole space $L^2(\mathbb{R}^{n-1})$ the adjoint and inverse operators are identical.

Lemma 5.7 *The adjoint operators to $R^{\alpha,\beta}$, D^{δ} and T^{ξ} are given by*

$$(R^{\alpha,\beta})^* = R^{\overline{\alpha},-\beta}, \qquad (D^{\delta})^* = D^{\frac{1}{\delta}}, \qquad \textit{and} \qquad (T^{\xi})^* = T^{-\xi}. \tag{5.15}$$

Proposition 5.8 *The operator M admits the following factorization*

$$MF = R^{\alpha,-\beta} D^{\frac{1}{\delta}} T^{-\xi} F. \tag{5.16}$$

Proof: We have

$$MF(y) = \left(\frac{4(1-|a|^2)}{|-(a-a_n e_n)y + 2(1+a_n)|^2}\right)^{\frac{n-1}{2}} F(\widetilde{\varphi}^{-1}(y))$$

and

$$R^{\alpha,-\beta} D^{\frac{1}{\delta}} T^{-\xi} F(y) = \left(\frac{\delta}{|\alpha-\beta y|^2}\right)^{\frac{n-1}{2}} F\left(\delta\,\frac{\alpha y - \beta}{\overline{\beta} y + \alpha} - \xi\right).$$

First we prove the equality between arguments. On the one hand,

$$\widetilde{\varphi}^{-1}(y) = \frac{c_4 y - c_2}{-c_3 y + c_1} = \frac{2(1-a_n)y - 4(-a+a_n e_n)}{(-a+a_n e_n)y + 2(1+a_n)}.$$

On the other hand,

$$\delta\,\frac{\alpha y - \beta}{\overline{\beta} y + \alpha} - \xi = \frac{(\delta\alpha - \overline{\beta}\xi)y - (\delta\beta + \alpha\xi)}{\overline{\beta} y + \alpha} = \frac{2(1-a_n)y - 4(-a+a_n e_n)}{(-a+a_n e_n)y + 2(1+a_n)}.$$

Finally, since $\left(\frac{\delta}{|\alpha-\beta y|^2}\right)^{\frac{n-1}{2}} = \left(\frac{4(1-|a|^2)}{|-(a-a_n e_n)y+2(1+a_n)|^2}\right)^{\frac{n-1}{2}}$ we have the desired factorization.

■

Corollary 5.9 *The intertwining relation (5.9) can be written as*

$$\Theta D_a \psi = R^{\alpha,-\beta} D^{\frac{1}{\delta}} T^{-\xi} \Theta\psi. \tag{5.17}$$

Corollary 5.10 *For $a = te_n \in L_{e_n}$ (restriction to the Spin$(1,1)$ case) we obtain the well-known intertwining relation*

$$\Theta D_{te_n} \psi = D^{\frac{1+t}{1-t}} \Theta\psi. \tag{5.18}$$

Relation (5.17) is the generalization of relation (5.18). The first case correspond to the isotropic case of Antoine and Vandergheynst and the second case correspond to the anisotropic case, for $a \in B^n \backslash L_{e_n}$.

For an arbitrary global left section $\sigma(te_n) = te_n \oplus g(t)e_{n-1}$ (4.4), the parameters (5.14) become

$$\alpha_t = \frac{2(1+tg(t)^2)}{\sqrt{(1+t^2+6t)g(t)^2+4(1+t^2g(t)^4)}},$$

$$\beta_t = \frac{(1-t)g(t)}{\sqrt{(1+t^2+6t)g(t)^2+4(1+t^2g(t)^4)}}\, e_{n-1},$$

$$\delta_t = \frac{4(1-t)(1-g(t)^2)(1+t^2g(t)^2)}{(t+1)((1+t^2+6t)g(t)^2+4(1+t^2g(t)^4))},$$

$$\xi_t = \frac{2(t-1)g(t)((5t^2+3t)g(t)^2+3t+5)}{(t+1)((1+t^2+6t)g(t)^2+4(1+t^2g(t)^4))}\, e_{n-1}\,. \tag{5.19}$$

When $g(t) \equiv 0$, (restriction to the fundamental section - Spin(1,1) case) we obtain $\alpha = 1$, $\beta = 0$, $\xi = 0$ and $\delta = \frac{1-t}{1+t}$, which reflects again the fact that we obtain a pure dilation.

Lemma 5.11 *The parameter δ_t can be written as a deformation of the pure dilation parameter, i.e. $\delta_t = \frac{1-t}{1+t}\delta_t^*$, with $\delta_t^* = \frac{4(1-g(t)^2)(1+t^2g(t)^2)}{(1+t^2+6t)g(t)^2+4(1+t^2g(t)^4)}$, and where δ_t^* satisfies the estimate $0 < \delta_t^* \leq \frac{2(3-2\sqrt{3})}{3(-2+\sqrt{3})}$, for all $t \in]-1,1[$.*

Proof: As $g(t) \in]-1,1[$, for every $t \in]-1,1[$, the study of the behavior of the parameter δ_t^* is equivalent to the study of the behavior of the function of two variables $h(t,\lambda) = \frac{4(1-\lambda^2)(1+t^2\lambda^2)}{(1+t^2+6t)\lambda^2+4(1+t^2\lambda^4)}$, with $t, \lambda \in]-1,1[$. Since for each $\lambda \in]-1,1[$ the function h is strictly decreasing in t we obtain the estimate

$$0 \leq h(t,\lambda) \leq \frac{1-\lambda^4}{1+\lambda^4-\lambda^2}, \quad \forall\, t \in]-1,1[. \tag{5.20}$$

Maximizing the right-hand side of (5.20) we obtain

$$0 \leq h(t,\lambda) \leq \frac{2(3-2\sqrt{3})}{3(\sqrt{3}-2)}, \quad \forall\, t,\lambda \in]-1,1[.$$

Therefore,

$$0 \leq \delta_t^* \leq \frac{2(3-2\sqrt{3})}{3(\sqrt{3}-2)}, \quad \forall\, t \in]-1,1[.$$

■

Now, we will prove the main theorem of this paper concerning the square-integrability of the representation U over an arbitrary continuous section W_σ.

Theorem 5.12 *Let $\psi \in L^2(S^{n-1})$ such that the family $\{R_s D_{te_n}\psi, s \in \mathrm{Spin}(n), t \in]-1,1[\}$ is a continuous frame, that is, there exist constants $0 < A \leq B < \infty$ such that*

$$A||f||^2 \leq \int_{\mathrm{Spin}(n)} \int_{-1}^{1} |\langle f, R_s D_{te_n}\psi\rangle|^2 d\mu(te_n)\, d\mu(s) \leq B||f||^2, \quad \forall f \in L^2(S^{n-1}). \tag{5.21}$$

Then ψ is an admissible function for any continuous section W_σ and the system $\{R_s D_{\sigma(te_n)}\psi, s \in \mathrm{Spin}(n), t \in]-1,1[\}$ is also a continuous frame with frame bounds A and B.

Proof: For every $a \in B^n$ and $f \in L^2(S^{n-1})$ arbitrary we have

$$\begin{aligned}
&\int_{\mathrm{Spin}(n)} \int_{-1}^{1} \left| \langle f, R_s D_a \psi \rangle_{L^2(S^2)} \right|^2 d\mu(te_n) d\mu(s) \\
&= \int_{\mathrm{Spin}(n)} \int_{-1}^{1} \left| \langle R_{\overline{s}} f, D_a \psi \rangle_{L^2(S^2)} \right|^2 d\mu(te_n) d\mu(s) \\
&= \int_{\mathrm{Spin}(n)} \int_{-1}^{1} \left| \langle \Theta R_{\overline{s}} f, \Theta D_a \psi \rangle_{L^2(\mathbb{R}^2)} \right|^2 d\mu(te_n) d\mu(s) \quad \text{(by Lemma 5.5)} \\
&= \int_{\mathrm{Spin}(n)} \int_{-1}^{1} \left| \left\langle \Theta R_{\overline{s}} f, R^{\alpha,-\beta} D^{\frac{1}{\delta}} T^{-\xi} \Theta \psi \right\rangle_{L^2(\mathbb{R}^{n-1})} \right|^2 d\mu(te_n) d\mu(s) \quad \text{(by (5.17))} \\
&= \int_{\mathrm{Spin}(n)} \int_{-1}^{1} \left| \left\langle \Theta R_{\overline{s}} f, R^{\alpha,-\beta} T^{-\frac{\xi}{\delta}} D^{\frac{1}{\delta}} \Theta \psi \right\rangle_{L^2(\mathbb{R}^{n-1})} \right|^2 d\mu(te_n) d\mu(s) \\
&= \int_{\mathrm{Spin}(n)} \int_{-1}^{1} \left| \left\langle T^{\frac{\xi}{\delta}} R^{\alpha,\beta} \Theta R_{\overline{s}} f, D^{\frac{1}{\delta}} \Theta \psi \right\rangle_{L^2(\mathbb{R}^{n-1})} \right|^2 d\mu(te_n) d\mu(s) \quad \text{(by (5.7))}. \qquad (5.22)
\end{aligned}$$

Now we consider $a = \sigma(te_n)$ and the parameters (5.19). By Lemma 5.11, the integral (5.22) becomes

$$\int_{\mathrm{Spin}(n)} \int_{-1}^{1} \left| \left\langle D^{\delta_t^*} T^{\frac{\xi_t}{\delta_t}} R^{\alpha_t,\beta_t} \Theta R_{\overline{s}} f, D^{\frac{1+t}{1-t}} \Theta \psi \right\rangle_{L^2(\mathbb{R}^{n-1})} \right|^2 d\mu(te_n) d\mu(s). \qquad (5.23)$$

For each $s \in \mathrm{Spin}(n)$ and $t \in]-1,1[$ we have

$$\left| \left\langle D^{\delta_t^*} T^{\frac{\xi_t}{\delta_t}} R^{\alpha_t,\beta_t} \Theta R_{\overline{s}} f, D^{\frac{1+t}{1-t}} \Theta \psi \right\rangle \right|^2 \leq \underbrace{\sup_{t' \in]-1,1[} \left| \left\langle D^{\delta_{t'}^*} T^{\frac{\xi_{t'}}{\delta_{t'}}} R^{\alpha_{t'},\beta_{t'}} \Theta R_{\overline{s}} f, D^{\frac{1+t}{1-t}} \Theta \psi \right\rangle \right|^2}_{I}$$

and

$$\underbrace{\inf_{t' \in]-1,1[} \left| \left\langle D^{\delta_{t'}^*} T^{\frac{\xi_{t'}}{\delta_{t'}}} R^{\alpha_{t'},\beta_{t'}} \Theta R_{\overline{s}} f, D^{\frac{1+t}{1-t}} \Theta \psi \right\rangle \right|^2}_{II} \leq \left| \left\langle D^{\delta_t^*} T^{\frac{\xi_t}{\delta_t}} R^{\alpha_t,\beta_t} \Theta R_{\overline{s}} f, D^{\frac{1+t}{1-t}} \Theta \psi \right\rangle \right|^2.$$

Let $t_1, t_2 \in [-1,1]$ such that

$$I := \left| \left\langle D^{\delta_{t_2}^*} T^{\frac{\xi_{t_2}}{\delta_{t_2}}} R^{\alpha_{t_2},\beta_{t_2}} \Theta R_{\overline{s}} f, D^{\frac{1+t}{1-t}} \Theta \psi \right\rangle \right|^2$$

and

$$II := \left| \left\langle D^{\delta_{t_1}^*} T^{\frac{\xi_{t_1}}{\delta_{t_1}}} R^{\alpha_{t_1},\beta_{t_1}} \Theta R_{\overline{s}} f, D^{\frac{1+t}{1-t}} \Theta \psi \right\rangle \right|^2.$$

Thus, we obtain

$$\int_{\mathrm{Spin}(n)}\int_{-1}^{1}\left|\left\langle D^{\delta_{t_1}^*}T^{\frac{\xi_{t_1}}{\delta_{t_1}}}R^{\alpha_{t_1},\beta_{t_1}}\Theta R_{\overline{s}}\,f, D^{\frac{1+t}{1-t}}\Theta\psi\right\rangle_{L^2(\mathbb{R}^{n-1})}\right|^2 d\mu(te_n)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(n)}\int_{-1}^{1}\left|\left\langle D^{\delta_t^*}T^{\frac{\xi_t}{\delta_t}}R^{\alpha_t,\beta_t}\Theta R_{\overline{s}}\,f, D^{\frac{1+t}{1-t}}\Theta\psi\right\rangle_{L^2(\mathbb{R}^{n-1})}\right|^2 d\mu(te_n)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(n)}\int_{-1}^{1}\left|\left\langle D^{\delta_{t_2}^*}T^{\frac{\xi_{t_1}}{\delta_{t_1}}}R^{\alpha_{t_1},\beta_{t_1}}\Theta R_{\overline{s}}\,f, D^{\frac{1+t}{1-t}}\Theta\psi\right\rangle_{L^2(\mathbb{R}^{n-1})}\right|^2 d\mu(te_n)d\mu(s).$$

Since for $t\in[-1+\epsilon,1-\epsilon]$, $\epsilon>0$, the operators $D^{\delta_t^*}$, $T^{\frac{\xi_t}{\delta_t}}$ and R^{α_t,β_t} are unitary and bijective mappings we only need to study the case of $t\to\pm1$. In this case, the parameter $\frac{\xi_t}{\delta_t}$ associated to the operator $T^{\frac{\xi_t}{\delta_t}}$ can become infinity. But, it is easy to see that the composition of the operators $D^{\delta_t^*}$ and $T^{\frac{\xi_t}{\delta_t}}$ is well behaved. Thus, for each $f\in L^2(S^{n-1})$ we can find $f_1,f_2\in L^2(S^{n-1})$ such that $D^{\delta_{t_1}^*}T^{\frac{\xi_{t_1}}{\delta_{t_1}}}R^{\alpha_{t_1},\beta_{t_1}}\Theta R_{\overline{s}}\,f=\Theta R_{\overline{s}}\,f_1$ and $D^{\delta_{t_2}^*}T^{\frac{\xi_{t_2}}{\delta_{t_2}}}R^{\alpha_{t_2},\beta_{t_2}}\Theta R_{\overline{s}}\,f=\Theta R_{\overline{s}}\,f_2$ with $||f_1||=||f_2||=||f||$.

Therefore, we have

$$\int_{\mathrm{Spin}(n)}\int_{-1}^{1}\left|\left\langle \Theta R_{\overline{s}}\,f_1, D^{\frac{1+t}{1-t}}\Theta\psi\right\rangle_{L^2(\mathbb{R}^{n-1})}\right|^2 d\mu(te_n)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(n)}\int_{-1}^{1}\left|\left\langle D^{\delta_t^*}T^{\frac{\xi_t}{\delta_t}}R^{\alpha_t,\beta_t}\Theta R_{\overline{s}}\,f, D^{\frac{1+t}{1-t}}\Theta\psi\right\rangle_{L^2(\mathbb{R}^{n-1})}\right|^2 d\mu(te_n)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(n)}\int_{-1}^{1}\left|\left\langle \Theta R_{\overline{s}}\,f_2, D^{\frac{1+t}{1-t}}\Theta\psi\right\rangle_{L^2(\mathbb{R}^{n-1})}\right|^2 d\mu(te_n)d\mu(s). \tag{5.24}$$

By (5.17), (5.22), and Lemma 5.5, condition (5.24) becomes

$$\int_{\mathrm{Spin}(n)}\int_{-1}^{1}\left|\langle f_1, R_sD_{te_n}\psi\rangle_{L^2(S^{n-1})}\right|^2 d\mu(te_n)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(n)}\int_{-1}^{1}\left|\left\langle f, R_sD_{\sigma(te_n)}\psi\right\rangle_{L^2(S^{n-1})}\right|^2 d\mu(te_n)d\mu(s)\leq$$
$$\int_{\mathrm{Spin}(n)}\int_{-1}^{1}\left|\langle f_2, D_{te_n}\Theta\psi\rangle_{L^2(S^{n-1})}\right|^2 d\mu(te_n)d\mu(s). \tag{5.25}$$

As, by hypothesis, ψ satisfies condition (5.21) there exist constants $0<A\leq B<\infty$ such that

$$A||f_1||^2\leq\int_{\mathrm{Spin}(n)}\int_{-1}^{1}|\left\langle f, R_sD_{\sigma(te_n)}\psi\right\rangle|^2 d\mu(te_n)d\mu(s)\leq B||f_2||^2,\ \forall f\in L^2(S^{n-1}),$$

and so, since $||f_1||=||f_2||=||f||$, the result holds.

■

As a consequence of this theorem we conclude that there exists $\psi \in L^2(S^{n-1})$ such that the condition (5.6) is satisfied.

For an arbitrary section W_σ and an admissible function $\psi \in L^2(S^{n-1})$ we define the **generalized spherical CWT**

$$W_\psi[f](s, \sigma(te_n)) = \langle R_s D_{\sigma(te_n)}\psi, f \rangle = \int_{S^{n-1}} \overline{R_s D_{\sigma(te_n)}\psi(x)} f(x) dS. \tag{5.26}$$

For each section σ and each function $\psi \neq 0$ we define an operator A_σ^ψ by

$$\left\langle A_\sigma^\psi f, f \right\rangle = \int_{\mathrm{Spin}(n)} \int_{-1}^{1} | \left\langle f, R_s D_{\sigma(te_n)}\psi \right\rangle |^2 d\mu(te_n) d\mu(s).$$

It is a Fourier multiplier given by $\widehat{A_\sigma^\psi h}(l, M) = C_\sigma^\psi(l)\widehat{h}(l, M)$, with

$$C_\sigma^\psi(l) := \frac{1}{N(n,l)} \sum_{M=1}^{N(n,l)} \int_{-1}^{1} |\widehat{\psi}_{\sigma(te_n)}(l, M)|^2 d\mu(te_n).$$

By Theorem 5.12, for an admissible wavelet the operator A_σ is bounded with bounded inverse, since there exist constants $0 < C_1 \leq C_2 < \infty$ such that $C_1 \leq C_\sigma^\psi(l) \leq C_2$.

The wavelet transform (5.26) is a mapping from $L^2(S^{n-1}, dS)$ to $L^2(\mathrm{Spin(n)} \times \sigma(te_n), d\mu(s) d\mu(te_n))$. Moreover, there exists a reconstruction formula and also a Plancherel Theorem.

Proposition 5.13 *Let $f \in L^2(S^{n-1})$. If ψ is an admissible wavelet then*

$$f(x) = \int_{-1}^{1} \int_{\mathrm{Spin}(n)} W_\psi[f](s, \sigma(te_n))[R_s (A_\sigma^\psi)^{-1} D_{\sigma(te_n)}\psi](x)\, d\mu(s)\, d\mu(te_n). \tag{5.27}$$

Proof: Let $\psi_{\sigma(te_n)} := D_{\sigma(te_n)}\psi$. Then we have

$$[R_s (A_\sigma^\psi)^{-1} \psi_{\sigma(te_n)}](x) = \sum_{l \in \mathbb{N}} \sum_M \sum_N \frac{1}{C_\sigma^\psi(l)} D_{MN}^l(s) \widehat{\psi}_{\sigma(te_n)}(l, M) Y_l^N(x). \tag{5.28}$$

The wavelet coefficients $W_\psi[f](s, \sigma(te_n))$ defined in (5.26) may be written as

$$W_\psi[f](s, \sigma(te_n)) = \sum_{l \in \mathbb{N}} \sum_M \sum_N \overline{D_{MN}^l(s)} \overline{\widehat{\psi}_{\sigma(te_n)}(l, M)} \widehat{f}(l, N). \tag{5.29}$$

Inserting expressions (5.28) and (5.29) in (5.27) and using the orthogonality relation for Wigner

$D-$functions, we obtain

$$\begin{aligned}
&\int_{-1}^{1}\int_{\mathrm{Spin}(n)} W_\psi[f](s,\sigma(te_n))[R_s(A_\sigma^\psi)^{-1}D_{\sigma(te_n)}\psi](x)\,d\mu(s)\,d\mu(te_n) = \\
&= \int_{-1}^{1}\int_{\mathrm{Spin}(n)} \sum_{l\in\mathbb{N}}\sum_{M}\sum_{N} \overline{D^l_{MN}(s)\widehat{\psi}_{\sigma(te_n)}(l,M)}\widehat{f}(l,N) \\
&\quad \sum_{l'\in\mathbb{N}}\sum_{M'}\sum_{N'} \frac{1}{C^\psi_\sigma(l')} D^{l'}_{M'N'}(s)\widehat{\psi}_{\sigma(te_n)}(l',M')Y^{N'}_{l'}(x)d\mu(s)d\mu(te_n) \\
&= \sum_{l,l'}\sum_{M,M'}\sum_{N,N'} \int_{-1}^{1} \overline{\widehat{\psi}_{\sigma(te_n)}}(l,M)\widehat{\psi}_{\sigma(te_n)}(l',M')d\mu(te_n)\ \widehat{f}(l,N)Y^{N'}_{l'}(x) \\
&\quad \frac{1}{C^\psi_\sigma(l')}\int_{\mathrm{Spin}(n)} \overline{D^l_{MN}(s)}D^{l'}_{M'N'}(s)d\mu(s) \\
&= \sum_{l\in\mathbb{N}}\sum_{N} \widehat{f}(l,N)Y^N_l(x)\frac{1}{C^\psi_\sigma(l)}\left[\frac{1}{N(n,l)}\sum_M\int_{-1}^{1}|\widehat{\psi}_{\sigma(te_n)}(l,M)|^2 d\mu(te_n)\right] \quad (by\,(5.5)) \\
&= \sum_{l\in\mathbb{N}}\sum_{N} \widehat{f}(l,N)Y^N_l(x) \\
&= f(x).
\end{aligned}$$

■

Corollary 5.14 *For an admissible wavelet ψ the following Plancherel relation is satisfied*

$$||f||^2 = \int_{-1}^{1}\int_{\mathrm{Spin}(n)} \overline{\widetilde{W}_\psi[f](s,\sigma(te_n))}W_\psi[f](s,\sigma(te_n))\,d\mu(s)\,d\mu(te_n) \tag{5.30}$$

with

$$\widetilde{W}_\psi[f](s,\sigma(te_n)) = \left\langle \widetilde{\psi}_{(s,\sigma(te_n))}, f\right\rangle = \left\langle R_s(A^\psi_\sigma)^{-1}D_{\sigma(te_n)}\psi, f\right\rangle. \tag{5.31}$$

As a consequence of Theorem 5.12, every admissible function for the fundamental section W_{σ_0} is also an admissible function for any global continuous section W_σ. We propose the following definition for the new wavelets arising from the action of the conformal group.

Definition 5.15 *For a given global continuous section W_σ and an admissible $\psi \in L^2(S^{n-1})$ the wavelets (or system of coherent states) obtained as*

$$\psi_{(s,\sigma(te_n))}(x) := U(s,\sigma(te_n))\psi(x) = R_s D_{\sigma(te_n)}\psi, \quad s\in \mathrm{Spin}(n),\ t\in]-1,1[$$

*are called **spherical conformlets**.*

6 Anisotropy and covariance of the generalized SCWT

The generalized spherical CWT depends on the chosen section, where the pure dilation operator is replaced by an anisotropic dilation operator varying continuously from scale to scale. One essential question is to understand what kind of dilations are obtained when the Möbius transformation $\varphi_{te_n}(x)$

is replaced by a Möbius transformation $\varphi_{\sigma(te_n)(x)}$. The study made in [10] of the influence of the parameter $a \in B^n$ on a given spherical cap centered at the North Pole allow us to understand the local character of anisotropic dilations. By the Unique Decomposition Theorem (Theorem 4.1) we obtain the following factorization of the dilation operator D_c, for each $c \in B^n$.

Proposition 6.1 *Let $c \in B^n$ such that $c = b \oplus a$, with $a \in D_{e_n}^{n-1}$ and $b \in L_{e_n}$. Then*

$$D_c f(x) = D_b D_a R_q f(x), \quad \textit{with} \quad q = \frac{1-ab}{|1-ab|}. \tag{6.1}$$

Proof: On the one hand we have,

$$\begin{aligned} D_c f(x) = D_{b\oplus a} f(x) &= \left(\frac{1-|b\oplus a|^2}{|1-(b\oplus a)x|^2} \right)^{\frac{n-1}{2}} f(\varphi_{-b\oplus a}(x)) \\ &= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ab-(a+b)x|^2} \right)^{\frac{n-1}{2}} f(\varphi_{-b\oplus a}(x)). \end{aligned}$$

As $-\varphi_a(x) = \varphi_{-a}(-x)$ then $-b \oplus a = -\varphi_{-b}(a) = \varphi_b(-a) = (-b) \oplus (-a)$.
Moreover, by (3.5) we have that $\varphi_{-a} \circ \varphi_{-b}(x) = q\varphi_{(-b)\oplus(-a)}(x)\overline{q}$, with $q = \frac{1-ab}{|1-ab|}$, and thus

$$\varphi_{-b\oplus a}(x) = \overline{q}(\varphi_{-a}(\varphi_{-b}(x)))q. \tag{6.2}$$

Therefore, we have

$$\begin{aligned} D_b D_a R_q f(x) &= \left(\frac{1-|b|^2}{|1-bx|^2} \frac{1-|a|^2}{|1-a\varphi_{-b}(x)|^2} \right)^{\frac{n-1}{2}} f(\overline{q}\varphi_{-a}(\varphi_{-b}(x))q) \\ &= \left(\frac{(1-|a|^2)(1-|b|^2)}{|1-ab-(a+b)x|^2} \right)^{\frac{n-1}{2}} f(\varphi_{-b\oplus a}(x)), \text{ by (6.2)}. \end{aligned}$$

■

By (6.1) we conclude that the operator $D_a R_q$ express the anisotropy character of the dilation.

Definition 6.2 *We define the anisotropy of a section $\sigma(te_n) = te_n \oplus g(t)e_{n-1}$ as*

$$\epsilon_g := \int_{-1}^{1} ||D_{g(t)e_{n-1}} R_{q(t)} - I|| dt. \tag{6.3}$$

Since the operators $D_{g(t)e_{n-1}}$ and $R_{q(t)}$ are unitary we have an upper bound for our anisotropy, $0 \leq \epsilon_g \leq 4$. The concept of anisotropy has an important meaning, however, it is very difficult to compute the quantity ϵ_g.

The question of covariance of the spherical CWT (5.26) under rotations and dilations is very important. In the flat case, the usual $n-$dimensional CWT is fully covariant with respect to translations, rotations, and dilations, and this property is essential for applications, in particular, the covariance under translations. In fact, covariance is a general feature of all coherent state systems directly derived from a square integrable representation [3]. However, when the representation is only square integrable over a quotient of the group then no general theorem is available. In the case of conformlets the results are the following:

- The spherical CWT (5.26) is covariant under rotations on S^{n-1} : for any $s_1 \in \mathrm{Spin}(n)$, the transform of the rotated signal $f(\overline{s_1} x s_1)$ is the function $W_\psi[f](\overline{s_1} s, \sigma(te_n))$.
- The spherical CWT (5.26) is not covariant under dilations. The wavelet transform of the dilated signal $D_{\sigma(t_1 e_n)} f(x)$, is of the form $W_\psi[f](s, (\overline{s}\sigma(t_1 e_n)s) \oplus (-\sigma(te_n)))$ as we can see by direct calculations.

In fact, if we consider the change of variables $\varphi_{-b}(x) = y$ ($\Leftrightarrow x = \varphi_b(y)$), with $dS_x = \left(\frac{1-|b|^2}{|1+by|^2}\right)^{n-1} dS_y$ we obtain

$$
\begin{aligned}
W_\psi[D_b f](s,a) &= \\
&= \int_{S^{n-1}} \left(\frac{1-|a|^2}{|1-a\overline{s}xs|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{-a}(\overline{s}xs)) \left(\frac{1-|b|^2}{|1-bx|^2}\right)^{\frac{n-1}{2}} f(\varphi_{-b}(x)) dS_x \\
&= \int_{S^{n-1}} \left(\frac{(1-|a|^2)(1-|b|^2)^3}{|1-sa\overline{s}\varphi_b(y)|^2|1-b\varphi_b(y)|^2|1+by|^4}\right)^{\frac{n-1}{2}} \psi(\varphi_{-a}(\overline{s}\varphi_b(y)s)) f(y) dS_y \\
&= \int_{S^{n-1}} \left(\frac{(1-|a|^2)(1-|b|^2)}{|1+by-sa\overline{s}(y-b)|^2}\right)^{\frac{n-1}{2}} \psi(\varphi_{(\overline{s}bs)\oplus(-a)}(\overline{s}ys)) f(y) dS_y \\
&= W_\psi[f](s, (\overline{s}bs) \oplus (-a)), \quad \forall s \in \mathrm{Spin}(n), \quad \forall a, b \in B^n.
\end{aligned}
$$

As in general $(\overline{s}\sigma(t_1 e_n)s) \oplus (-\sigma(te_n))$ is not an element of the section $\sigma(te_n)$, covariance under dilation does not hold. This is a consequence of the parameter space not being a group and is a general feature of coherent systems based on homogeneous spaces. For applications, of course, it is the covariance under rotations that is essential.

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