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# New Wide Classes of Weakly Mal'tsev Categories

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**Abstract** The following classes of categories are shown to be weakly Mal'tsev in the sense of the author: (i) a suitable class of algebras with cancellation; (ii) the dual of any quasi-adhesive category; (iii) the dual of any extensive category with pullback-stable epimorphisms; (iv) the dual of any solid quasi-topos. The examples in (i) include all the Mal'tsev varieties of algebras such as groups, rings, Lie algebras, etc., but also distributive lattices and commutative monoids with cancellation. The examples in (ii)-(iv) capture many of the familiar aspects of topological spaces.

**Keywords** Extensive category · Weakly Mal'tsev category · Quasi-adhesive category · Pullback-stable epimorphism · Stable coproduct · Local coproduct · Topological space · Van Kampen square · Solid quasi-topos

**Mathematics Subject Classifications (2010)** 18B30 · 18A20 · 18F99 (2010)

## 1 Introduction

In this paper we continue the study of weakly Mal'tsev categories. After having studied internal categorical structures, namely internal categories and internal groupoids [27], the connection with the classical definition of Mal'tsev category [6, 7] via strong relations [13], and considered the particular example of distributive lattices [28], we now turn our attention to the dual notion of a weakly Mal'tsev category. Before that we provide a general algebraic setting which unifies all the main examples which are known to share the weak Mal'tsev property.

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Recall that a category $\mathcal{C}$ is said to be weakly Mal'tsev [27] if it has pullbacks of split epimorphisms along split epimorphisms and if every two morphisms into a pullback of split epimorphisms, which are induced by the given sections of the respective split epimorphisms, form a jointly epimorphic cospan. More specifically, in any commutative square of split epimorphisms

$$\begin{array}{ccc} E & \underset{p_2}{\overset{e_2}{\leftrightarrows}} & C \\ {\scriptstyle e_1}\uparrow\downarrow{\scriptstyle p_1} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle s} \\ A & \underset{r}{\overset{f}{\rightleftarrows}} & B, \end{array}$$

that is, satisfying the identities

$$\begin{aligned} fp_1 = gp_2 \quad &, \quad e_1 r = e_2 s, \\ p_1 e_1 = 1_A \quad &, \quad p_2 e_2 = 1_C, \\ p_1 e_2 = rg \quad &, \quad p_2 e_1 = sf, \\ fr = 1_B &= gs, \end{aligned}$$

if $fp_1 = gp_2$ is a pullback square then the pair of morphisms $(e_1, e_2)$ is jointly epimorphic.

This property is indeed a weakening of the Mal'tsev property, which says that every internal reflexive relation is an internal equivalence relation. As proved in [2], the Mal'tsev property for regular categories is equivalent to the requirement that the pair $(e_1, e_2)$ above is always jointly strongly epimorphic (see also [1], p.151).

This means that any Mal'tsev category is "in particular" an example of a weakly Mal'tsev category, and moreover, every category admitting a pullback-preserving and faithful functor into a Mal'tsev category (or even a weakly Mal'tsev one) is an example of a weakly Mal'tsev category. That is the case for instance for continuous Mal'cev algebras ([17, 33])

The two main examples of algebraic structures which are weakly Mal'tsev but not Mal'tsev are the category of distributive lattices and the category of commutative monoids with cancellation. In Section 2 we present a general context, called *weakly Mal'tsev algebras with cancellation* which unifies all the known examples of algebraic structures sharing the weak Mal'tsev property.

In the remaining sections we turn our attention to the dual notion of the weak Mal'tsev property and show that the duals of the following classes of categories are weakly Mal'tsev:

1. any quasi-adhesive category [19]
2. any extensive category [5] with pullback-stable epimorphisms
3. any solid quasi-topos [15]

These categories capture many familiar aspects of topological spaces, in fact the idea of this work started with the simple observation, due to Zurab Janelidze, that the dual category of topological spaces is weakly Mal'tsev. Moreover, every *concrete* category whose (faithful) *forgetful* functor to the category of sets preserves finite limits and finite colimits is trivially co-weakly Mal'tsev, since so is the category of sets. This includes what categorical topologists call categories topological over sets [11].

The paper is organized as follows.

In the next section, after a few words on the weakly Mal'tsev categories in general, we describe a wide class of (quasi-)varieties of universal algebras that are weakly Mal'tsev categories, including all Mal'tsev (quasi-)varieties, and all quasi-varieties of commutative magmas with cancellation and of distributive lattices.

Section 3 contains the main result stating that the dual of a category with pullbacks and pushouts of split monomorphisms along split monomorphisms, in which a cospan is jointly epimorphic whenever it is obtained by pulling back a local coproduct diagram, is weakly Mal'tsev (Theorem 1). We use the term *local coproduct diagram* to denote a cospan which is the result of a pushout of a split monomorphism along a split monomorphism, see diagram (1).

The main result is then used in Section 4 to prove that the dual of any quasi-adhesive category [20] is weakly Mal'tsev. The proof is based on the notion of Van Kampen square [3] and uses the fact that any local coproduct diagram, indeed any pushout along a regular monomorphism in a quasi-adhesive category, is a part of a Van Kampen square (Proposition 2).

Finally, in the last section we show that the dual of any category with pullback-stable epimorphisms and stable coproducts (in the sense of [10] or [25], p.574) is weakly Mal'tsev. Examples include the dual of any extensive category [5] with pullback-stable epimorphisms [18] or the dual of any solid quasi-topos [15]. Keeping in mind that the original motivation was the case of topological spaces, we give specific references showing that many familiar categories of spaces fit into the above setting, such as the dual of any lax algebra, or $(T, V)$-category in the sense of [9].

## 2 Weakly Mal'tsev Algebras with Cancellation

The notion of weakly Mal'tsev category was introduced in [27] in order to provide a new, more general, setting where an internal reflexive graph can have at most one multiplicative graph structure and every multiplicative graph is automatically an internal category. On the other hand, contrary to the well-known case of Mal'tsev categories [6, 7], not every internal category in such a category is an internal groupoid. For instance the linearly ordered set of natural numbers is an internal category in the category of commutative monoids with cancellation (a weakly Mal'tsev category) and it is obviously not an internal groupoid. The category of distributive lattices is another important example of a weakly Mal'tsev category which is not Mal'tsev [28]. However, the category of modular lattices is not weakly Mal'tsev [28]. Thus, in this way it is possible to cover a wider range of examples while still keeping some of the useful properties desirable for internal categorical structures. As proved in [30], in the context of weakly Mal'tsev categories, groupoids and internal categories coincide if and only if every reflexive and transitive relation (i.e. a preorder) is an equivalence relation. Remarkably, when the category is regular the weak Mal'tsev property is not necessary and groupoids coincide with internal categories as soon as preorders coincide with equivalence relations [29].

Mal'tsev categories are characterized by the fact that every reflexive relation is an equivalence relation. From [13] we now know that the weak Mal'tsev property can be characterized by the fact that every strong relation is difunctional, or equivalently that every reflexive and strong relation is an equivalence relation.

Another characterization of a Mal'tsev category, due to Bourn [2], is that every pair of local product injections is jointly strongly epimorphic. By definition a weakly Mal'tsev category is one where local product injections are jointly epimorphic (further details can be found in [13] but will not be needed here). A general example of a class of categories with the weak Mal'tsev property, including "in particular" every Mal'tsev variety of universal algebras (such as groups, rings, Lie-algebras, etc.), the category of distributive lattices or the category of commutative magmas with cancellation, is presented next.

Let $I$ be a fixed set of indices and consider a quasi-variety, say $\mathcal{Q}$, of algebras having (among others) ternary terms $p_i$ $(i \in I)$, satisfying the following two conditions:

(i) $p_i(x, y, y) = p_i(y, y, x)$
(ii) $(\exists a, \forall i \in I, p_i(x, a, a) = p_i(y, a, a)) \Rightarrow x = y$

The morphisms are the expected ones.

**Proposition 1** *For any set of indices $I$, a quasi-variety such as $\mathcal{Q}$, from above, is a weakly Mal'tsev category.*

*Proof* The proof is a small variation of a similar result, involving only one ternary operation $p(x, y, z)$ and can be found in the introduction of [28]. We only observe that since we need to show that for every diagram of the form

$$A \underset{r}{\overset{f}{\rightleftarrows}} B \underset{s}{\overset{g}{\leftrightarrows}} C,$$

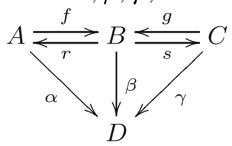
with $fr = 1_B = gs$, the morphisms

$$\langle 1, sf \rangle : A \to A \times_B C \quad , \quad \langle rg, 1 \rangle : C \to A \times_B C$$

are jointly epimorphic, we begin with arbitrary

$$\varphi, \varphi' : A \times_B C \to D$$

with $\varphi\langle 1, sf \rangle = \varphi'\langle 1, sf \rangle$ and $\varphi\langle rg, 1 \rangle = \varphi'\langle rg, 1 \rangle$, and have to show that $\varphi = \varphi'$. In order to simplify the calculations we introduce $\alpha, \beta, \gamma$, as illustrated,




with $\alpha r = \beta = \gamma s$, and for every $a \in A$ and $c \in C$,

$$\begin{aligned} \alpha(a) = \varphi(a, sf(a)) &= \varphi'(a, sf(a)) \\ \gamma(c) = \varphi(rg(c), c) &= \varphi'(rg(c), c). \end{aligned}$$

We will use condition (i) to show that $p_i(\varphi(a, c), \beta(b), \beta(b))$ does not depend on $\varphi$, which, together with condition (ii), implies $\varphi = \varphi'$, as desired.

Indeed, for $f(a) = b = g(c)$, we have:

$$\begin{aligned} p_i(\varphi(a, c), \beta(b), \beta(b)) &= p_i(\varphi(a, c), \alpha rf(a), \gamma sg(c)) \\ &= p_i(\varphi(a, c), \varphi(rf(a), sg(c)), \varphi(rf(a), sg(c))) \\ &= \varphi(p_i(a, rf(a), rf(a)), p_i(c, sg(c), sg(c))) \\ &= \varphi(p_i(a, rf(a), rf(a)), p_i(sg(c), sg(c), c)) \\ &= p_i(\varphi(a, sg(c)), \varphi(rf(a), sg(c)), \varphi(rf(a), c)) \\ &= p_i(\varphi(a, s(b)), \varphi(r(b), s(b)), \varphi(r(b), c)) \\ &= p_i(\alpha(a), \beta(b), \gamma(c)). \end{aligned}$$

□

Every Mal'tsev variety with a Mal'tsev term $m(x, y, y) = m(y, y, x) = x$ is an instance of the case above: choose $I$ to be a one element set and put $p = m$. The case of distributive lattices is another instance of the above: take $I = \{1, 2\}$ and define $p_1(x, y, z) = x \wedge z$ and $p_2(x, y, z) = x \vee z$. The case of commutative magmas with cancellation may be captured by choosing again $I$ as a one element set and defining $p(x, y, z) = x \cdot z$.

Furthermore, it is clear that in any category with finite limits, $\mathcal{C}$, we may consider the category of internal weakly Mal'tsev algebras with cancellation, say $\mathcal{Q}(\mathcal{C})$, which will again be a weakly Mal'tsev category (indeed the arguments used in the proof above easily extend to arguments involving generalized elements), and moreover any category with a pullback preserving and faithful functor to $\mathcal{Q}(\mathcal{C})$ is weakly Mal'cev.

## 3 The Main Result

In this section we show that the dual of a category with pullbacks and pushouts of split monomorphisms along split monomorphisms, in which pullbacks of local coproducts are jointly epimorphic, is weakly Mal'tsev.

The diagram that, from now on, we shall always have in mind is

$$\begin{array}{ccc} B & \underset{g}{\overset{s}{\leftrightarrows}} & C \\ {\scriptstyle r}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle \iota_C}\downarrow\uparrow{\scriptstyle p_2} \\ A & \underset{p_1}{\overset{\iota_A}{\rightleftarrows}} & Q \end{array} \tag{1}$$

where $rf = 1_B = sg$, $Q = A +_B C$, $\iota_A$ and $\iota_C$ are the pushout injections, $p_1 = [1, fs]$ and $p_2 = [gr, 1]$; that is, $p_1$ and $p_2$ are defined by

$$p_1\iota_A = 1_A, \quad p_1\iota_C = fs \quad \text{and} \quad p_2\iota_A = gr, \quad p_2\iota_C = 1_C.$$

It will be convenient for us to say that a cospan

$$A \xrightarrow{\iota_A} Q \xleftarrow{\iota_C} C$$

is a *local coproduct* whenever it is part of a pushout diagram as above.

**Theorem 1** *Let $\mathcal{C}$ be a category with pushouts of split monomorphisms along split monomorphisms, pullbacks, and such that for every commutative diagram*

$$\begin{array}{ccccc} E & \longrightarrow & F & \longleftarrow & G \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & Q & \longleftarrow & C \end{array} \tag{2}$$

*where both squares are pullback squares, if the bottom cospan is a local coproduct then the top cospan is jointly epimorphic. Then $\mathcal{C}^{op}$ is a weakly Mal'tsev category.*

*Proof* We need to prove that any two morphisms $u$ and $v$ from $D$ to $Q$ with $p_1u = p_1v$ and $p_2u = p_2v$, where $p_1$ and $p_2$ as in Eq. 1, are equal to each other.

First consider the commutative diagram

$$\begin{array}{ccccc} D_A & \xrightarrow{i} & D & \xleftarrow{j} & D_C \\ {\scriptstyle u_A}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u_C} \\ A & \xrightarrow{\iota_A} & Q & \xleftarrow{\iota_C} & C \end{array}$$

whose both squares are pullbacks. By our assumption, the morphism $u$ is determined by its composites with $i$ and $j$, which we shall express by writing $u = (ui, uj)$. Next, in order to involve the morphism $v$, let us construct two other diagrams, namely

$$\begin{array}{ccccc} D_1 & \xrightarrow{k_1} & D_A & \xleftarrow{k_2} & D_2 \\ {\scriptstyle v_1}\downarrow & & \downarrow{\scriptstyle vi} & & \downarrow{\scriptstyle v_2} \\ A & \xrightarrow{\iota_A} & Q & \xleftarrow{\iota_C} & C \end{array} \qquad \begin{array}{ccccc} D_3 & \xrightarrow{k_3} & D_C & \xleftarrow{k_4} & D_4 \\ {\scriptstyle v_3}\downarrow & & \downarrow{\scriptstyle vj} & & \downarrow{\scriptstyle v_4} \\ A & \xrightarrow{\iota_A} & Q & \xleftarrow{\iota_C} & C \end{array}$$

where again, all squares are pullbacks. We can now write

$$u = ((uik_1, uik_2), (ujk_3, ujk_4))$$

and $v = ((vik_1, vik_2), (vjk_3, vjk_4))$, and so in order to prove $u = v$ it suffices to prove the four equalities

$$uik_1 = vik_1, \quad uik_2 = vik_2, \quad ujk_3 = vjk_3, \quad ujk_4 = vjk_4.$$

We have:

$$\begin{aligned} p_1u &= ((p_1uik_1, p_1uik_2), (p_1ujk_3, p_1ujk_4)) \\ &= ((p_1\iota_Au_Ak_1, p_1\iota_Au_Ak_2), (p_1\iota_Cu_Ck_3, p_1\iota_Cu_Ck_4)) \\ &= ((u_Ak_1, u_Ak_2), (fsu_Ck_3, fsu_Ck_4)) \end{aligned}$$

$$\begin{aligned} p_1v &= ((p_1vik_1, p_1vik_2), (p_1vjk_3, p_1vjk_4)) \\ &= ((p_1\iota_Av_1, p_1\iota_Cv_2), (p_1\iota_Av_3, p_1\iota_Cv_4)), \\ &= ((v_1, fsv_2), (v_3, fsv_4)), \end{aligned}$$

and so

$$u_Ak_1 = v_1, \quad u_Ak_2 = fsv_2, \quad fsu_Ck_3 = v_3, \quad fsu_Ck4 = fsv_4.$$

Similarly

$$\begin{aligned} p_2u &= ((p_2uik_1, p_2uik_2), (p_2ujk_3, p_2ujk_4)) \\ &= ((p_2\iota_Au_Ak_1, p_2\iota_Au_Ak_2), (p_2\iota_Cu_Ck_3, p_2\iota_Cu_Ck_4)) \\ &= ((gru_Ak_1, gru_Ak_2), (u_Ck_3, u_Ck_4)) \end{aligned}$$

$$\begin{aligned} p_2v &= ((p_2vik_1, p_2vik_2), (p_2vjk_3, p_2vjk_4)) \\ &= ((p_2\iota_Av_1, p_2\iota_Cv_2), (p_2\iota_Av_3, p_2\iota_Cv_4)), \\ &= ((grv_1, v_2), (grv_3, v_4)), \end{aligned}$$

and so

$$gru_Ak_1 = grv_1, \quad gru_Ak_2 = v_2, \quad u_Ck_3 = grv_3, \quad u_Ck_4 = fsv_4.$$

Using these four equalities, four equalities above, and the commutativity of the diagrams we constructed, we obtain:

$$\begin{aligned} uik_1 &= \iota_A u_A k_1 = \iota_A v_1 = vik_1, \\ uik_2 &= \iota_A u_A k_2 = \iota_A fsv_2 = \iota_C gsv_2 = \iota_C g 1_B sv_2 \\ &= \iota_C grfsv_2 = \iota_C gru_2 = \iota_C v_2 = vik_2, \end{aligned}$$

and similarly

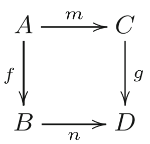
$$\begin{aligned} ujk_3 &= \iota_C u_C k_3 = \iota_C grv_3 = \iota_A frv_3 = \iota_A f 1_B rv_3 \\ &= \iota_A fsgrv_3 = \iota_A fsu_C k_3 = \iota_A v_3 = vjk_3, \\ ujk_4 &= \iota_C u_C k_4 = \iota_C v_4 = vjk_4, \end{aligned}$$

as desired. □

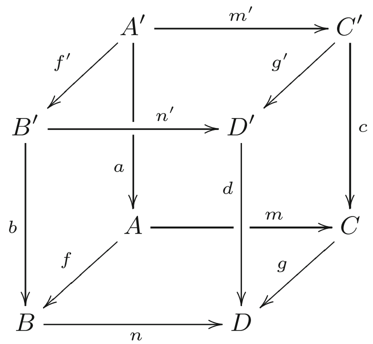
## 4 Adhesive and Quasi-Adhesive Categories

Adhesive categories were introduced in [19] and further generalised to quasi-adhesive categories [20], see also [16] and [21].

A category is quasi-adhesive when it has pullbacks, pushouts along regular monomorphisms and such pushouts are Van Kampen squares. Recall that a commutative square




is Van Kampen ([19], see also [3]) when for each commutative cube (of which it is the bottom face)




that has pullback squares as rear faces, *its top face is a pushout square if and only if its front faces are pullbacks*.

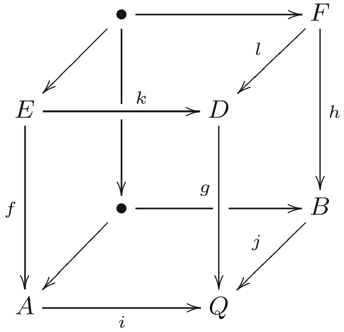
As is clear from the proof of the following result, if restricting the class of strong monomorphisms to the class of split monomorphisms in the definition of quasi-adhesive category, the result is still valid and hence the dual of such categories are weakly Mal'tsev.

**Proposition 2** *The dual of any quasi-adhesive category is weakly Mal'tsev.*

*Proof* Consider a commutative diagram of the form

$$\begin{array}{ccccc} E & \xrightarrow{k} & D & \xleftarrow{l} & F \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ A & \xrightarrow{i} & Q & \xleftarrow{j} & B \end{array} \tag{3}$$

in which both squares are pullback squares and the cospan $(i, j)$ is a local coproduct (see diagram (1)). We will see that the cospan $(k, l)$ is the pushout of its pullback and hence in particular jointly epimorphic. For, complete diagram (3) as the cube diagram



(4)

in which all faces are pullback squares. Since $(i, j)$ is a local coproduct diagram, the bottom square in Eq. 4 is a pushout of split (hence, regular) monomorphisms, and so the top square of Eq. 4 is a pushout by quasi-adhesivity. The result in Theorem 1 concludes the proof. □

## 5 Stable Coproducts, Quasi-toposes and Extensive Categories

In a category with pullbacks and binary coproducts we say that coproducts are stable [10] (see also [25], p. 574) if, given any commutative diagram

$$\begin{array}{ccccc} D & \xrightarrow{k} & F & \xleftarrow{l} & E \\ {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{i} & C & \xleftarrow{j} & B \end{array}$$

in which both squares are pullback squares, the square

$$\begin{array}{ccc} D+E & \xrightarrow{[k,l]} & F \\ {\scriptstyle f+g}\downarrow & & \downarrow{\scriptstyle h} \\ A+B & \xrightarrow{[i,j]} & C \end{array}$$

also is a pullback square.

**Proposition 3** *Let $\mathcal{C}$ be a category with pullbacks, binary coproducts and pushouts of split monomorphisms. If in addition $\mathcal{C}$ has*

(a) *pullback-stable epimorphisms, and*
(b) *stable coproducts,*

*then $\mathcal{C}^{\mathrm{op}}$ is weakly Mal'tsev.*

*Proof* Given a commutative diagram of the form

$$\begin{array}{ccccc} D & \xrightarrow{k} & F & \xleftarrow{l} & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{i} & C & \xleftarrow{j} & B \end{array}$$

in which both squares are pullbacks and the bottom row is a local coproduct (see diagram (1)), the induced morphism $[i, j]\colon A + B \to C$ is a regular epimorphism. Also, since epimorphisms are stable under pullback and the square

$$\begin{array}{ccc} D + E & \xrightarrow{[k,l]} & F \\ {\scriptstyle f+g}\downarrow & & \downarrow{\scriptstyle h} \\ A + B & \xrightarrow{[i,j]} & C \end{array}$$

is a pullback, the induced morphism $[k, l]\colon D + E \to F$ is an epimorphism. The result in Theorem 1 concludes the proof. □

Recall from [5] that a category is extensive if and only if it has disjoint and universal finite coproducts. Coproducts are universal when the pullback of a coproduct diagram is also a coproduct diagram. Coproducts are said to be disjoint when coproduct inclusions are monomorphisms and the pullback of any coproduct diagram is the initial object.

From Proposition 3, we obtain:

**Corollary 1** *Let $\mathcal{C}$ be an extensive category with pullbacks, pushouts of split monomorphisms along split monomorphisms, and pullback stable epimorphisms. Then $\mathcal{C}^{\mathrm{op}}$ is a weakly Mal'tsev category.*

*Proof* An extensive category with pullbacks always has stable coproducts, see for instance Proposition 1.2 in [22]. The result in Proposition 3 concludes the proof. □

In particular the category Top of topological spaces and continuous maps is extensive and epimorphisms (i.e. surjections) are pullback stable (see for instance [8] and [18]).

Many other familiar categories of spaces share these properties, for instance $(T, V)$-categories [9], which include approach spaces [24], preordered sets and metric spaces [23], probabilistic metric spaces [12, 31], or closure spaces [34]. Indeed, any $(T, V)$-category, or lax $(T, V)$-algebra, is extensive ([26], Corollary 8) and has pullback stable epimorphisms [9].

In [21] it is proved that any topos is adhesive. Combining the results from that paper, in particular the theorem by Brown and Janelidze on Van Kampen squares [3], we also see that the dual of any extensive and locally cartesian closed category is weakly Mal'tsev. Indeed, in a locally cartesian closed category, a morphism is effective for descent if and only if it is a regular epimorphism ([21], Lemma 12). In an extensive category any local coproduct gives rise to a Van Kampen square. This is just a particular case of Theorem 23 as stated in [21]; see also [3], where the morphisms are not arbitrary monomorphisms but are split monomorphisms.

In [4] it is proved that the category of Kelley spaces is a regular category, it is also extensive because *coproducts coincide with topological sums*, hence its dual is weakly Mal'tsev.

Finally, it is worth noting that every solid quasi-topos [14, 15] is extensive and has pullback stable epimorphisms, hence its dual is weakly Mal'tsev.

**Corollary 2** *The dual of a solid quasi-topos (one with disjoint coproducts) is a weakly Mal'tsev category.*

*Proof* A quasi-topos is a category with finite limits and finite colimits, which is locally cartesian closed and has a strong subobject classifier. Hence it has pullback-stable epimorphisms (see for instance [15]). Moreover, a quasi-topos is solid if and only if it has disjoint coprodutcs ([32]) and hence it also has stable coproducts, indeed it is extensive. Then the result of Proposition 3 completes the proof. □

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