A meshfree method with domain decomposition for Helmholtz boundary value problems

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***Abstract*—In the framework of meshfree methods, we address the numerical solution of boundary value problems (BVP) for the non-homogeneous modified Helmholtz partial differential equation (PDE). In particular, the unknown solution of the BVP is calculated in two steps. First, a particular solution of the PDE is approximated by superposition of plane wave functions with different wavenumbers and directions of propagation. Then, the corresponding homogeneous BVP is solved, for the homogeneous part of the solution, using the classical method of fundamental solutions (MFS). The combination of these two meshfree techniques shows excellent numerical results for non-homogeneous BVPs posed in simple geometries and when the source term of the PDE is sufficiently regular. However, for more complex domains or when the source term is piecewise defined, the MFS fails to converge. We overcome this problem by coupling the MFS with Lions non-overlapping domain decomposition method. The proposed technique is tested for the modified Helmholtz PDE with a discontinuous source term, posed in an L-shaped domain.**

***Index Terms*—meshfree method, plane wave functions, method of fundamental solutions, domain decomposition, modified Helmholtz equation, non-homogeneous PDE, L-shaped domain.**

I. Introduction

Meshfree methods are becoming an increasingly popular alternative to classical element-based numerical methods, e.g. FEM or BEM, for the approximate solution of boundary value problems (BVP) for partial differential equations (PDE). Avoiding the time consuming and computationally demanding mesh generation and numerical integration, especially for three-dimensional problems, are major advantages of the meshfree methods. Furthermore, most meshfree methods are simple to implement and, in smooth settings, outperform the classical schemes in terms of convergence and accuracy.

The method of fundamental solutions (MFS) is one of the most commonly used meshfree methods for the numerical solution of elliptic PDE. Although originally formulated in 1964 as a Trefftz type method for homogeneous linear elliptic PDEs posed in smooth settings [1], the MFS has undergone a significant development in the last five decades [2]–[4] and it has now been extended to non-homogeneous Helmholtz-type PDEs [5]–[7], non-homogeneous Cauchy-Navier equations of elastodynamics [8], BVPs posed in domains with geometric singularities [9]–[11], harmonic problems with discontinuous boundary conditions [12], second order nonlinear Dirichlet problems [13], [14], large scale problems [15], [16], etc.

In [6] the unknown solution of the Helmholtz BVP is approximated by superposition of spherical acoustic waves (fundamental solutions) with different source points and different test frequencies. More precisely, discrete collocation is used to impose simultaneously the PDE (in the domain) and the boundary conditions (on the boundary) of the BVP. The resulting numerical method is known in the literature as the domain version of the MFS, or MFS-D, and its applicability has been justified in terms of density results for fundamental solutions in the solution space of the PDE. One clear drawback of the MFS-D is that it requires the solution of a large and fully-populated ill-conditioned linear system. For more complex domains and in non-smooth settings, coupling MFS-D with an iterative domain decomposition technique, see [7], can increase the computational cost of the method even further.

Here we propose an improved version of the method presented in [7], with a reduced run time of the algorithm. In particular, first we calculate a particular solution for the non-homogeneous PDE using a basis of plane wave functions instead of fundamental solutions. Note that, in 2D, exponential functions are evaluated much faster than Bessel functions. Also, the long standing open problem on how and where to distribute the source points for the fundamental solutions is not present here. Second, the corresponding homogeneous BVP is solved using the classical MFS. The domain decomposition method will be used only in the second step of the scheme, leading to the solution of a sequence of small-dimension boundary-only collocation linear systems.

We consider the following boundary value problem for the modified Helmholtz partial differential equation

$$\begin{cases} \mathcal{L}u := (\Delta - \kappa^2)u = f & \text{in } \Omega \\ \mathcal{B}u = g & \text{on } \Gamma \end{cases}, \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) is a bounded domain with boundary $\Gamma = \partial\Omega$, Δ is the Laplace operator, $\kappa > 0$ and f is a given source function. The operator $\mathcal{B}$ defines the boundary conditions, which may be of Dirichlet, Neumann, Robin or mixed type, and g is the prescribed boundary data. These BVPs

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arise naturally, after applying the Laplace transform, in the solution of transient heat conduction problems, e.g. [17].

In order to solve problem (1), we represent its solution u in the form

$$u = u^P + u^H,$$

where u^P is a particular solution of the PDE

$$\mathcal{L}u^P = f, \text{ in } \Omega \tag{2}$$

and u^H satisfies the associated homogeneous BVP

$$\begin{cases} \mathcal{L}u^H = 0 & \text{in } \Omega \\ \mathcal{B}u^H = g - \mathcal{B}u^P & \text{on } \Gamma \end{cases}. \tag{3}$$

Since the solution of problem (3) depends directly on u^P, an accurate approximation for the particular solution is crucial for the global accuracy of the scheme. In section II, we approximate u^P by a linear combination of plane wave functions, whose high quality approximation properties are well known from the Fourier analysis. In section III, we present a brief review of the classical MFS formulation, which is used to solve problem (3). In section IV, we couple the presented two stage method with Lions non-overlapping domain decomposition method, e.g. [18]. Finally, in section V, we illustrate the accuracy of the proposed method and present some numerical results for a particularly challenging case of problem (1), posed in a L-shaped domain, with a piecewise defined source term, where the classical MFS fails to converge.

II. A Particular Solution for the PDE

For a set of N_0 distinct unitary directions

$$\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_{N_0}\} \subset S^{d-1}$$

and a set o P_0 positive test frequencies

$$\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_{P_0}\} \subset \mathbb{R}^+,$$

we approximate the particular solution u^P of the PDE by a linear combination of plane waves

$$u^P(x) \approx \tilde{u}^P(x) = \sum_{p=1}^{P_0}\sum_{n=1}^{N_0} \alpha_{p,n} W_{\lambda_p}(x, \mathbf{d}_n), \tag{4}$$

where $W_\lambda(x, \mathbf{d}) = \exp(i\lambda\, x \cdot \mathbf{d})$ and $\alpha_{p,n} \in \mathbb{C}$.

The unknown coefficients $\alpha_{p,n}$ are calculated by collocating the PDE on a set of M_0 domain collocation points

$$\mathcal{X}_0 = \{x_1, x_2, \dots, x_{M_0}\} \subset \Sigma,$$

where $\Sigma \supset \bar{\Omega}$. Note that u^P is not required to satisfy any boundary conditions and therefore the collocation of the PDE can be performed on a larger but simpler, usually rectangular, domain Σ. Also, numerical results show that the error is usually lower in the interior of the collocation domain.

Using the property $\Delta W_\lambda(x, \mathbf{d}) = -\lambda^2 W_\lambda(x, \mathbf{d})$, the collocation process leads to the solution of the linear system

$$\left[-(\lambda_1^2 + \kappa^2)\mathbb{W}_{\lambda_1} \quad \dots \quad -(\lambda_{P_0}^2 + \kappa^2)\mathbb{W}_{\lambda_{P_0}}\right][\mathbf{a}] = [\mathbf{f}], \tag{5}$$

where $\mathbf{a} = \{\alpha_{1,1}, \dots, \alpha_{P_0,N_0}\}$, $\mathbf{f} = f(\mathcal{X}_0)$ and $\mathbb{W}_\lambda$ is the matrix block defined by

$$\mathbb{W}_\lambda = \begin{bmatrix} W_\lambda(x_1, \mathbf{d}_1) & \dots & W_\lambda(x_1, \mathbf{d}_{N_0}) \\ \vdots & \ddots & \vdots \\ W_\lambda(x_{M_0}, \mathbf{d}_1) & \dots & W_\lambda(x_{M_0}, \mathbf{d}_{N_0}) \end{bmatrix}. \tag{6}$$

Note that no differentiation is requires during the collocation of the PDE, which represents a major advantage of the plane wave approximation in comparison with other meshfree methods based on collocation of general purpose radial basis functions (RBF), e.g. [19].

System (5) is fully populated and ill-conditioned, as it is characteristic for meshfree methods based on global collocation of smooth functions. Also, the system is usually over-determined, with $M_0 > P_0 \times N_0$, and we will use a least squares approach for its solution. To deal with the ill-conditioning, we will consider the truncated singular value decomposition (TSVD) regularization technique [20].

From a theoretical point of view, the approximation we use here may be justified in terms of a density result for plane acoustic waves in the $L^2(\Omega)$ space, see [21].

III. Solving the Homogeneous Problem

The fundamental solution for the modified Helmholtz equation is given by

$$\Phi(x) = \begin{cases} \dfrac{1}{2\pi} K_0(\kappa\|x\|), & \text{in 2D} \\ \dfrac{1}{4\pi}\dfrac{\exp(-\kappa\|x\|)}{\|x\|}, & \text{in 3D} \end{cases},$$

where $\|x\|$ is the Euclidian norm of $x \in \mathbb{R}^d$ and K_0 is the Bessel function of the third kind and order zero. Note that $\Phi(x)$ has radial symmetry and a singularity at $x = 0$.

The classical method of fundamental solutions consists in approximating the solution u^H of problem (3) by a linear combination of translates of the fundamental solution, $\Phi(x - y_j)$, also known as point sources, with singularities y_j laying in the exterior of the domain Ω.

More precisely, for a set of N_1 distinct (source) points

$$\mathcal{Y} = \{y_1, y_2, \dots, y_{N_1}\} \subset \hat{\Gamma},$$

selected on an admissible [22], [23] fictitious boundary $\hat{\Gamma}$, e.g. the boundary of a larger domain $\hat{\Omega} \supset \bar{\Omega}$, we seek for an approximate solution of the form

$$u^H(x) \approx \tilde{u}^H(x) = \sum_{j=1}^{N_1} \beta_j \Phi(x - y_j), \tag{7}$$

where $\beta_j \in \mathbb{C}$ are unknown coefficients.

Note that, according its definition, $\tilde{u}^H$ is an analytic function in $\bar{\Omega}$ which satisfies the homogeneous modified Helmholtz PDE. In order to calculate the coefficients β_j we impose the boundary condition of problem (3) on a set of M_1 distinct boundary collocation points

$$\mathcal{X}_1 = \{x_1, x_2, \dots, x_{M_1}\} \subset \Gamma,$$

which, in matrix form, corresponds to the linear system

$$\begin{bmatrix} \mathcal{B}\Phi(x_1 - y_1) & \dots & \mathcal{B}\Phi(x_1 - y_{N_1}) \\ \vdots & \ddots & \vdots \\ \mathcal{B}\Phi(x_{M_1} - y_1) & \dots & \mathcal{B}\Phi(x_{M_1} - y_{N_1}) \end{bmatrix} [\mathbf{b}] = [\mathbf{h}], \quad (8)$$

where $\mathbf{b} = \{\beta_1, \dots, \beta_{N_1}\}$ and $\mathbf{h} = g(\mathcal{X}_1) - \mathcal{B}\tilde{u}^P(\mathcal{X}_1)$.

We will usually consider more collocation points than singularities, $M_1 > N_1$, in order to deal with the ill-conditioning and the non-smoothness of the boundary Γ, and system (8) will be solved in the least squares sense. Again, Tikhonov or TSVD regularization may be required.

The convergence of the MFS may be justified in terms of density results for linear combinations of fundamental solutions in appropriate Sobolev spaces defined on Γ, see [7].

IV. Iterative Domain Decomposition Method

As we will illustrate in section V, the method described in sections II–III generates an accurate numerical solution when problem (1) is posed in a simple domain and when the source function f and the boundary data g are smooth functions. However, when f or g is a discontinuous function or when Ω has certain geometric singularities, the method shows low accuracy, no convergence and the approximate solution exhibits inappropriate oscillations near the singularities, as a consequence of the Gibbs phenomenon [24]. This behaviour is due to the analyticity (in $\bar{\Omega}$) of the basis functions used in (4) and (7), which are unable to fit accurately the singular solutions.

To alleviate the problem described in the previous paragraph, we will couple the presented numerical scheme with Lions non-overlapping domain decomposition method.

Consider a partition of the domain Ω into two non-overlapping sub-domains Ω_1 and Ω_2, with a common interface $\bar{\gamma} = \bar{\Omega}_1 \cup \bar{\Omega}_2$ and let $\Gamma_i = \partial\Omega \cup \partial\Omega_i$, $i = 1, 2$, see Fig. 1. Also, we denote by ν the unit normal vector pointing outwards with respect to Ω_1.

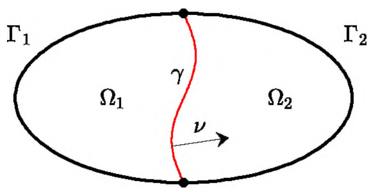

Fig. 1. Example of a domain decomposition.

In the general case, f and g may be discontinuous functions on γ and $\partial\gamma$, respectively

$$f = \begin{cases} f_1 & \text{in } \Omega_1 \\ f_2 & \text{in } \Omega_2 \end{cases} \quad \text{and} \quad g = \begin{cases} g_1 & \text{on } \Gamma_1 \\ g_2 & \text{on } \Gamma_2 \end{cases}$$

and the solution of problem (1) can be represented as

$$u = \begin{cases} u_1 = u_1^P + u_1^H & \text{in } \Omega_1 \\ u_2 = u_2^P + u_2^H & \text{in } \Omega_2 \end{cases},$$

where the total solution u and its normal derivative $\partial_\nu u$ should be continuous across γ.

Using the method described in section II, we begin by approximating a particular solution of the PDE in each subdomain

$$\mathcal{L}u_i^P = f_i, \text{ in } \Omega_i, \; i = 1, 2.$$

Note that these two problems can be solved independently and, in general, u_1^P and u_2^P do not match on γ. Denote by $[u^P] = u_2^P - u_1^P$ and $[\partial_\nu u^P] = \partial_\nu u_2^P - \partial_\nu u_1^P$ the jumps of the particular solution and its normal derivative across γ.

Next, we rewrite problem (3) in the form of two coupled BVPs with Robin-type transmission conditions on γ

$$\begin{cases} \mathcal{L}u_1^H = 0 & \text{in } \Omega_1 \\ \mathcal{B}u_1^H = g_1 - \mathcal{B}u_1^P & \text{on } \Gamma_1 \\ (\partial_\nu + \mu)u_1^H = (\partial_\nu + \mu)u_2^H + [\partial_\nu u^P] + \mu[u^P] & \text{on } \gamma \end{cases} \quad (9)$$

and

$$\begin{cases} \mathcal{L}u_2^H = 0 & \text{in } \Omega_2 \\ \mathcal{B}u_2^H = g_2 - \mathcal{B}u_2^P & \text{on } \Gamma_2 \\ (\partial_\nu - \mu)u_2^H = (\partial_\nu - \mu)u_1^H - [\partial_\nu u^P] + \mu[u^P] & \text{on } \gamma \end{cases} \quad (10)$$

where $\mu > 0$ is a parameter. In this formulation, the imposed transmission conditions guarantee continuity for the total solution u and its normal derivative $\partial_\nu u$ across γ.

Lions method consists in the iterative matching of u_1 and u_2 on γ, by the successive solution of problem (9) and problem (10), starting with an initial guess for u_2^H and $\partial_\nu u_2^H$ on γ. In each iteration, we use the classical MFS for the solution of the two problems, as described in section III. Here, μ acts as a convergence parameter for the iterative scheme.

V. Numerical Simulations

The accuracy of the method presented in sections II–IV will be illustrated for two examples of problem (1). We will consider only Dirichlet BVPs posed in 2D simply connected domains but the application of the proposed method to 3D problems and other boundary conditions is analogous. In the case of BVPs posed in multiply-connected domains, the only difference is that, for the MFS, a set of source points should be selected in every connected component of the exterior of the domain, see [23].

In order to evaluate the accuracy of the approximate solution

$$\tilde{u} = \begin{cases} \tilde{u}_1 = \tilde{u}_1^P + \tilde{u}_1^H, & \text{in } \Omega_1 \\ \tilde{u}_2 = \tilde{u}_2^P + \tilde{u}_2^H, & \text{in } \Omega_2 \end{cases},$$

and since we assume that problem (1) is well posed, it is sufficient to measure the PDE collocation errors

$$\varepsilon_{\Omega_i} := \|f_i - \mathcal{L}\tilde{u}_i^P\|_{L^2(\Omega_i)}, \; i = 1, 2,$$

the collocation errors of the Dirichlet boundary condition

$$\varepsilon_{\Gamma_i} := \|g_i - \tilde{u}_i\|_{L^2(\Gamma_i)}, \; i = 1, 2,$$

and also to check how well $\tilde{u}_1$ and $\tilde{u}_2$ match on the interface γ, by evaluating the continuity error

$$\varepsilon_\gamma^0 := \|\tilde{u}_2 - \tilde{u}_1\|_{L^2(\gamma)}$$

and the differentiability error (continuity error of $\partial_\nu \tilde{u}$)

$$\varepsilon_\gamma^1 := \|\partial_\nu \tilde{u}_2 - \partial_\nu \tilde{u}_1\|_{L^2(\gamma)}.$$

We measure these errors using the discrete l^2 norm, equivalent to the root-mean-square (RMS) error, calculated on a large set of domain or boundary error test points.

A. Example 1 – square domain

For $\Omega = [0,1]^2$ we consider problem (1) with $\kappa = 3$, null Dirichlet boundary data, i.e. $g = 0$ on Γ, and source function

$$f(x) = -20\sin(7x_1 - 7x_2)\exp(x_1 + x_2),\ x \in \Omega,$$

for which the exact solution of the BVP is not known. In our opinion, the BVPs (considered by many authors) where functions g and f are calculated from a given exact solution u are not appropriate for testing the numerical method. In such settings, the method proposed here shows numerical results with accuracy close to the machine precision.

In order to approximate the particular solution u^P of the BVP we considered a set of $N_0 = 100$ unitary directions, uniformly distributed on the unit sphere S^1 and $P_0 = 21$ integer test frequencies from the set $\Lambda = \{5, 6, \ldots, 25\}$. We collocated the PDE using $M_0 = 2601$ uniformly distributed points on the larger domain $\Sigma = [-0.1, 1.1]^2$.

The resulting linear system, with dimension 2601×2100, was solved, using the mldivide Matlab function, with a residual RMS error of order 10^{-12}. On a uniform grid of 40401 error test points in $\bar{\Omega}$, we measured $\varepsilon_\Omega = 6.19 \times 10^{-12}$ for the PDE collocation error. No absolute error accumulation was observed near the corners or near the boundary of Ω.

To approximate the homogeneous part of the solution, u^H, we considered $N_1 = 400$ source points, on the boundary of the square $[-0.1, 1.1]^2$ and $M_1 = 800$ collocation points on Γ. Both sets contained uniformly distributed points. The corresponding MFS linear system was solved with a residual RMS error of order 10^{-7} and we measured $\varepsilon_\Gamma = 6.40 \times 10^{-7}$, using 6000 boundary error test points. In terms of the absolute error, marginal accumulation was observed in the neighborhood of the corners of the domain, which did not affect the global performance of the method.

In Fig. 2 we present the plot of the approximate total solution $\tilde{u} = \tilde{u}^P + \tilde{u}^H$, in Ω.

We also tested the method for the boundary condition

$$g(x) = (x_1 - 0.25)^2 + (x_2 - 0.25)^2,\ x \in \Gamma.$$

The remaining parameters for the BVP and the method were kept unchanged. We measured $\varepsilon_\Gamma = 6.46 \times 10^{-7}$ for this simulation and the results can be improved by increasing the number of collocation and source points or by varying the position of the fictitious boundary $\hat{\Gamma}$, as it is characteristic for the MFS.

The situation changes dramatically when g or f are non-smooth functions. For example, for

$$f(x) = H(x_2 - 0.5),\ x \in \Omega,$$

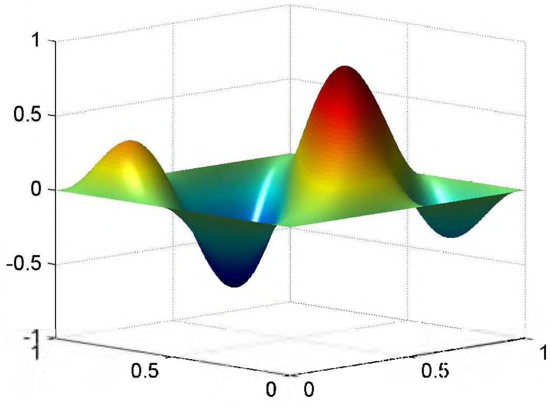

Fig. 2. Approximate solution for example 1 with $g(x) = 0$.

where $H(x)$ is the Heaviside step function, the lowest value of the PDE collocation error we observed was $\varepsilon_\Omega = 8.66 \times 10^{-2}$. The discontinuous source function could not be approximated accurately, using analytic plane wave functions, in the neighborhood of the line $x_2 = 0.5$. Naturally, a low precision particular solution invalidates the application of the remaining part of the numerical scheme and domain decomposition is one of the possible approaches to deal with this problem.

B. Example 2 – L-shaped domain

Consider the L-shaped domain $\Omega = [-1,1]^2 \backslash]-1,0[\times]0,1[$ and its partition into three sub-domains, as shown in Fig. 3. In the following, we use the notation from section IV.

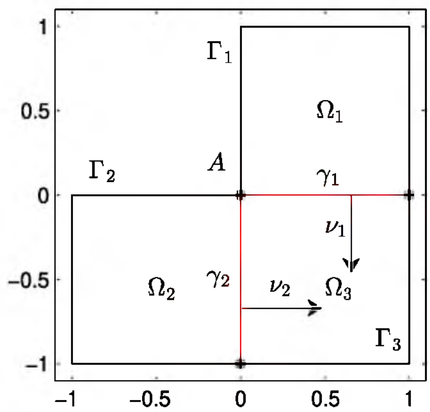

Fig. 3. Domain decomposition for example 2.

Note that, even for smooth data g and f, the MFS (and also many other numerical methods) presents serious difficulties in approximating the solution of the BVP in the neighborhood of the corner point $A = (0,0)$. Usually, the Gibbs phenomenon that occurs near that point affects the overall precision of the solution in Ω. This problem is due to the singularity of the solution u, induced by the singular corner A, and one way to deal with it is to enrich the MFS basis with a set of appropriate singular corner-adapted particular solutions of the PDE, as it has been done in [11] for the Helmholtz equation.

Another possible way to alleviate the approximation problem in the neighborhood of point A is to consider an appropriate domain partition. Also, if f is a piecewise defined and discontinuous function, domain decomposition is the most commonly used approach.

Consider problem (1) with $\kappa = 5$ and $g = 0$ on Γ. For the source function we took

$$f(x) = \begin{cases} \dfrac{-10}{\sqrt{\|x - C_i\|^2 + 0.5}} & \text{in } \Omega_i,\ i = 1,2 \\ 100\cos(3\pi x_1)\cos(3\pi x_2) & \text{in } \Omega_3 \end{cases},$$

which is symmetric with respect to the line $x_2 = -x_1$ and discontinuous on $\gamma_1 =]0,1[\times\{0\}$ and $\gamma_2 = \{0\}\times]-1,0[$. Here $C_1 = (0.5, 0.5)$ and $C_2 = -C_1$.

To approximate the particular solutions u_i^P, $i = 1,2,3$, we considered $N_0 = 100$ unitary directions and $P_0 = 16$ test frequencies from the set $\Lambda = \{5, 6, \ldots, 20\}$. For u_1^P, we collocated the PDE using a set $\mathcal{X}_0$ of $M_0 = 2601$ uniformly distributed points on the larger square $\Sigma = [-0.1, 1.1]^2$. The collocation points for the calculation of u_2^P and u_3^P were obtained by reflection of the set $\mathcal{X}_0$ with respect to the coordinate axis. We measured $\varepsilon_{\Omega_1} = \varepsilon_{\Omega_2} = 2.53 \times 10^{-10}$ and $\varepsilon_{\Omega_3} = 1.86 \times 10^{-12}$, using 40401 error test points in Ω_i.

For Lions method, in each iteration, we apply the MFS to approximate u_i^H, $i = 1,2$, from the BVP

$$\begin{cases} \mathcal{L}u_i^H = 0 & \text{in } \Omega_i \\ \mathcal{B}u_i^H = g - \mathcal{B}u_i^P & \text{on } \Gamma_i \\ (\partial_{\nu_i} + \mu)u_i^H = (\partial_{\nu_i} + \mu)u_3^H + [\partial_{\nu_i} u^P] + \mu[u^P] & \text{on } \gamma_i \end{cases} \quad (11)$$

and then calculate an approximation for u_3^H by solving

$$\begin{cases} \mathcal{L}u_3^H = 0 & \text{in } \Omega_3 \\ \mathcal{B}u_3^H = g - \mathcal{B}u_3^P & \text{on } \Gamma_3 \\ (\partial_{\nu_1} - \mu)u_3^H = (\partial_{\nu_1} - \mu)u_1^H - [\partial_{\nu_1} u^P] + \mu[u^P] & \text{on } \gamma_1 \\ (\partial_{\nu_2} - \mu)u_3^H = (\partial_{\nu_2} - \mu)u_2^H - [\partial_{\nu_2} u^P] + \mu[u^P] & \text{on } \gamma_2 \end{cases}. \quad (12)$$

This formulation of the iterative method requires an initial guess for u_3^H and $\partial_{\nu_i} u_3^H$ on γ_i, in problem (11). For u_3^H we consider the interpolating polynomial defined by the values of the Dirichlet boundary condition on the end points of γ_i. For the normal derivative we take $\partial_{\nu_i} u_3^H = 0$.

In Ω_1, the MFS was applied with $N_1 = 600$ source points, uniformly distributed on the boundary of the square $[-0.01, 1.01]^2$ and $M_1 = 800$ boundary collocation points, from which 199 on γ_1. For Ω_2 and Ω_3, the corresponding knot distributions were obtained by reflection with respect to the coordinate axis.

For $\mu = 100$, after calculating 70 iterations by Lions method, the solutions of the sub-problems matched accurately on γ_1 and γ_2 and we measures $\varepsilon^0_{\gamma_1} = \varepsilon^0_{\gamma_2} = 3.96 \times 10^{-6}$, $\varepsilon^1_{\gamma_1} = 3.60 \times 10^{-3}$ and $\varepsilon^1_{\gamma_2} = 3.85 \times 10^{-3}$. Since the gradient of the solution is singular at point A, we excluded a small neighborhood of this point when measuring $\varepsilon^1_{\gamma_i}$, $i = 1,2$. The Dirichlet boundary condition was also well fitted by the method, with $\varepsilon_{\Gamma_1} = 4.41 \times 10^{-6}$, $\varepsilon_{\Gamma_2} = 4.43 \times 10^{-6}$ and $\varepsilon_{\Gamma_3} = 3.23 \times 10^{-6}$. All errors were calculated using 2000 test points per linear boundary segment.

In Fig. 4 we present the contour plot of the total solution of problem (1) which exhibits the expected symmetry with respect to the line $x_2 = -x_1$, due to the same symmetry of the domain Ω and the functions g and f.

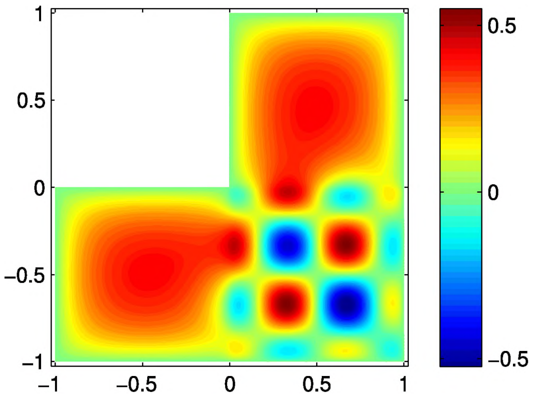

Fig. 4. Contour plot of the approximate solution for example 2.

The convergence of the continuity and differentiability errors, $\varepsilon^0_{\gamma_i}$ and $\varepsilon^1_{\gamma_i}$, $i = 1,2$, during the iterative process is illustrated in Fig. 5. The two plots are practically identical due to the symmetry of the problem. It can be observed that the method remains stable after reaching its optimal precision, which, in this case, occurs in approximately 40 iterations.

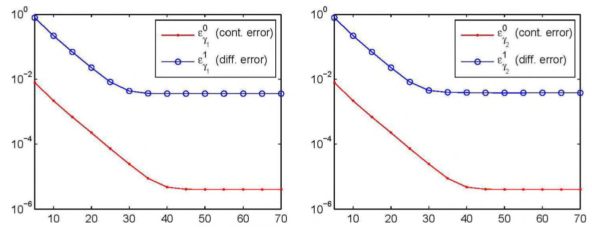

Fig. 5. Convergence of $\varepsilon^0_{\gamma_i}$ and $\varepsilon^1_{\gamma_i}$ for $i = 1$ (left) and $i = 2$ (right).

Numerical tests show that the value of the parameter $\mu > 1$ is inversely related to the speed of convergence of the method. For example, practically the same accuracy as before can be achieved after 30 iterations, for $\mu = 50$ and after 65 iterations, for $\mu = 150$. However, decreasing the value of μ corresponds to increasing the importance given to the differentiability in the transmission conditions, and this may lead to instability of the method, due to the singularity at point A. A trade-off between speed of convergence and stability should be carefully considered, when selecting the value of μ.

If the particular solutions u_i^P are accurately approximated and if an appropriate value of μ is selected, the maximum achievable precision of the method depends solely on the knot configuration considered for the MFS. For example, taking three times more collocation and source points and keeping all the remaining parameters of the method unchanged, the optimal precision of the method was achieved after 70 iterations and we measures $\varepsilon^0_{\gamma_1} = 1.03 \times 10^{-6}$, $\varepsilon^0_{\gamma_2} = 1.11 \times 10^{-6}$,

$\varepsilon^1_{\gamma_1} = 7.92 \times 10^{-7}$ and $\varepsilon^1_{\gamma_2} = 1.12 \times 10^{-5}$. Here, the Dirichlet boundary condition was approximated with $\varepsilon_{\Gamma_1} = 1.31 \times 10^{-6}$, $\varepsilon_{\Gamma_2} = 1.30 \times 10^{-6}$ and $\varepsilon_{\Gamma_3} = 8.81 \times 10^{-8}$. These results corresponds to a decrease of the differentiability error with more than 2 orders of magnitude. A plot of the approximate solution $\tilde{u}$ is shown in Fig. 6.

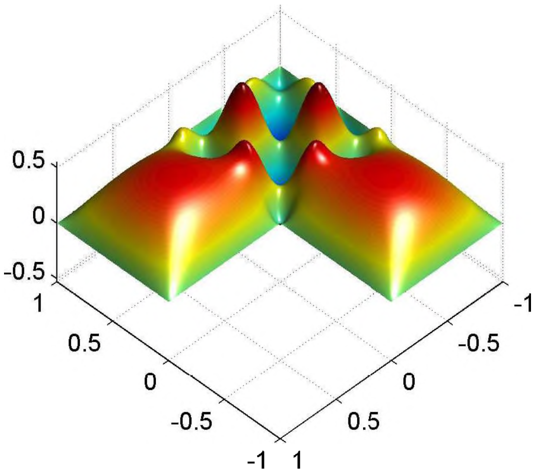

Fig. 6. Approximate solution for example 2.

VI. Conclusions

We have presented a meshfree method with domain decomposition for the numerical solution of BVPs for the non-homogeneous modified Helmholtz PDE. After approximating the particular solution of the PDE in each subdomain by a linear combination of plane wave functions, Lions non-overlapping domain decomposition iterative method was used to match the solution on the sub-domain interfaces. In each iteration, the homogeneous BVPs were solved using the classical formulation of the MFS. The proposed method was tested for a Dirichlet BVP with a discontinuous source functions, posed in an L-shaped domain, and it showed convergence and generated accurate numerical results.

Due to the discontinuity of some partial derivatives of the solution on the boundary of the domain, induced by the geometric singularities of the domain and also by the change in the type of boundary conditions in the sub-domains, also known as the Motz problem [25], further improvement of the numerical results was not possible. Following [11], an enrichment of the MFS approximation basis is one possible way to address this issue.

References

[1] V. D. Kupradze and M. A. Aleksidze, "The method of fundamental equations for an approximate solution of certain boundary value problems," *Comput Math Math Phys*, vol. 4, pp. 82–126, 1964.

[2] G. Fairweather and A. Karageorghis, "The method of fundamental solutions for elliptic boundary value problems," *Adv Comput Math*, vol. 9, pp. 69–95, 1998.

[3] G. Fairweather, A. Karageorghis, and P. A. Martin, "The method of fundamental solutions for scattering and radiation problems," *Engineering Analysis with Boundary Elements*, vol. 27, no. 7, pp. 759–769, jul 2003.

[4] A. Karageorghis, D. Lesnic, and L. Marin, "A survey of applications of the MFS to inverse problems," *Inverse Problems in Science and Engineering*, vol. 19, no. 3, pp. 309–336, apr 2011.

[5] C. J. S. Alves and C. S. Chen, "A new method of fundamental solutions applied to nonhomogeneous elliptic problems," *Advances in Computational Mathematics*, vol. 23, no. 1-2, pp. 125–142, jul 2005.

[6] C. J. S. Alves and S. S. Valtchev, "A Kansa type method using fundamental solutions applied to elliptic PDEs," in *Advances in meshfree techniques, Computational methods in applied sciences, vol. 5*. Springer, 2006.

[7] C. J. S. Alves, N. F. M. Martins, and S. S. Valtchev, "Domain decomposition methods with fundamental solutions for Helmholtz problems with discontinuous source terms," *Computers & Mathematics with Applications*, dec 2018.

[8] ——, "Extending the method of fundamental solutions to non-homogeneous elastic wave problems," *Applied Numerical Mathematics*, vol. 115, pp. 299–313, may 2017.

[9] C. J. S. Alves and V. M. Leitão, "Crack analysis using an enriched MFS domain decomposition technique," *Engineering Analysis with Boundary Elements*, vol. 30, no. 3, pp. 160–166, mar 2006.

[10] L. Marin, "Treatment of singularities in the method of fundamental solutions for two-dimensional Helmholtz-type equations," *Applied Mathematical Modelling*, vol. 34, no. 6, pp. 1615–1633, jun 2010.

[11] P. R. S. Antunes and S. S. Valtchev, "A meshfree numerical method for acoustic wave propagation problems in planar domains with corners and cracks," *Journal of Computational and Applied Mathematics*, vol. 234, no. 9, pp. 2646–2662, sep 2010.

[12] C. J. S. Alves and S. S. Valtchev, "On the application of the method of fundamental solutions to boundary value problems with jump discontinuities," *Applied Mathematics and Computation*, vol. 320, pp. 61–74, mar 2018.

[13] J. K. Grabski and A. Karageorghis, "Moving pseudo-boundary method of fundamental solutions for nonlinear potential problems," *Engineering Analysis with Boundary Elements*, vol. 105, pp. 78–86, aug 2019.

[14] C. J. S. Alves and A. L. Silvestre, "On the application of the method of fundamental solutions to nonlinear partial differential equations," *Engineering Analysis with Boundary Elements*, vol. 92, pp. 267–274, jul 2018.

[15] Y. J. Liu, N. Nishimura, and Z. H. Yao, "A fast multipole accelerated method of fundamental solutions for potential problems," *Engineering Analysis with Boundary Elements*, vol. 29, no. 11, pp. 1016–1024, nov 2005.

[16] C. M. Fan, Y. K. Huang, C. S. Chen, and S. R. Kuo, "Localized method of fundamental solutions for solving two-dimensional Laplace and biharmonic equations," *Engineering Analysis with Boundary Elements*, vol. 101, pp. 188–197, apr 2019.

[17] H. A. Cho, M. A. Golberg, A. S. Muleshkov, and X. Li, "Trefftz methods for time dependent partial differential equations," *Comput. Mat. Cont*, pp. 1–37, 2004.

[18] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, ser. Numerical mathematics and scientific computing. Clarendon Press, Oxford, 1999.

[19] E. Kansa, "Multiquadrics—a scattered data approximation scheme with applications to computational fluid-dynamics—II solutions to parabolic, hyperbolic and elliptic partial differential equations," *Computers & Mathematics with Applications*, vol. 19, no. 8-9, pp. 147–161, 1990.

[20] P. C. Hansen, *Rank-Deficient and Discrete Ill-Posed Problems*. SIAM, 1998.

[21] C. J. S. Alves and S. S. Valtchev, "Numerical comparison of two meshfree methods for acoustic wave scattering," *Engineering Analysis with Boundary Elements*, vol. 29, no. 4, pp. 371–382, apr 2005.

[22] F. Dou, L.-P. Zhang, Z.-C. Li, and C. S. Chen, "Source nodes on elliptic pseudo-boundaries in the method of fundamental solutions for Laplace's equation; selection of pseudo-boundaries," *Journal of Computational and Applied Mathematics*, vol. 377, p. 112861, oct 2020.

[23] C. J. Alves, "On the choice of source points in the method of fundamental solutions," *Engineering Analysis with Boundary Elements*, vol. 33, no. 12, pp. 1348–1361, dec 2009.

[24] J.-H. Jung, "A note on the Gibbs phenomenon with multiquadric radial basis functions," *Applied Numerical Mathematics*, vol. 57, no. 2, pp. 213–229, feb 2007.

[25] H. Motz, "The treatment of singularities of partial differential equations by relaxation methods," *Quarterly of Applied Mathematics*, vol. 4, no. 4, pp. 371–377, jan 1947.