Application of the fractional Sturm-Liouville theory to a fractional Sturm-Liouville telegraph equation*

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Abstract

In this paper, we consider a non-homogeneous time-space-fractional telegraph equation in n-dimensions, which is obtained from the standard telegraph equation by replacing the first- and second-order time derivatives by Caputo fractional derivatives of corresponding fractional orders, and the Laplacian operator by a fractional Sturm-Liouville operator defined in terms of right and left fractional Riemann-Liouville derivatives. Using the method of separation of variables, we derive series representations of the solution in terms of Wright functions, for the homogeneous and non-homogeneous cases. The convergence of the series solutions is studied by using well known properties of the Wright function. We show also that our series can be written using the bivariate Mittag-Leffler function. In the end of the paper some illustrative examples are presented.

Keywords: Caputo fractional derivatives; Riemann-Liouville fractional derivatives; Fractional Sturm-Liouville operator; Time-space-fractional telegraph equation; Mittag-Leffler functions; Wright functions.

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1 Introduction

The telegraph equation is a second order linear hyperbolic equation given by

$$\frac{\partial^2 u}{\partial t^2} + a\frac{\partial u}{\partial t} + bu = c\frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in \mathbb{R}, \quad t > 0, \quad a,b,c > 0, \tag{1}$$

together with initial conditions, and/or boundary conditions when restricted to a closed interval. Equations of the form (1) arise in the study of propagation of electrical signals in a cable of transmission line and wave phenomena, but serve also as mathematical model for several other phenomena such as random walks, solar particle transport, traffic jams, population dynamics, and oceanic diffusion (see [1] for a list of references about these applications). In fact, the telegraph equation is more suitable than the diffusion equation in modeling reaction diffusion since it has the potential to describe both diffusive and wave-like phenomena, due to the simultaneous presence of first and second order time derivatives.

Analytical and numerical methods for solving the telegraph equation were studied in the last decades and are an active area of research (see [1, 16, 17, 19] and the list of references therein). In our work we are interested

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in the fractional version of the telegraph equation. During the past decade, several generalizations of (1) appeared where time- and/or space-fractional derivatives were considered instead of the integer derivatives. The time-space-fractional telegraph equation is given by

$$D_t^\alpha u(x,t) + aD_t^\beta u(x,t) + b\, u(x,t) = c\, D_x^\gamma u(x,t) + f(x,t), \tag{2}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}^+$, D_t^α and D_t^β are time-fractional derivatives of order $\alpha \in]1,2[$ and $\beta \in]0,1[$, and D_x^γ is a space-fractional derivative of order $\gamma \in]1,2[$. Equation (2) and some of its variants in the one and the multi-dimensional cases were studied in several works (see e.g. [1, 4–7, 15, 16, 21]). In [16], it was obtained the fundamental solution for a time-fractional telegraph equation in the case $\alpha = 2\beta$, while in [1], the authors found the fundamental solution for the neutral-fractional telegraph equation and discussed its properties. In [14], using the Green function method solutions to boundary value problems for the time-fractional telegraph equation were derived. In [15], the author used the Adomain decomposition method to obtain analytic and approximate solutions of (2). In [6], the authors considered a non-homogeneous version of (2) in $\mathbb{R}^n \times \mathbb{R}^+$. They discussed and derived the analytical solutions under non-homogeneous Dirichlet and Neuman boundary conditions in terms of multivariate Mittag-Leffler functions and using the method of separation of variables. Also in the multidimensional setting, the fundamental solution for a time-fractional telegraph equation was obtained in [4,5]. In [7,21] the authors considered a time-space-fractional telegraph equation in $\mathbb{R} \times \mathbb{R}^+$ with Hilfer time-fractional derivative and Riesz-Feller space-fractional derivative. In [21], it was proved that the solutions for the Cauchy problem for equation (2) can be represented as a linear combination of two-parameter Mittag-Leffler functions, which allowed to a probabilistic interpretation of the solution. In [7], the authors used the Fourier and Laplace integral transform to obtain the Fourier transform of the solutions of the non-homogeneous time-space-fractional telegraph equation in $\mathbb{R} \times \mathbb{R}^+$.

In this work, we consider the following time-space-fractional telegraph equation in $\mathbb{R}^n \times \mathbb{R}^+$:

$$ {}^C_{0^+}\partial_t^\beta f(x,t) - \theta\ {}^C_{0^+}\partial_t^\gamma f(x,t) = -\frac{1}{r(x)}\left[-\left({}^{RL}\nabla^\alpha_{b^-}\cdot\left(\mu(x)\,{}^{RL}\nabla^\alpha_{a^+} f(x,t)\right)\right) + \nu(x)\, f(x,t)\right] + h(x,t) \tag{3}$$

where $x \in \Omega \subset \mathbb{R}^n$, $\theta, t \in \mathbb{R}^+$, $1 < \beta < 2$, $0 < \gamma < 1$, the time-fractional derivatives are in the Caputo sense, and the space-fractional derivative is a fractional Sturm-Liouville operator defined in terms of right and left fractional Riemann-Liouville derivatives. One of the reasons for the consideration of the space-fractional derivative in terms of a fractional Sturm-Liouville operator in $\mathbb{R}^n$ is the fact that the orthogonal eigenfunctions' system of the fractional Sturm-Liouville problem can be used to solve fractional partial differential equations that are related with anomalous diffusion processes (see [11]). In [3], the authors studied the fractional Sturm-Liouville problem in $\mathbb{R}^n$ subject to mixed Dirichlet and Neuman boundary conditions and proved several properties of the eigenvalues and eigenfunctions associated to the fractional Sturm-Liouville problem. In particular, using fractional variational calculus, it was shown in [3], the existence of a countable set of orthogonal solutions and corresponding eigenvalues.

The aim of this paper is to present a series representation for the solution of (3) in the homogeneous and non-homogeneous cases. The derivation of the solution is made using the method of separation of variables. Moreover, we obtain conditions for which the series solutions representations are convergent. These series are represented in terms of Wright functions of the type ${}_1\Psi_1$, however, we show that our representation coincides with the correspondent one presented in [6] in terms of the bivariate Mittag-Leffler function. In fact, the series representation involving Wright functions is more convenient for the analysis of the convergence. Our results generalize part of the results presented in [6] in the sense that the space-fractional derivative is represented in terms of a fractional Sturm-Liouville operator, and the convergence conditions for the series solution are presented.

The structure of the paper reads as follows: in the preliminaries section we recall some basic definitions and results about fractional calculus and special functions that are needed for the development of this work. In Section 3 we present some auxiliary results in the context of fractional Sturm-Liouville theory which are very important in the proof of the main results presented in the following section. In Section 4, we obtain and prove the convergence of the series representation of the solution of the homogeneous and non-homogeneous time-space-fractional telegraph equation with homogeneous boundary conditions. Moreover, we establish the connection with the results presented in [6]. In the end of the paper some illustrative examples are presented.

2 Preliminaries

In this section we recall some basic facts about fractional calculus and special functions that are needed for the understanding of this work. For a more detail revision of the fractional calculus literature we refer [10, 18, 20], for example. Let $a, b, \alpha \in \mathbb{R}$ with $a < b$ and $\alpha > 0$. The left and right Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ of order α are given by (see [10])

$$\left(I^{\alpha}_{a^+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \qquad x > a \tag{4}$$

$$\left(I^{\alpha}_{b^-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\, dt, \qquad x < b. \tag{5}$$

By ${}^{RL}D^{\alpha}_{a^+}$ and ${}^{RL}D^{\alpha}_{b^-}$ we denote the left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ on $[a, b] \subset \mathbb{R}$, which are defined by (see [10])

$$\left({}^{RL}D^{\alpha}_{a^+} f\right)(x) = \left(D^m I^{m-\alpha}_{a^+} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a \tag{6}$$

$$\left({}^{RL}D^{\alpha}_{b^-} f\right)(x) = (-1)^m \left(D^m I^{m-\alpha}_{b^-} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(t-x)^{\alpha-m+1}}\, dt, \qquad x < b. \tag{7}$$

Here, $m = [\alpha] + 1$ and $[\alpha]$ means the integer part of α. Let ${}^{C}D^{\alpha}_{a^+}$ denote the left Caputo fractional derivative of order $\alpha > 0$ on $[a, b] \subset \mathbb{R}$, which is defined by (see [10])

$$\left({}^{C}D^{\alpha}_{a^+} f\right)(x) = \left(I^{m-\alpha}_{a^+} D^m f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a. \tag{8}$$

Remark 2.1 *When dealing with functions of several variables, the definitions (4)-(8) are adapted with partial derivatives (see [10], for example).*

We denote by $I^{\alpha}_{a^+}(L_p)$, $p \geq 1$ the class of functions f that are represented by the fractional integral (4) of a summable function, that is $f = I^{\alpha}_{a^+}\varphi$, with $\varphi \in L_p(a, b)$. A description of the space $I^{\alpha}_{a^+}(L_1)$ is given in [20].

Theorem 2.2 *(cf. [20]) A function f belongs to $I^{\alpha}_{a^+}(L_1)$, with $\alpha > 0$, if and only if $I^{m-\alpha}_{a^+} f$ belongs to $AC^m([a, b])$, $m = [\alpha] + 1$ and $(I^{m-\alpha}_{a^+} f)^{(k)}(a) = 0$, $k = 0, \ldots, m-1$.*

In Theorem 2.2, $AC^m([a, b])$ denotes the class of functions f which are continuously differentiable on the segment $[a, b]$ up to the order $m-1$ and $f^{(m-1)}$ is absolutely continuous on $[a, b]$. We note that the conditions $(I^{m-\alpha}_{a^+} f)^{(k)}(a) = 0$, $k = 0, \ldots, m-1$, imply that $f^{(k)}(a) = 0$, $k = 0, \ldots, m-1$ (see [18, 20]). Removing the last condition in Theorem 2.2 we obtain the class of functions that admit a summable fractional derivative.

Definition 2.3 *(see [20]) A function $f \in L_1(a, b)$ has a summable fractional derivative $\left(D^{\alpha}_{a^+} f\right)(x)$ if $\left(I^{m-\alpha}_{a^+} f\right)(x)$ belongs to $AC^m([a, b])$, where $m = [\alpha] + 1$.*

If a function f admits a summable fractional derivative, then we have the following composition rules (see [18,20])

$$\left(I^{\alpha}_{a^+}\, {}^{RL}D^{\alpha}_{a^+} f\right)(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(I^{m-\alpha}_{a^+} f\right)^{(m-k-1)}(a), \qquad m = [\alpha] + 1 \tag{9}$$

$$\left(I^{\alpha}_{b^-}\, {}^{RL}D^{\alpha}_{b^-} f\right)(x) = f(x) - \sum_{k=0}^{m-1} \frac{(b-x)^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(I^{m-\alpha}_{a^+} f\right)^{(m-k-1)}(b), \qquad m = [\alpha] + 1. \tag{10}$$

We remark that if $f \in I^{\alpha}_{a^+}(L_1)$ then (9) and (10) reduce to

$$\left(I^{\alpha}_{a^+}\, {}^{RL}D^{\alpha}_{a^+} f\right)(x) = \left(I^{\alpha}_{b^-}\, {}^{RL}D^{\alpha}_{b^-} f\right)(x) = f(x). \tag{11}$$

Nevertheless, we note that

$${}^{RL}D^{\alpha}_{a^+}\ I^{\alpha}_{a^+} f = {}^{RL}D^{\alpha}_{b^-}\ I^{\alpha}_{b^-} f = f. \tag{12}$$

This is a particular case of a more general property (see expression (2.114) in [18])

$$D_{a^+}^{\alpha}\left(I_{a^+}^{\gamma} f\right) = D_{a^+}^{\alpha-\gamma} f, \qquad \alpha \geq \gamma > 0. \tag{13}$$

It is important to remark that the semigroup property for the composition of fractional derivatives does not hold in general (see [18, Sect. 2.3.6]). In fact, the property

$$D_{a^+}^{\alpha}\left(D_{a^+}^{\beta} f\right) = D_{a^+}^{\alpha+\beta} f \tag{14}$$

holds whenever $f \in AC^{m-1}([a,b])$, $f^{(m)} \in L_1(a,b)$ with $m = [\beta]+1$, and

$$f^{(j)}(a^+) = 0, \qquad j = 0, 1, \ldots, m-1. \tag{15}$$

Moreover, for $m-1 < \alpha < m$ with $m \in \mathbb{N}$ and $\beta > 0$, we have (see [10])

$${}^{RL}D_{a^+}^{\alpha}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, \qquad {}^{RL}D_{b^-}^{\alpha}(b-x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1}, \tag{16}$$

$$I_{a^+}^{\alpha}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}, \qquad I_{b^-}^{\alpha}(b-x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1}. \tag{17}$$

In particular, it is verified

$${}^{RL}D_{a^+}^{\alpha}(x-a)^{\alpha-j} = 0, \qquad {}^{RL}D_{b^-}^{\alpha}(b-x)^{\alpha-j} = 0, \qquad j = 1, \ldots, m. \tag{18}$$

Now, we present some special functions that are used in this work, together with some of their properties. The three-parameter Mittag-Leffler function (or the Prabhakar function) is defined as (see [8])

$$E_{\alpha,\beta}^{\gamma}(z) := \sum_{p=0}^{+\infty} \frac{(\gamma)_p}{p!\,\Gamma(\alpha p+\beta)} z^p, \qquad \Re(\alpha) > 0,\ \Re(\beta) > 0,\ \ \gamma > 0, \tag{19}$$

where $(\gamma)_p = \gamma(\gamma+1)\cdots(\gamma+p-1)$. For $\gamma = 1$ we recover the two-parametric Mittag-Leffler function

$$E_{\alpha,\beta}(z) := \sum_{p=0}^{+\infty} \frac{z^p}{\Gamma(\alpha p+\beta)}, \tag{20}$$

and for $\gamma = \beta = 1$ we recover the classical Mittag-Leffler function

$$E_{\alpha}(z) := \sum_{p=0}^{+\infty} \frac{z^p}{\Gamma(\alpha p+1)}. \tag{21}$$

For the three-parameter Mittag-Leffler function we have the following differentiation rule (see formula (5.1.15) in [8]):

$$\frac{d^m}{dz^m}\left[z^{\beta-1} E_{\alpha,\beta}^{\gamma}(\tau z^{\alpha})\right] = z^{\beta-m-1} E_{\alpha,\beta-m}^{\gamma}(\tau z^{\alpha}), \qquad \tau \in \mathbb{C}, \quad m \in \mathbb{N}. \tag{22}$$

Due to its series representation, the three-parametric Mittag-Leffler function can be considered as a special case of the Wright generalized hypergeometric function ${}_1\Psi_1$ (see formula (5.1.37) in [8])

$$E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)}\, {}_1\Psi_1\left[\begin{array}{c|c} (\gamma,1) & \\ & z \\ (\beta,\alpha) & \end{array}\right] = \frac{1}{\Gamma(\gamma)} \sum_{p=0}^{+\infty} \frac{\Gamma(\gamma+p)}{\Gamma(\beta+\alpha p)} \frac{z^p}{p!}. \tag{23}$$

Taking into account (22) with $m = 1$ and relation (23), we have the following differentiation rule for ${}_1\Psi_1$

$$\frac{1}{\Gamma(\gamma)} \frac{d}{dz}\left[z^{\beta-1}\, {}_1\Psi_1\left[\begin{array}{c|c} (\gamma,1) & \\ & \tau z^{\alpha} \\ (\beta,\alpha) & \end{array}\right]\right] = \frac{1}{\Gamma(\gamma)} z^{\beta-2}\, {}_1\Psi_1\left[\begin{array}{c|c} (\gamma,1) & \\ & \tau z^{\alpha} \\ (\beta-1,\alpha) & \end{array}\right]. \tag{24}$$

Considering the following auxiliar function (see [22])

$$e_{\alpha,\beta,\omega}^{\gamma}(t) := t^{\beta-1} E_{\alpha,\beta}^{\gamma}(\omega t^{\alpha}), \qquad t \in \mathbb{R},\ \alpha, \beta, \gamma, \omega \in \mathbb{C},\ \Re(\alpha) > 0, \tag{25}$$

we have from (23) that

$${}_1\Psi_1\left[\begin{array}{c|c} (\gamma,1) & \\ & \omega t^{\alpha} \\ (\beta,\alpha) & \end{array}\right] = \frac{\Gamma(\gamma)}{t^{\beta-1}} e_{\alpha,\beta,\omega}^{\gamma}(t). \tag{26}$$

Moreover, in [22] (see Theorem 3), it is proved the following result for the auxiliar function defined in (25).

Theorem 2.4 *For all $\alpha \in (0,1)$, $\gamma, \omega > 0$, $\alpha\gamma > \beta - 1 > 0$, the following uniform bound holds true:*

$$\left|e^{\gamma}_{\alpha,\beta,\omega}(t)\right| \leq \frac{\Gamma\left(\gamma - \frac{\beta-1}{\alpha}\right)\Gamma\left(\frac{\beta-1}{\alpha}\right)}{\pi\,\alpha\,\omega^{\frac{\beta-1}{\alpha}}\,\Gamma(\gamma)\,\left(\cos\left(\frac{\pi\alpha}{2}\right)\right)^{\gamma-\frac{\beta-1}{\alpha}}}, \qquad t > 0. \tag{27}$$

Another generalization of the Mittag-Leffler function is the multivariate Mittag-Leffler function (see [13]).

Definition 2.5 *The multivariate Mittag-Leffler function $E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n)$ of n complex variables $z_1,\ldots,z_n \in \mathbb{C}$ with complex parameters $a_1,\ldots,a_n, b \in \mathbb{C}$ (with positive real parts) is defined by*

$$E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n) = \sum_{k=0}^{+\infty} \sum_{\substack{l_1+\ldots+l_n=k\\ l_1,\ldots,l_n\geq 0}} \binom{k}{l_1,\ldots,l_n} \frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma\left(b+\sum_{i=1}^{n} a_i l_i\right)}, \tag{28}$$

where the multinomial coefficients are given by

$$\binom{k}{l_1,\ldots,l_n} := \frac{k!}{l_1!\ldots l_n!}.$$

When $n = 2$ we obtain the bivariate Mittag-Leffler function, which can be written as

$$E_{(a_1,a_2),b}(z_1,z_2) = \sum_{l_1=0}^{+\infty}\sum_{l_2=0}^{+\infty} \frac{(l_1+l_2)!}{l_1!\,l_2!}\frac{z_1^{l_1}\,z_2^{l_2}}{\Gamma\left(b+a_1l_1+a_2l_2\right)}. \tag{29}$$

From (29) we can deduce, after straightforward calculations, an addition formula for the bivariate Mittag-Leffler function (see Lemma 2.2 in [4]).

Lemma 2.6 *Let $z_1, z_2 \in \mathbb{C}$, and $a_1, a_2, b \in \mathbb{C}$ (with positive real parts). Then it holds*

$$E_{(a_1,a_2),b}(z_1,z_2) = \frac{1}{\Gamma(b)} + z_1\,E_{(a_1,a_2),b+a_1}(z_1,z_2) + z_2\,E_{(a_1,a_2),b+a_2}(z_1,z_2). \tag{30}$$

Moreover, we have the following differentiation formula

$$\frac{d^m}{dz^m}\left[z^{b-1}\,E_{(a_1,a_2),b}\left(\tau_1\,z^{a_1},\,\tau_2\,z^{a_2}\right)\right] = z^{b-m-1}\,E_{(a_1,a_2),b-m}\left(\tau_1\,z^{a_1},\,\tau_2\,z^{a_2}\right), \tag{31}$$

where $\tau_1, \tau_2, z \in \mathbb{C}$ and $m \in \mathbb{N}$. For general properties of the Mittag-Leffler function see [9, 13].

Now, we consider the following particular Wright function

$${}_1\Psi_1\left[\begin{array}{c|c} (p+1,1) & \\ & \theta\,t^{\beta-\gamma} \\ (\tau,\beta-\gamma) & \end{array}\right] = \sum_{q=0}^{\infty}\frac{\Gamma(p+1+q)}{\Gamma(\tau+(\beta-\gamma)q)}\,\frac{\left(\theta\,t^{\beta-\gamma}\right)^q}{q!} \tag{32}$$

where $p \in \mathbb{Z}_0^+$, $1 < \beta < 2$, $0 < \gamma < 1$, $\tau > 1$, and $t, \theta, \lambda \in \mathbb{R}^+$. This special function will appear in the next sections. Concerning its convergence, taking into account Theorem 1.5 in [10], we can guarantee that the series (32) is absolutely convergent for all possible values of $\theta\,t^{\beta-\gamma}$. This conclusion is due to the fact that $\Delta = \beta - \gamma - 1 > -1$, where the definition of the quantity Δ is given by formula (1.11.15) in [10]. From (26), we have

$${}_1\Psi_1\left[\begin{array}{c|c} (p+1,1) & \\ & \theta\,t^{\beta-\gamma} \\ (\tau,\beta-\gamma) & \end{array}\right] = \frac{p!}{t^{\tau-1}}\,e^{p+1}_{\beta-\gamma,\tau,\theta}(t).$$

Hence, by Theorem 2.4, we have the following estimate for (32)

$$\left|{}_1\Psi_1\left[\begin{array}{c|c} (p+1,1) & \\ & \theta\,t^{\beta-\gamma} \\ (\tau,\beta-\gamma) & \end{array}\right]\right| \leq t^{1-\tau}\,\underbrace{\frac{\Gamma\left(p+1-\frac{\tau-1}{\beta-\gamma}\right)\Gamma\left(\frac{\tau-1}{\beta-\gamma}\right)}{\pi\,(\beta-\gamma)\,\theta^{\frac{\tau-1}{\beta-\gamma}}\,\left(\cos\left(\frac{\pi(\beta-\gamma)}{2}\right)\right)^{p+1-\frac{\tau-1}{\beta-\gamma}}}}_{\mathcal{M}(\beta,\gamma,p,\theta,\tau)}. \tag{33}$$

Due to the convergence of (32) we can guarantee that the right-hand side of (33) is finite for every $t \in \mathbb{R}^+$, and therefore we can write

$$\left| {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\tau, \beta-\gamma) \end{array} \middle| \, \theta\, t^{\beta-\gamma} \right] \right| \leq t^{1-\tau}\, \mathcal{M}(\beta,\gamma,p,\theta,\tau), \tag{34}$$

where $\mathcal{M}(\beta,\gamma,p,\theta,\tau)$ is given in (33) and is a finite positive constant depending only on the parameters β, γ, p, θ and τ.

In [10] it is presented the solution of several partial fractional differential equations. Here we recall Corollary 5.9 and Theorem 5.16 in [10], where the parameters α, β, μ and λ where replaced by β, γ, $-\lambda$ and θ, respectively, and $l = 2$ in Theorem 5.16.

Theorem 2.7 *(cf. [10, Cor. 5.9]) The equation*

$$\left({}^C D^{\beta}_{0^+} u\right)(t) - \theta \left({}^C D^{\gamma}_{a^+} u\right)(t) + \lambda\, u(t) = 0,$$

where $t > 0$, $1 < \beta \leq 2$, $0 < \gamma \leq 1$, $\theta, \lambda \in \mathbb{R}^+$ has one solution $u(t)$, given by

$$\begin{aligned} u(t) = & \sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!}\, t^{\beta p}\, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+1, \beta-\gamma) \end{array} \middle| \, \theta\, t^{\beta-\gamma} \right] \\ & - \theta \sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!}\, t^{\beta p+\beta-\gamma}\, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+1+\beta-\gamma, \beta-\gamma) \end{array} \middle| \, \theta\, x^{\beta-\gamma} \right], \end{aligned} \tag{35}$$

and a second solution $v(t)$ given by

$$v(t) = \sum_{p=0}^{\infty} \frac{(-\lambda)^p}{p!}\, t^{\beta p+1}\, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+2, \beta-\gamma) \end{array} \middle| \, \theta\, t^{\beta-\gamma} \right]. \tag{36}$$

Remark 2.8 *Taking into account the series expansion of the Wright function given in (23), the series expansion of the bivariate Mittag-Leffler function given in (29), and Lemma 2.6, we can rewrite, after straightforward calculations, functions u and v presented in Theorem 2.7 in the following way (cf. analogous formulas in [4])*

$$u(t) = 1 - \lambda\, t^{\beta}\, E_{(\beta-\gamma,\,\beta),\,\beta+1}\left(\theta\, t^{\beta-\gamma},\, -\lambda\, t^{\beta}\right) \quad \text{and} \quad v(t) = t\, E_{(\beta-\gamma,\,\beta),\,2}\left(\theta\, t^{\beta-\gamma},\, -\lambda\, t^{\beta}\right). \tag{37}$$

Theorem 2.9 *(cf. [10, Thm. 5.16]) Let $1 < \beta < 2$, $0 < \gamma < \beta$ be such that $\gamma \leq \beta - 1$. Let $\theta, \lambda \in \mathbb{R}^+$, and $h(t)$ be a given real function defined on $\mathbb{R}^+$. Then the equation*

$$\left({}^C D^{\beta}_{0^+} u\right)(t) - \theta \left({}^C D^{\gamma}_{a^+} u\right)(t) + \lambda\, u(t) = h(t),$$

is solvable, and its general solution has the form

$$u(t) = c_1\, u(t) + c_2\, v(t) + \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma,\theta,-\lambda}(t-w)\, dw$$

where u and v are given by (35) and (36), respectively, and $G_{\beta,\gamma,\theta,-\lambda}(z)$ is given by

$$G_{\beta,\gamma,\theta,-\lambda}(z) = \sum_{p=0}^{+\infty} \frac{(-\lambda_k)^p}{p!}\, z^{\beta p}\, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta, \beta-\gamma) \end{array} \middle| \, \theta\, z^{\beta-\gamma} \right]. \tag{38}$$

Remark 2.10 *Taking into account the series expansion of the Wright function given in (23), and the series expansion of the bivariate Mittag-Leffler function given in (29), we can rewrite, after straightforward calculations, function $G_{\beta,\gamma,\theta,-\lambda}$ in Theorem 2.9 in the following way*

$$G_{\beta,\gamma,\theta,-\lambda}(z) = E_{(\beta-\gamma,\,\beta),\,\beta}\left(\theta\, t^{\beta-\gamma},\, -\lambda\, t^{\beta}\right).$$

The previous formula establishes a connection with the work of Luchko and Gorenflo (see [13]).

3 Auxiliary results

3.1 Fractional Sturm-Liouville problem in higher dimensions

In this work we want to obtain existence results for Sturm-Liouville telegraph equation (3) by using the method of separation of variables. This approach is based on the fractional Sturm-Liouville theory (see [3]), more precisely on the existence of eigenvalues and corresponding eigenfunctions to the following fractional differential equation:

$$-\left({}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu(x)\, {}^{RL}\nabla^{\alpha}_{a^+} y\right)\right)(x) + \nu(x)\, y(x) = \lambda\, r(x)\, y(x) \tag{39}$$

subject to the conditions

$$\beta_1^{[j]}\, y(x)\Big|_{x_j=a_j} + \beta_2^{[j]}\, I^{1-\alpha_j}_{b_j^-}\left(\mu\, {}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j} y\right)(x)\Big|_{x_j=a_j} = 0, \quad j = 1, \ldots, n, \tag{40}$$

$$\beta_3^{[j]}\, y(x)\Big|_{x_j=b_j} + \beta_4^{[j]}\, I^{1-\alpha_j}_{b_j^-}\left(\mu\, {}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j} y\right)(x)\Big|_{x_j=b_j} = 0, \quad j = 1, \ldots, n, \tag{41}$$

where:

(i) $x \in \Omega = \prod_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n$ and "$\cdot$" is the usual scalar product between two vectors in $\mathbb{R}^n$;

(ii) ${}^{RL}\nabla^{\alpha}_{b^-}$ and ${}^{RL}\nabla^{\alpha}_{a^+}$ are, respectively, the right and left Riemann-Liouville fractional gradient operators of order $\alpha = (\alpha_1, \ldots, \alpha_n)$ given by

$${}^{RL}\nabla^{\alpha}_{a^+} = \sum_{i=1}^n e_i\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} \qquad \text{and} \qquad {}^{RL}\nabla^{\alpha}_{b^-} = \sum_{i=1}^n e_i\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}, \tag{42}$$

where for $i = 1, \ldots, n$, e_i denotes the standard unit vector in the direction of x_i, the partial derivatives ${}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}$, ${}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}$, are the left and right Riemann-Liouville fractional derivatives of order $\alpha_i \in \left]\frac{1}{2}, 1\right]$ with respect to the variable $x_i \in [a_i, b_i]$;

(iii) $I^{1-\alpha_j}_{b_j^-}$ denotes the right Riemann-Liouville fractional integral of order $1 - \alpha_j$ with respect to the variable $x_j \in]a_j, b_j[$, where $\alpha_j \in \left]\frac{1}{2}, 1\right]$ and $j = 1, \ldots, n$;

(iv) μ, ν, and r are continuous scalar functions defined on Ω. The function r is called the weight or density function. Moreover, $\mu(x) > 0$ and $r(x) > 0$ for all $x \in \Omega$;

(v) the values of $\lambda \in \mathbb{C}$ for which there exists non-trivial solutions $y(x) \in I^{\alpha_j}_{a_j^+}(L_p(\Omega))$, $p > 1$ and $j = 1, \ldots, n$, are called the eigenvalues of the problem.

From now on until the end of the paper we consider that assumptions (i)-(v) over the fractional Sturm-Liouville problem (39)-(41) are satisfied. For the proofs of the main results presented in Section 4, we make use of the following theorem proved in [3] regarding the existence of solutions to the problem (39)-(41).

Theorem 3.1 *(cf. [3]) Under the assumptions (i)-(v), the fractional Sturm-Liouville problem (39)-(41) has an infinite increasing sequence of real eigenvalues $\lambda_1, \lambda_2, \ldots$, and to each eigenvalue λ_k there is a correspondent eigenfunction y_k which is unique up to a constant factor and satisfies the minimization problem of the following functional*

$$J(f) = \int_\Omega \left[\mu(x)\, \left({}^{RL}\nabla^{\alpha}_{a^+} f(x)\right) \cdot \left({}^{RL}\nabla^{\alpha}_{a^+} f(x)\right) + \nu(x)\, f^2(x)\right] dx \tag{43}$$

subject to the conditions in (40)-(41) and to the additional condition

$$I(f) = \int_\Omega r(x)\, f^2(x)\, dx = 1. \tag{44}$$

Furthermore, the eigenfunctions y_k form an orthogonal set of solutions, with respect to the inner product (48). Moreover, the eigenvalues are given by

$$\lambda_k = \int_\Omega \left[\mu(x)\, \left({}^{RL}\nabla^{\alpha}_{a^+} y_k(x)\right) \cdot \left({}^{RL}\nabla^{\alpha}_{a^+} y_k(x)\right) + \nu(x)\, \left(y_k(x)\right)^2\right] dx$$

$$= \sum_{i=1}^{n} \int_{\Omega} \left[\mu(x) \, \left({}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(x) \right)^2 + \frac{1}{n} \, \nu(x) \, (y_k(x))^2 \right] dx. \tag{45}$$

For $\Omega = \prod_{i=1}^{n} [a_i, b_i]$, let us introduce the following spaces

$$L^2_r(\Omega) := \left\{ g \in L^2(\Omega) : \left(\int_{\Omega} r(x) \, |g(x)|^2 \, dx \right)^{\frac{1}{2}} < \infty \right\} \tag{46}$$

and

$$C_B(\Omega) := \left\{ g \in C(\Omega) : \, g(x)\big|_{x_j = a_j} = 0 = g(x)|_{x_j = b_j} \, , \; j = 1, \ldots, n \right\}. \tag{47}$$

The space $L^2_r(\Omega)$ is a weighted Hilbert space of real-valued functions and $C_B(\Omega)$ is a space of continuous functions with homogeneous conditions on the boundary. Moreover, the space L^2_r is endowed with an inner product and norm given by

$$\langle f, \, g \rangle := \int_{\Omega} r(x) \, f(x) \, g(x) \, dx \qquad \text{and} \qquad \|f\|_{L^2_r} := \left(\int_{\Omega} r(x) \, |f(x)|^2 \, dx \right)^{\frac{1}{2}}. \tag{48}$$

In the case $r(x) = 1$ we denote the L^2-norm as $\|f\|_2$. In addition, we denote by $Y := \{y_k, \, k \in \mathbb{N}\}$ the set of eigenfunctions of (39) and its closed linear span by

$$\overline{span(Y)} := \left\{ f \in L^2_r(\Omega) : \, \forall \epsilon > 0, \; \forall n \geq n_0, \; \left\| f - \sum_{k=1}^{n} \langle f, \, y_k \rangle \right\|_{L^2_r} \leq \epsilon \right\}. \tag{49}$$

3.2 Properties of the eigenvalues and the eigenfunctions

Now, we prove some auxilar results about the eigenvalues and the correspondent eigenfunctions that are used in the following section to prove the main results of the paper. Let us start with a result where we obtain an estimation for the eigenfunctions and eigenvalues of the fractional Sturm-Liouville operator.

Lemma 3.2 *Let the assumptions over the fractional Sturm-Liouville problem (39)-(41) be fulfilled. There exists $k_0 \in \mathbb{N}$ such that eigenfunctions and eigenvalues of the fractional Sturm-Liouville problem (39)-(41) fulfill the inequality*

$$\frac{|y_k(x)|}{\sqrt{\lambda_k}} \leq M_0, \qquad \forall k \geq k_0, \quad \forall x \in \Omega,$$

for some $M_0 \in \mathbb{R}^+$.

Proof: Denoting by $\widehat{x} = (x_1, \ldots, x_{i-1}, \omega, x_{i+1}, \ldots, x_n)$, considering the composition rule (11), Hölder's inequality, and taking into account the definition of the left fractional integral (4), we have after straightforward calculations

$$\begin{aligned}
|y_k(x)|^2 &= \frac{1}{n^2} \left| \sum_{i=1}^{n} I^{\alpha_i}_{a_i^+} \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(x) \right|^2 \\
&\leq \frac{1}{n^2} \left(\sum_{i=1}^{n} \left| I^{\alpha_i}_{a_i^+} \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(x) \right| \right)^2 \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \left| I^{\alpha_i}_{a_i^+} \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(x) \right|^2 \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(\Gamma(\alpha_i))^2} \left(\int_{a_i}^{x_i} |x_i - \omega|^{\alpha_i - 1} \left| {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(\widehat{x}) \right| \, d\omega \right)^2 \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(\Gamma(\alpha_i))^2} \left(\int_{a_i}^{x_i} (x_i - \omega)^{2(\alpha_i - 1)} \, d\omega \right) \left(\int_{a_i}^{x_i} \left| {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(\widehat{x}) \right|^2 \, d\omega \right) \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \frac{1}{(\Gamma(\alpha_i))^2} \, \frac{(b_i - a_i)^{2\alpha_i - 1}}{2\alpha_i - 1} \int_{a_i}^{b_i} \left| {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(\widehat{x}) \right|^2 \, d\omega.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
|y_k(x)| &\le \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} \frac{1}{(\Gamma(\alpha_i))^2} \frac{(b_i-a_i)^{2\alpha_i-1}}{2\left(\alpha_i-\frac{1}{2}\right)} \int_{a_i}^{b_i} \left| {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(\widehat{x}) \right|^2 d\omega \right)^{\frac{1}{2}} \\
&\le \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} \frac{1}{(\Gamma(\alpha_i))^2} \frac{(b_i-a_i)^{2\alpha_i-1}}{2\left(\alpha_i-\frac{1}{2}\right)} M_1 \right)^{\frac{1}{2}}, \qquad (50)
\end{aligned}
$$

where

$$
0 < M_1 = \max_{i=1,\ldots,n} \left(\sup_{\widehat{\underline{x}}} \int_{a_i}^{b_i} \left| {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(\widehat{x}) \right|^2 d\omega \right) < +\infty
$$

and $\widehat{\underline{x}} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Since Ω is bounded, there exists $M_2 > 0$ such that $M_1 \le M_2 \left\| {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k \right\|_2^2$, and, hence, relation (50) becomes

$$
\begin{aligned}
|y_k(x)| &\le \sqrt{\frac{M_2}{n}} \left(\sum_{i=1}^{n} \frac{1}{(\Gamma(\alpha_i))^2} \frac{(b_i-a_i)^{2\alpha_i-1}}{2\left(\alpha_i-\frac{1}{2}\right)} \int_{\Omega} \left| {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(\widehat{x}) \right|^2 dx \right)^{\frac{1}{2}} \\
&\le \sqrt{\frac{M_2\, M_3}{2n}} \left(\sum_{i=1}^{n} \int_{\Omega} \left| {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(\widehat{x}) \right|^2 dx \right)^{\frac{1}{2}}, \qquad (51)
\end{aligned}
$$

where

$$
M_3 = \max_{i=1,\ldots,n} \frac{(b_i-a_i)^{2\alpha_i-1}}{(\Gamma(\alpha_i))^2 \left(\alpha_i-\frac{1}{2}\right)}.
$$

Now, considering (45) with $\nu(x) = 0$, relation (51) becomes

$$
\begin{aligned}
|y_k(x)| &\le \sqrt{\frac{M_2\, M_3 \left\| \frac{1}{\mu} \right\|}{2n}} \left(\sum_{i=1}^{n} \int_{\Omega} \mu(x) \left({}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(x) \right)^2 dx \right)^{\frac{1}{2}} \qquad (52) \\
&= \sqrt{\frac{M_2\, M_3 \left\| \frac{1}{\mu} \right\|}{2n}} \sqrt{\lambda_k},
\end{aligned}
$$

where $\|\cdot\|$ denotes the supremum norm in the $C_B(\Omega)$ space. Consequently, for $k \in \mathbb{N}$, we have

$$
\frac{|y_k(x)|}{\sqrt{\lambda_k}} \le \sqrt{\frac{M_2\, M_3 \left\| \frac{1}{\mu} \right\|}{2n}}.
$$

For $\nu(x) \neq 0$ we have from (52) and (44)

$$
\begin{aligned}
|y_k(x)| &\le \sqrt{\frac{M_2\, M_3 \left\| \frac{1}{\mu} \right\|}{2n}} \left(\sum_{i=1}^{n} \int_{\Omega} \left[\mu(x) \left({}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(x) \right)^2 + \frac{1}{n}\, \nu(x) (y_k(x))^2 \right] dx - \frac{1}{n} \int_{\Omega} \nu(x)\, (y_k(x))^2\, dx \right)^{\frac{1}{2}} \\
&\le \sqrt{\frac{M_2\, M_3 \left\| \frac{1}{\mu} \right\|}{2n}} \left(\sum_{i=1}^{n} \int_{\Omega} \left[\mu(x) \left({}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} y_k(x) \right)^2 + \frac{1}{n}\, \nu(x) (y_k(x))^2 \right] dx + \frac{1}{n} \left\| \frac{\nu}{r} \right\| \int_{\Omega} r(x)\, (y_k(x))^2\, dx \right)^{\frac{1}{2}} \\
&= \sqrt{\frac{M_2\, M_3 \left\| \frac{1}{\mu} \right\|}{2n}} \left(\lambda_k + \frac{1}{n} \left\| \frac{\nu}{r} \right\| \right)^{\frac{1}{2}},
\end{aligned}
$$

where we used again the variational formulation presented in [3] with $\nu(x) \neq 0$. Dividing both side of the previous relation by $\sqrt{\lambda_k}$ we get

$$
\frac{|y_k(x)|}{\sqrt{\lambda_k}} \le \sqrt{\frac{M_2\, M_3 \left\| \frac{1}{\mu} \right\|}{2n}} \left(1 + \frac{\left\| \frac{\nu}{r} \right\|}{n\, \lambda_k} \right)^{\frac{1}{2}}.
$$

As $\lambda_k \to +\infty$ for $k \to +\infty$ (see Theorem 3.1) we note that there exists k_0 such that

$$\frac{\left\| \frac{\nu}{r} \right\|}{n\,\lambda_k} < 1, \qquad \forall k \geq k_0.$$

Taking

$$M_0 = \sqrt{\frac{M_2\, M_3\, \left\| \frac{1}{\mu} \right\|}{n}},$$

we conclude that for all $k \geq k_0$ and an arbitrary continuous function ν we have that

$$\frac{|y_k(x)|}{\sqrt{\lambda_k}} \leq M_0\, \frac{1}{\sqrt{2}}\, (1+1)^{\frac{1}{2}} = M_0, \qquad \forall k \geq k_0, \quad \forall x \in \Omega.$$

■

In the following lemmas we obtain some important uniform convergence results.

Lemma 3.3 *Let* $0 < T < +\infty$ *and* ${}_{0^+}^{\;C}\partial_t^{\beta}$ *(resp.* ${}_{0^+}^{\;C}\partial_t^{\gamma}$*) be the left Caputo fractional partial derivative of order* $\beta > 0$ *(resp.* $\gamma > 0$*) with respect to* t *(see (8)). Assume that* $(g_k)_{k=1}^{+\infty}$ *is a sequence of functions uniformly convergent in* $[0,T]$, $\left({}_{0^+}^{\;C}\partial_t^{\beta} g_k \right)_{k=1}^{+\infty}$, $\left({}_{0^+}^{\;C}\partial_t^{\gamma} g_k \right)_{k=1}^{+\infty}$, *with* $1 < \beta < 2$ *and* $0 < \gamma < 1$, *are uniformly convergent in* $]0,T]$ *and* $g_k,\ {}_{0^+}^{\;C}\partial_t^{\beta} g_k,\ {}_{0^+}^{\;C}\partial_t^{\gamma} g_k \in C[0,T]$, *for any* $k \in \mathbb{N}$. *Then, for* $t \in \mathbb{R}^+$

$$\left({}_{0^+}^{\;C}\partial_t^{\beta} - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma} \right) \lim_{k\to+\infty} g_k(t) = \lim_{k\to+\infty} \left({}_{0^+}^{\;C}\partial_t^{\beta} - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma} \right) g_k(t).$$

Proof: In the conditions of the lemma we have that

$$\left({}_{0^+}^{\;C}\partial_t^{\beta} - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma} \right) \lim_{k\to+\infty} g_k(t) = {}_{0^+}^{\;C}\partial_t^{\beta} \lim_{k\to+\infty} g_k(t) - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma} \lim_{k\to+\infty} g_k(t).$$

Taking into account Lemma 3.4 in [12] we have that

$${}_{0^+}^{\;C}\partial_t^{\beta} \lim_{k\to+\infty} g_k(t) = \lim_{k\to+\infty} {}_{0^+}^{\;C}\partial_t^{\beta} g_k(t) \qquad \text{and} \qquad {}_{0^+}^{\;C}\partial_t^{\gamma} \lim_{k\to+\infty} g_k(t) = \lim_{k\to+\infty} {}_{0^+}^{\;C}\partial_t^{\gamma} g_k(t).$$

Hence, we conclude that

$$\begin{aligned}\left({}_{0^+}^{\;C}\partial_t^{\beta} - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma} \right) \lim_{k\to+\infty} g_k(t) &= \lim_{k\to+\infty} {}_{0^+}^{\;C}\partial_t^{\beta} g_k(t) - \lim_{k\to+\infty} \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma} g_k(t) \\ &= \lim_{k\to+\infty} \left({}_{0^+}^{\;C}\partial_t^{\beta} - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma} \right) g_k(t).\end{aligned}$$

This gives the desired result.

■

Let us denote the fractional Sturm-Liouville operator in the right-hand side of (39) by

$${}^{RL}\widehat{L}_{\nu}^{\alpha} := \frac{1}{r(x)} \left[- \left({}^{RL}\nabla_{b^-}^{\alpha} \cdot \left(\mu(x)\, {}^{RL}\nabla_{a^+}^{\alpha} \right) \right) + \nu(x) \right]. \tag{53}$$

Lemma 3.4 *Assume that* $(f_k)_{k=1}^{+\infty}$ *and* $\left({}^{RL}\widehat{L}_{\nu}^{\alpha} f_k \right)_{k=1}^{+\infty}$ *are uniformly convergent in* Ω, *let say to* g *and* ${}^{RL}\widehat{L}_{\nu}^{\alpha} f$, *respectively. Assume also that* $f_k, {}^{RL}\widehat{L}_{\nu}^{\alpha} f_k \in C_B(\Omega)$ *for any* $k \in \mathbb{N}$. *Then,*

$$\lim_{k\to+\infty} {}^{RL}\widehat{L}_{\nu}^{\alpha} f_k(x) = {}^{RL}\widehat{L}_{\nu}^{\alpha} \lim_{k\to+\infty} f_k(x).$$

Proof: We have by (53) that

$${}^{RL}\widehat{L}_{\nu}^{\alpha} f_k(x) = -\frac{1}{r(x)} \sum_{i=1}^{n} {}_{b_i^-}^{RL}\partial_{x_i}^{\alpha_i} \left(\mu(x)\, {}_{a_i^+}^{RL}\partial_{x_i}^{\alpha_i} f_k(x) \right) + \frac{\nu(x)}{r(x)}\, f_k(x). \tag{54}$$

Multiplying each member of (54) by $r(x)$, applying $I^{\alpha_j}_{b_j^-}$, taking into account (10) and making straightforward calculations, we get

$$I^{\alpha_j}_{b_j^-}\, r(x)\, {}^{RL}\widehat{L}^{\alpha}_{\nu} f_k(x)$$

$$= -\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f_k(x) + \frac{(b_j - x_j)^{\alpha_j - 1}}{\Gamma(\alpha_j)}\ \xi_2^{[j]}\Big|_{x_j = b_j} - \sum_{\substack{i=1\\ i\neq j}}^{n} I^{\alpha_j}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f_k(x)\right) + I^{\alpha_j}_{b_j^-} \left(\nu(x)\, f_k(x)\right), \quad (55)$$

where the constant $\xi_2^{[j]}\Big|_{x_j=b_j}$, with respect to the variable x_j, is given by

$$\xi_2^{[j]}\Big|_{x_j=b_j} = I^{1-\alpha_j}_{b_j^-} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f_k(x)\right)\Big|_{x_j=b_j}.$$

Multiplying each member of (55) by $-\frac{1}{\mu(x)}$, applying $I^{\alpha_j}_{a_j^+}$, taking into account (9) and making straightforward calculations, we arrive to

$$-I^{\alpha_j}_{a_j^+} \frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, r(x)\, {}^{RL}\widehat{L}^{\alpha}_{\nu} f_k(x) = f_k(x) - \frac{(x_j - a_j)^{\alpha_j - 1}}{\Gamma(\alpha_j)}\ \xi_1^{[j]}\Big|_{x_j=a_j} - \xi_2^{[j]}\Big|_{x_j=b_j}\ I^{\alpha_j}_{a_j^+} \left(\frac{(b_j - x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}\right)$$
$$+ \sum_{\substack{i=1\\ i\neq j}}^{n} I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f_k(x)\right)\right) - I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-} \left(\nu(x)\, f_k(x)\right)\right), \quad (56)$$

for any $k \in \mathbb{N}$ and where the constant $\xi_1^{[j]}\Big|_{x_j=a_j}$, with respect to the variable x_j, is given by

$$\xi_1^{[j]}\Big|_{x_j=a_j} = I^{1-\alpha_j}_{a_j^+} f(x)\Big|_{x_j=a_j}.$$

Taking the limit of (56), when $k \to +\infty$, we get

$$\lim_{k\to+\infty} \left(-I^{\alpha_j}_{a_j^+} \frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, r(x)\, {}^{RL}\widehat{L}^{\alpha}_{\nu} f_k(x)\right)$$
$$= \lim_{k\to+\infty} \Bigg[f_k(x) - \frac{(x_j - a_j)^{\alpha_j-1}}{\Gamma(\alpha_j)}\ \xi_1^{[j]}\Big|_{x_j=a_j} - \xi_2^{[j]}\Big|_{x_j=b_j}\ I^{\alpha_j}_{a_j^+} \left(\frac{(b_j - x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}\right)$$
$$+ \sum_{\substack{i=1\\ i\neq j}}^{n} I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f_k(x)\right)\right) - I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-} \left(\nu(x)\, f_k(x)\right)\right)\Bigg]. \quad (57)$$

By the assumptions stated we know that the following limits exist

$$\lim_{k\to+\infty} f_k(x) = g(x),$$

$$\lim_{k\to+\infty} {}^{RL}\widehat{L}^{\alpha}_{\nu} f_k(x) = f(x),$$

$$\lim_{k\to+\infty} \left(-I^{\alpha_j}_{a_j^+} \frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, r(x)\, {}^{RL}\widehat{L}^{\alpha}_{\nu} f_k(x)\right) = -I^{\alpha_j}_{a_j^+} \frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, r(x)\, {}^{RL}\widehat{L}^{\alpha}_{\nu} f(x),$$

$$\lim_{k\to+\infty} \left(I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f_k(x)\right)\right)\right) = I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-}\, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x)\, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right)\right),$$

$$\lim_{k\to+\infty} \left(-I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-} \left(\nu(x)\, f_k(x)\right)\right)\right) = -I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)}\, I^{\alpha_j}_{b_j^-} \left(\nu(x)\, f(x)\right)\right).$$

Moreover, functions f and g are continuous in Ω. Therefore

$$\lim_{k\to+\infty} f_k(x)\big|_{x_j=a_j\vee x_j=b_j} = g(x)\big|_{x_j=a_j\vee x_j=b_j} < \infty,$$

$$\lim_{k\to+\infty}\left(-I^{\alpha_j}_{a_j^+}\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,r(x)\,{}^{RL}\widehat{L}^{\alpha}_{\nu}f_k(x)\right)\bigg|_{x_j=a_j\vee x_j=b_j} = \left(-I^{\alpha_j}_{a_j^+}\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,r(x)\,{}^{RL}\widehat{L}^{\alpha}_{\nu}f(x)\right)\bigg|_{x_j=a_j\vee x_j=b_j} < \infty,$$

$$\lim_{k\to+\infty}\left(-I^{\alpha_j}_{a_j^+}\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,r(x)\,{}^{RL}\widehat{L}^{\alpha}_{\nu}f_k(x)\right)\bigg|_{x_j=a_j\vee x_j=b_j} = \left(-I^{\alpha_j}_{a_j^+}\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,r(x)\,{}^{RL}\widehat{L}^{\alpha}_{\nu}f(x)\right)\bigg|_{x_j=a_j\vee x_j=b_j} < \infty.$$

The above three pointwise convergences together with

$$\begin{aligned}&\lim_{k\to+\infty}\left(-I^{\alpha_j}_{a_j^+}\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,r(x)\,{}^{RL}\widehat{L}^{\alpha}_{\nu}f_k(x)\right)\bigg|_{x_j=a_j\vee x_j=b_j}\\ &=\lim_{k\to+\infty}\Bigg[f_k(x)-\frac{(x_j-a_j)^{\alpha_j-1}}{\Gamma(\alpha_j)}\;\xi_1^{[j]}\Big|_{x_j=a_j}-\xi_2^{[j]}\Big|_{x_j=b_j}\;I^{\alpha_j}_{a_j^+}\left(\frac{(b_j-x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}\right)\\ &\qquad+\sum_{\substack{i=1\\ i\neq j}}^{n}I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}f_k(x)\right)\right)-I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\left(\nu(x)\,f_k(x)\right)\right)\Bigg]\Bigg|_{x_j=a_j\vee x_j=b_j},\end{aligned}\tag{58}$$

give

$$\lim_{k\to+\infty}\left[-\,\xi_2^{[j]}\Big|_{x_j=b_j}\;I^{\alpha_j}_{a_j^+}\left(\frac{(b_j-x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}\right)\bigg|_{x_j=a_j}+\sum_{\substack{i=1\\ i\neq j}}^{n}I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}f_k(x)\right)\right)\bigg|_{x_j=a_j}\right]<\infty,$$

$$\begin{aligned}&\lim_{k\to+\infty}\Bigg[-\frac{(b_j-a_j)^{\alpha_j-1}}{\Gamma(\alpha_j)}\;\xi_1^{[j]}\Big|_{x_j=a_j}-\xi_2^{[j]}\Big|_{x_j=b_j}\;I^{\alpha_j}_{a_j^+}\left(\frac{(b_j-x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}\right)\bigg|_{x_j=b_j}\\ &\qquad+\sum_{\substack{i=1\\ i\neq j}}^{n}I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}f_k(x)\right)\right)\bigg|_{x_j=b_j}\Bigg]<\infty.\end{aligned}$$

Consequently, we have from (57)

$$\begin{aligned}&-I^{\alpha_j}_{a_j^+}\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,r(x)\,f(x)+\xi_1^{[j]}\Big|_{x_j=a_j}\;\frac{(x_j-a_j)^{\alpha_j-1}}{\Gamma(\alpha_j)}+\xi_2^{[j]}\Big|_{x_j=b_j}\;I^{\alpha_j}_{a_j^+}\left(\frac{(b_j-x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}\right)\\ &\qquad=g(x)+\sum_{\substack{i=1\\ i\neq j}}^{n}I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}g(x)\right)\right)-I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\left(\nu(x)\,g(x)\right)\right).\end{aligned}\tag{59}$$

Applying $-\frac{1}{r(x)}\,{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}$ to each term of (59), taking into account (12) and (18), and making straightforward calculations, we get for each term of (59)

$$\begin{aligned}&-\frac{1}{r(x)}\,{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}\left(-I^{\alpha_j}_{a_j^+}\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,r(x)\,f(x)\right)\\ &\qquad=\frac{1}{r(x)}\,{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}\,I^{\alpha_j}_{a_j^+}\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,r(x)\,f(x)=f(x)\end{aligned}\tag{60}$$

$$\begin{aligned}&-\frac{1}{r(x)}\,{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}\left(\xi_1^{[j]}\Big|_{x_j=a_j}\;\frac{(x_j-a_j)^{\alpha_j-1}}{\Gamma(\alpha_j)}\right)\\ &\qquad=-\frac{1}{r(x)\,\Gamma(\alpha_j)}\;\xi_1^{[j]}\Big|_{x_j=a_j}\;{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}(x_j-a_j)^{\alpha_j-1}=0\end{aligned}\tag{61}$$

$$-\frac{1}{r(x)}\,{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}\left(\xi_2^{[j]}\Big|_{x_j=b_j}\;I^{\alpha_j}_{a_j^+}\left(\frac{(b_j-x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}\right)\right)$$

$$=-\frac{1}{r(x)\,\Gamma(\alpha_j)}\;\xi_2^{[j]}\Big|_{x_j=b_j}\;{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}(b_j-x_j)^{\alpha_j-1}=0 \tag{62}$$

$$-\frac{1}{r(x)}\,{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}\left(\sum_{\substack{i=1\\i\neq j}}^{n}I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\Big(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}g(x)\Big)\right)\right)$$

$$=-\frac{1}{r(x)}\sum_{\substack{i=1\\i\neq j}}^{n}{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}\,I^{\alpha_j}_{a_j^+}\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\Big(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}g(x)\Big)$$

$$=-\frac{1}{r(x)}\sum_{\substack{i=1\\i\neq j}}^{n}{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\Big(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}g(x)\Big) \tag{63}$$

$$-\frac{1}{r(x)}\,{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}\left(-I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,(\nu(x)\,g(x))\right)\right)$$

$$=\frac{1}{r(x)}\,{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}\,I^{\alpha_j}_{a_j^+}\,\frac{1}{\mu(x)}\,I^{\alpha_j}_{b_j^-}\,(\nu(x)\,g(x))=\frac{\nu(x)}{r(x)}\,g(x), \tag{64}$$

i.e., expression (59) becomes

$$f(x)=-\frac{1}{r(x)}\,{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}-\frac{1}{r(x)}\sum_{\substack{i=1\\i\neq j}}^{n}{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\Big(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}g(x)\Big)+\frac{\nu(x)}{r(x)}\,g(x).$$

Summing up each member from $j=1,\dots,n$ we obtain

$$\sum_{j=1}^{n}f(x)=-\frac{1}{r(x)}\sum_{j=1}^{n}{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\,\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}g(x)-\frac{1}{r(x)}\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n}{}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\Big(\mu(x)\,{}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i}g(x)\Big)+\sum_{j=1}^{n}\frac{\nu(x)}{r(x)}\,g(x)$$

$$\Leftrightarrow n\,f(x)=-\frac{1}{r(x)}\left({}^{RL}\nabla^{\alpha}_{b^-}\cdot\big(\mu(x)\,{}^{RL}\nabla^{\alpha}_{a^+}g(x)\big)\right)-\frac{1}{r(x)}\,(n-1)\sum_{j=1}^{n}{}^{RL}_{b_j^-}\partial^{\alpha_j}_{x_j}\Big(\mu(x)\,{}^{RL}_{a_j^+}\partial^{\alpha_j}_{x_j}g(x)\Big)+n\,\frac{\nu(x)}{r(x)}\,g(x)$$

$$\Leftrightarrow n\,f(x)=-\frac{1}{r(x)}\left({}^{RL}\nabla^{\alpha}_{b^-}\cdot\big(\mu(x)\,{}^{RL}\nabla^{\alpha}_{a^+}g(x)\big)\right)-\frac{1}{r(x)}\,(n-1)\left({}^{RL}\nabla^{\alpha}_{b^-}\cdot\big(\mu(x)\,{}^{RL}\nabla^{\alpha}_{a^+}g(x)\big)\right)+n\,\frac{\nu(x)}{r(x)}\,g(x)$$

$$\Leftrightarrow n\,f(x)=-\frac{n}{r(x)}\left({}^{RL}\nabla^{\alpha}_{b^-}\cdot\big(\mu(x)\,{}^{RL}\nabla^{\alpha}_{a^+}g(x)\big)\right)+n\,\frac{\nu(x)}{r(x)}\,g(x)$$

$$\Leftrightarrow f(x)=-\frac{1}{r(x)}\left({}^{RL}\nabla^{\alpha}_{b^-}\cdot\big(\mu(x)\,{}^{RL}\nabla^{\alpha}_{a^+}g(x)\big)\right)+\frac{\nu(x)}{r(x)}\,g(x)$$

$$\Leftrightarrow f(x)={}^{RL}\widehat{L}^{\alpha}_{\nu}\,g(x).$$

Hence, we conclude that

$$\lim_{k\to+\infty}{}^{RL}\widehat{L}^{\alpha}_{\nu}f_k(x)={}^{RL}\widehat{L}^{\alpha}_{\nu}\lim_{k\to+\infty}f_k(x),$$

and therefore the proof is completed.

■

4 Main results

Let $1<\beta<2$ and $0<\gamma<1$, and let us consider the following non-homogeneous fractional Sturm-Liouville telegraph equation:

$${}^{C}_{0^+}\partial^{\beta}_{t}f(x,t)-\theta\;{}^{C}_{0^+}\partial^{\gamma}_{t}f(x,t)=-{}^{RL}\widehat{L}^{\alpha}_{\nu}f(x,t)+h(x,t),\qquad (x,t)\in\Omega\times\mathbb{R}^+,\quad \theta>0, \tag{65}$$

where ${}_{0^+}^{\;C}\partial_t^{\beta}$ (resp. ${}_{0^+}^{\;C}\partial_t^{\gamma}$) is the Caputo fractional partial derivative (8) of order $\beta \in]1,2[$ (resp. $\gamma \in]0,1[$) with respect to t, such that $\gamma \leq \beta - 1$, ${}^{RL}\widehat{L}_{\nu}^{\alpha}$ is the fractional Sturm-Liouville operator (53), and subject to the conditions (40), (41) and

$$f(x,0) = g_0(x), \qquad \frac{\partial f}{\partial t}(x,0) = g_1(x) \qquad g_0, g_1 \in C_B(\Omega), \tag{66}$$

where the space $C_B(\Omega)$ is defined in (47). In our first main result, we study the existence of solution of the homogeneous equation associated to (65), i.e., in the case when $h(x,t) = 0$.

Theorem 4.1 *Let us assume that the fractional Sturm-Liouville problem (39)-(41), described in Section 3.1, has eigenfunctions y_k and correspondent eigenvalues λ_k obeying the following conditions*

$$\sum_{k=1}^{+\infty} \left(\sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p-\frac{1}{2}}}{p!} \left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1) + \theta\, \mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma) \right) \right)^2 < +\infty, \tag{67}$$

$$\sum_{k=1}^{+\infty} \left(\sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p-\frac{1}{2}}}{p!}\, \mathcal{M}(\beta,\gamma,p,\theta,\beta p+2) \right)^2 < +\infty, \tag{68}$$

$$\sum_{k=1}^{+\infty} \left(\sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{1}{2}}}{p!} \left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1) + \theta\, \mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma) \right) \right)^2 < +\infty, \tag{69}$$

$$\sum_{k=1}^{+\infty} \left(\sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{1}{2}}}{p!}\, \mathcal{M}(\beta,\gamma,p,\theta,\beta p+2) \right)^2 < +\infty, \tag{70}$$

where $\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1)$, $\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)$, and $\mathcal{M}(\beta,\gamma,p,\theta,\beta p+2)$ are finite positive constants given in (33) and depending only on β, γ, p, and θ. Then, the homogeneous fractional Sturm-Liouville telegraph equation associated to (65) and subject to the conditions (40), (41), and (66) has a continuous solution $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ given by the series

$$f(x,t) = \sum_{k=1}^{+\infty} \left(\langle y_k, g_0 \rangle\; u_k(t) + \langle y_k, g_1 \rangle\; v_k(t) \right)\, y_k(x), \tag{71}$$

where u and v are given by (35) and (36) with $\lambda = \lambda_k$, respectively, and providing that for $i = 0,1$ and $j = 1, \dots, n$

$$g_i \in \overline{span(Y)} \subseteq L_r^2(\Omega), \qquad {}^{RL}\widehat{L}_{\nu}^{\alpha}\, g_i \in L_r^2(\Omega), \qquad I_{a_j^+}^{1-\alpha_j} g_i(x) \Big|_{x_j = b_j} = 0. \tag{72}$$

Proof: In the proof we use the method of separation of variables, i.e., we seek for a particular solution of the equation

$${}_{0^+}^{\;C}\partial_t^{\beta} f(x,t) - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma} f(x,t) = -{}^{RL}\widehat{L}_{\nu}^{\alpha} f(x,t), \tag{73}$$

subject to the conditions (40), (41) and (66), in the form

$$f(x,t) = T(t)\, y(x), \qquad (x,t) \in \Omega \times \mathbb{R}^+. \tag{74}$$

Plugging (74) into (73) leads to

$$
\begin{aligned}
& {}_{0^+}^{\;C}\partial_t^{\beta}\, T(t)\, y(x) - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma}\, T(t)\, y(x) = -\frac{1}{r(x)} \left[-\left({}^{RL}\nabla_{b^-}^{\alpha} \cdot \left(\mu(x)\, {}^{RL}\nabla_{a^+}^{\alpha}\, T(t)\, y(x) \right) \right) + \nu(x)\, T(t)\, y(x) \right] \\
\Leftrightarrow\; & y(x) \left[{}_{0^+}^{\;C}\partial_t^{\beta}\, T(t) - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma}\, T(t) \right] = -\frac{T(t)}{r(x)} \left[-\left({}^{RL}\nabla_{b^-}^{\alpha} \cdot \left(\mu(x)\, {}^{RL}\nabla_{a^+}^{\alpha}\, y(x) \right) \right) + \nu(x)\, y(x) \right] \\
\Leftrightarrow\; & \frac{1}{T(t)} \left[{}_{0^+}^{\;C}\partial_t^{\beta}\, T(t) - \theta\; {}_{0^+}^{\;C}\partial_t^{\gamma}\, T(t) \right] = -\frac{1}{r(x)\, y(x)} \left[-\left({}^{RL}\nabla_{b^-}^{\alpha} \cdot \left(\mu(x)\, {}^{RL}\nabla_{a^+}^{\alpha}\, y(x) \right) \right) + \nu(x)\, y(x) \right] = -\lambda,
\end{aligned}
$$

where $\lambda > 0$ is the separation constant not depending on the variables x and t. We obtain the following fractional differential equations

$$\left[{}^{C}_{0+}\partial_t^{\beta} T(t) - \theta\ {}^{C}_{0+}\partial_t^{\gamma} T(t) \right] = -\lambda T(t), \tag{75}$$

$$\left[- \left({}^{RL}\nabla^{\alpha}_{b-} \cdot \left(\mu(x)\, {}^{RL}\nabla^{\alpha}_{a+}\, y(x) \right) \right) + \nu(x)\, y(x) \right] = \lambda\, r(x)\, y(x). \tag{76}$$

Taking into account Theorem 2.7, we have that the solution of (75) is given by

$$T(t) = c\, u(t) + d\, v(t), \tag{77}$$

where c and d are real constants, and u and v are given by (35) and (36), respectively. Moreover, equation (76) is the fractional Sturm-Liouville equation (39) and by Theorem 3.1 there exists an infinite increasing sequence of eigenvalues $\lambda_1 < \lambda_2 < \ldots$ and the correspondent sequence of eigenfunctions $y_1(x), y_2(x), \ldots$ for (76). Therefore,

$$f_k(x,t) = \left(c_k\, u_k(t) + d_k\, v_k(t) \right)\, y_k(x), \qquad k = 1, 2, \ldots,$$

where u_k and v_k are given by (35) and (36), respectively, with $\lambda = \lambda_k$. Substituting in (74), we get

$$f(x,t) = \sum_{k=1}^{+\infty} \left(c_k\, u_k(t) + d_k\, v_k(t) \right) y_k(x).$$

Now, we determine the coefficients c_k. Since $u_k(0) = 1$ and $v_k(0) = 0$ (see (37)), we obtain from the first initial condition in (66) that

$$g_0(x) = f(x,0) = \sum_{k=1}^{+\infty} c_k\, y_k(x). \tag{78}$$

Multiplying (78) by $y_l(x)\, r(x)$, integrating over Ω and using the orthogonality condition for the eigenfunctions we obtain

$$\begin{aligned} & y_l(x)\, g_0(x)\, r(x) = \sum_{k=1}^{+\infty} c_k\, y_k(x)\, y_l(x)\, r(x) \\ \Leftrightarrow\ & \int_\Omega y_l(x)\, g_0(x)\, r(x)\, dx = \sum_{k=1}^{+\infty} c_k \int_\Omega y_k(x)\, y_l(x)\, r(x)\, dx \\ \Leftrightarrow\ & \langle y_l,\, g_0 \rangle = c_l. \end{aligned}$$

Since l is arbitrary, we conclude that $c_k = \langle y_k,\, g_0 \rangle$. For the case of d_k, we have from the differentiation rule (31) that

$$u_k'(t) = -\lambda_k\, t^{\beta-1}\, E_{(\beta-\gamma,\beta),\,\beta}\left(\theta\, t^{\beta-\gamma},\, -\lambda_k\, t^{\beta} \right) \qquad \text{and} \qquad v_k'(t) = E_{(\beta-\gamma,\beta),\,1}\left(\theta\, t^{\beta-\gamma},\, -\lambda_k\, t^{\beta} \right),$$

and hence $u_k'(0) = 0$ and $v_k'(0) = 1$. Then, proceeding in a similar way as for the coefficients c_k, but considering the second initial condition in (66), we get that $\langle y_k,\, g_1 \rangle = d_k$. We want to show that under assumptions (67), (68), (69), (70), and (72) the series representing f is convergent in $\Omega \times \mathbb{R}^+$. We start obtaining some important relations for the coefficients $\langle y_k,\, g_0 \rangle$ and $\langle y_k,\, g_1 \rangle$. For the coefficients $\langle y_k,\, g_0 \rangle$, we have

$$\begin{aligned} \langle y_k,\, g_0 \rangle &= \int_\Omega y_k(x)\, g_0(x)\, r(x)\, dx \\ &= \frac{1}{\lambda_k} \int_\Omega \frac{1}{r(x)} \left(- \left({}^{RL}\nabla^{\alpha}_{b-} \cdot \left(\mu(x)\, {}^{RL}\nabla^{\alpha}_{a+} y_k(x) \right) \right) + \nu(x)\, y_k(x) \right)\, g_0(x)\, r(x)\, dx \\ &= \frac{1}{\lambda_k} \int_\Omega {}^{RL}L^{\alpha}_{\nu} y_k(x)\, g_0(x)\, dx \\ &= \frac{1}{\lambda_k} \int_\Omega y_k(x)\, {}^{RL}L^{\alpha}_{\nu}\, g_0(x)\, dx, \end{aligned}$$

where the last equality is due to the fact that in [3] it is proved that ${}^{RL}L^{\alpha}_{\nu}$ is a self-adjoint operator. Multiplying and dividing the right-hand side of the last equality by $r(x)$ leads to

$$\langle y_k,\, g_0\rangle = \frac{1}{\lambda_k}\int_{\Omega} y_k(x)\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_0(x)\, r(x)\, dx \;=\frac{1}{\lambda_k}\left\langle y_k,\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_0\right\rangle.$$

Thus, the following relation is valid for the coefficients c_k

$$|c_k| \;=|\langle y_k,\, g_0\rangle| \;=\frac{\left|\left\langle y_k,\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_0\right\rangle\right|}{|\lambda_k|}. \tag{79}$$

In a similar way, we obtain the following relation for the coefficients d_k

$$|d_k| \;=|\langle y_k,\, g_1\rangle| \;=\frac{\left|\left\langle y_k,\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_1\right\rangle\right|}{|\lambda_k|}. \tag{80}$$

Moreover, taking into account the inequality (34), we have the following estimates for u_k and v_k

$$\begin{aligned}
|u_k(t)| \le& \sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\, t^{\beta p}\, t^{-\beta p}\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1)\\
&+\theta\sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\, t^{\beta p+\beta-\gamma}\, t^{-\beta p-\beta+\gamma}\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\\
=& \sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\left[\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1)+\theta\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\right],
\end{aligned}\tag{81}$$

$$\begin{aligned}
|v_k(t)| \le& \sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\, t^{\beta p+1}\, t^{-\beta p-1}\mathcal{M}(\beta,\gamma,p,\theta,\beta p+2)\\
=& \sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+2).
\end{aligned}\tag{82}$$

Now we are ready to prove the convergence of the series (71). Taking into account (79), (80), (81), (82), Lemma 3.2, and considering $k\ge k_0$, we have

$$\begin{aligned}
&|(\langle y_k,\, g_0\rangle\; u_k(t)+\langle y_k,\, g_1\rangle\; v_k(t))\; y_k(x)|\\
&\le |\langle y_k,\, g_0\rangle|\; |u_k(t)|\; |y_k(x)| + |\langle y_k,\, g_1\rangle|\; |v_k(t)|\; |y_k(x)|\\
&\le \frac{\left|\left\langle y_k,\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_0\right\rangle\right|}{|\lambda_k|}\sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1)+\theta\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\right)|y_k(x)|\\
&\quad+\frac{\left|\left\langle y_k,\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_1\right\rangle\right|}{|\lambda_k|}\sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+2)\; |y_k(x)|\\
&= \frac{\left|\left\langle y_k,\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_0\right\rangle\right|}{\sqrt{|\lambda_k|}}\;\frac{|y_k(x)|}{\sqrt{|\lambda_k|}}\sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1)+\theta\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\right)\\
&\quad+\frac{\left|\left\langle y_k,\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_1\right\rangle\right|}{\sqrt{|\lambda_k|}}\;\frac{|y_k(x)|}{\sqrt{|\lambda_k|}}\sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+2)\\
&\le M_0\,\left|\left\langle y_k,\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_0\right\rangle\right|\sum_{p=0}^{+\infty}\frac{|\lambda_k|^{p-\frac{1}{2}}}{p!}\left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1)+\theta\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\right)\\
&\quad+M_0\,\left|\left\langle y_k,\, {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_1\right\rangle\right|\sum_{p=0}^{+\infty}\frac{|\lambda_k|^{p-\frac{1}{2}}}{p!}\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+2).
\end{aligned}\tag{83}$$

By the Cauchy-Schwarz inequality for series we can prove the convergence of the following series

$$\sum_{k=k_0}^{+\infty}\left[M_0\left|\left\langle y_k,\,{}^{RL}\widehat{L}_\nu^\alpha\,g_0\right\rangle\right|\sum_{p=0}^{+\infty}\frac{|\lambda_k|^{p-\frac{1}{2}}}{p!}\left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1)+\theta\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\right)\right]$$

$$+\sum_{k=k_0}^{+\infty}\left[M_0\left|\left\langle y_k,\,{}^{RL}\widehat{L}_\nu^\alpha\,g_1\right\rangle\right|\sum_{p=0}^{+\infty}\frac{|\lambda_k|^{p-\frac{1}{2}}}{p!}\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+2)\right]$$

$$\leq M_0\left(\sum_{k=k_0}^{+\infty}\left|\left\langle y_k,\,{}^{RL}\widehat{L}_\nu^\alpha\,g_0\right\rangle\right|^2\right)^{\frac{1}{2}}\left(\sum_{k=k_0}^{+\infty}\left(\sum_{p=0}^{+\infty}\frac{|\lambda_k|^{p-\frac{1}{2}}}{p!}\left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1)+\theta\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\right)\right)^2\right)^{\frac{1}{2}}$$

$$+M_0\left(\sum_{k=k_0}^{+\infty}\left|\left\langle y_k,\,{}^{RL}\widehat{L}_\nu^\alpha\,g_1\right\rangle\right|^2\right)^{\frac{1}{2}}\left(\sum_{k=k_0}^{+\infty}\left(\sum_{p=0}^{+\infty}\frac{|\lambda_k|^{p-\frac{1}{2}}}{p!}\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+2)\right)^2\right)^{\frac{1}{2}},$$

where the series in the right-hand side are convergent because ${}^{RL}\widehat{L}_\nu^\alpha\,g_0, {}^{RL}\widehat{L}_\nu^\alpha\,g_1 \in L_r^2(\Omega)$ and due to assumptions (67) and (68). Hence, by the Weierstrass criterion for uniform convergence, the series defining the solution

$$f(x,t)=\sum_{k=1}^{+\infty}\left(\langle y_k,\,g_0\rangle\;u_k(t)+\langle y_k,\,g_1\rangle\;v_k(t)\right)\,y_k(x)$$

is uniformly convergent in any compact subset of $\Omega\times\mathbb{R}^+$. This fact implies that the function f is continuous in $\Omega\times\mathbb{R}^+$. Finally, we shall prove that the series defining f can be differentiated term by term using the Caputo fractional derivative with respect to the time variable, or using the fractional Sturm-Liouville operator ${}^{RL}\widehat{L}_\nu^\alpha$. From (75) and (76), we have that

$$\begin{aligned}\left({}_{0^+}^{\;C}\partial_t^\beta-\theta\;{}_{0^+}^{\;C}\partial_t^\gamma\right)f(x,t)&=\sum_{k=1}^{+\infty}y_k(x)\,\left({}_{0^+}^{\;C}\partial_t^\beta-\theta\;{}_{0^+}^{\;C}\partial_t^\gamma\right)\left(\langle y_k,\,g_0\rangle\;u_k(t)+\langle y_k,\,g_1\rangle\;v_k(t)\right)\\&=-\sum_{k=1}^{+\infty}\lambda_k\,\left(\langle y_k,\,g_0\rangle\;u_k(t)+\langle y_k,\,g_1\rangle\;v_k(t)\right)\,y_k(x)\end{aligned}\tag{84}$$

and

$$\begin{aligned}{}^{RL}\widehat{L}_\nu^\alpha f(x,t)&=\sum_{k=1}^{+\infty}\left(\langle y_k,\,g_0\rangle\;u_k(t)+\langle y_k,\,g_1\rangle\;v_k(t)\right)\,{}^{RL}\widehat{L}_\nu^\alpha y_k(x)\\&=\sum_{k=1}^{+\infty}\lambda_k\,\left(\langle y_k,\,g_0\rangle\;u_k(t)+\langle y_k,\,g_1\rangle\;v_k(t)\right)\,y_k(x).\end{aligned}\tag{85}$$

Since expression (84) and (85) are equal up to a sign, we study only the convergence of (84) (for expression (85) the analysis and conclusions are the same). As it was done previously, taking into account (79), (80), (81), (82), Lemma 3.2, and considering $k\geq k_0$, we have

$$\begin{aligned}&|\lambda_k\,\left(\langle y_k,\,g_0\rangle\;u_k(t)+\langle y_k,\,g_1\rangle\;v_k(t)\right)\,y_k(x)|\\&\quad\leq|\lambda_k|\;|\langle y_k,\,g_0\rangle|\;|u_k(t)|\;|y_k(x)|+|\lambda_k|\;|\langle y_k,\,g_1\rangle|\;|v_k(t)|\;|y_k(x)|\\&\quad\leq|\lambda_k|\;\frac{\left|\left\langle y_k,\,{}^{RL}\widehat{L}_\nu^\alpha\,g_0\right\rangle\right|}{|\lambda_k|}\sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1)+\theta\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\right)|y_k(x)|\\&\qquad+|\lambda_k|\;\frac{\left|\left\langle y_k,\,{}^{RL}\widehat{L}_\nu^\alpha\,g_1\right\rangle\right|}{|\lambda_k|}\sum_{p=0}^{+\infty}\frac{|\lambda_k|^p}{p!}\,\mathcal{M}(\beta,\gamma,p,\theta,\beta p+2)\;|y_k(x)|\end{aligned}$$

$$= \left|\left\langle y_k,\ {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_0\right\rangle\right| \frac{|y_k(x)|}{\sqrt{|\lambda_k|}} \sqrt{|\lambda_k|} \sum_{p=0}^{+\infty} \frac{|\lambda_k|^p}{p!} \left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1) + \theta\, \mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\right)$$

$$+ \left|\left\langle y_k,\ {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_1\right\rangle\right| \frac{|y_k(x)|}{\sqrt{|\lambda_k|}} \sqrt{|\lambda_k|} \sum_{p=0}^{+\infty} \frac{|\lambda_k|^p}{p!} \mathcal{M}(\beta,\gamma,p,\theta,\beta p+2)$$

$$\leq M_0 \left|\left\langle y_k,\ {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_0\right\rangle\right| \sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{1}{2}}}{p!} \left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+1) + \theta\, \mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta-\gamma)\right)$$

$$+ M_0 \left|\left\langle y_k,\ {}^{RL}\widehat{L}^{\alpha}_{\nu}\, g_1\right\rangle\right| \sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{1}{2}}}{p!} \mathcal{M}(\beta,\gamma,p,\theta,\beta p+2). \tag{86}$$

Again, applying similar arguments and calculations as before, we observe from (86) that series (84) and (85) are uniformly convergent in any compact subset of $\Omega \times \mathbb{R}^+$. By Lemmas 3.3 and 3.4, we can calculate $\left({}^{C}_{0^+}\partial^{\beta}_t - \theta\ {}^{C}_{0^+}\partial^{\gamma}_t \right) f(x,t)$ as well as ${}^{RL}\widehat{L}^{\alpha}_{\nu} f(x,t)$ term by term verifying that the series (71) fulfills (73).

■

Remark 4.2 *Due to Remark 2.8, the series representation (71) in Theorem 4.1 for the solution of the homogeneous space-time fractional telegraph equation (73) coincides with the correspondent one presented in [6] when we consider* $r(x) = \mu(x) = 1$, $\theta = -2\lambda$, $\beta = \alpha$, *and* $\gamma = \delta$ *in Theorem 4.1, and* $c = 1$, $\beta = 2$, *and* $f(x,t) = 0$ *in Theorem 1 in [6].*

In our second main result, we study the existence of a solution of the non-homogeneous equation (65). For that, we additionally assume that the function $h \in L^2(\Omega \times (0,T))$ for $T > 0$ arbitrary, is given, in the form of a series, by

$$h(x,t) = \sum_{k=1}^{+\infty} A_k(t)\, y_k(x). \tag{87}$$

Theorem 4.3 *Let us assume that the fractional Sturm-Liouville problem (39)-(41) described in Section 3.1, has eigenfunctions* y_k *and eigenvalues* λ_k *obeying to conditions (67)-(70), and also to the additional condition*

$$\sum_{k=1}^{+\infty} \left(\sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{3}{2}}}{p!} \mathcal{M}(\beta,\gamma,p,\theta,\beta p+1+\beta) \right)^2 < +\infty. \tag{88}$$

Moreover, let us assume that the function h *given by (87) is such that*

$$\sum_{k=1}^{+\infty} \|A'_k\|^2_{L_{(0,T)}} \tag{89}$$

is convergent in any interval $(0,T)$ *for* $T > 0$. *Then the non-homogeneous space-time fractional telegraph equation (65) with conditions (40), (41), and (66) has a continuous solution* $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ *given by the series*

$$f(x,t) = \sum_{k=1}^{+\infty} \left[\langle y_k,\, g_0\rangle\ u_k(t) + \langle y_k,\, g_1\rangle\ v_k(t) + \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\, A_k(w)\, dw \right] y_k(x), \tag{90}$$

where $u_k(t)$ *and* $v_k(t)$ *are given by (35) and (36), respectively, with* $\lambda = \lambda_k$, *and* $G_{\beta,\gamma;\theta,-\lambda_k}(z)$ *is given by (38), and provided that (72) is satisfied.*

Proof: Similarly to the proof of Theorem 4.1, we look for a series solution of the form

$$f(x,t) = \sum_{k=1}^{+\infty} T_k(t)\, y_k(x), \tag{91}$$

where y_k, with $k \in \mathbb{N}$, are the orthonormal eigenfunctions of the Sturm-Liouville problem (39)-(41). Substituting (91) into (65) we arrive to the following set of non-homogeneous linear fractional differential equations for the coefficients T_k

$$ {}^{C}_{0^+}\partial_t^{\beta} T_k(t) - \theta\ {}^{C}_{0^+}\partial_t^{\beta} T_k(t) = -\lambda_k\, T_k(t) + A_k(t). \tag{92} $$

From Theorem 2.9 we have that the solution of (92) is given by

$$ T_k(t) = c_k\, u_k(t) + d_k\, v_k(t) + \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\, A_k(w)\, dw, $$

where $u_k(t)$, $v_k(t)$ and $G_{\beta,\gamma;\theta,-\lambda_k}(z)$ are given by (35), (36), and (38) respectively, with $\lambda = \lambda_k$. Therefore, the solution f given by (91) takes the form

$$ f(x,t) = \sum_{k=1}^{+\infty} \left[c_k\, u_k(t) + d_k\, v_k(t) + \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\, A_k(w)\, dw \right] y_k(x). \tag{93} $$

By the initial conditions (66) we conclude, similarly as in the proof of Theorem 4.1, that $c_k = \langle g_0,\, y_k\rangle$ and $d_k = \langle g_1,\, y_k\rangle$. The first part of the series (93), corresponding to the solution of the homogeneous equation, is uniformly convergent on any compact subset of $\Omega \times \mathbb{R}^+$ and can be differentiated term by term using the fractional Caputo derivatives ${}^{C}_{0^+}\partial_t^{\beta} - \theta\ {}^{C}_{0^+}\partial_t^{\gamma}$ or using the fractional Sturm-Liouville operator ${}^{RL}\widehat{L}_{\nu}^{\alpha}$ in $\Omega \times \mathbb{R}^+$ (see proof of Theorem 4.1). Hence, we only have to prove the similar result for the second part of the series (93). Taking into account Theorem 2.7 in [2] about the interchange of limit operation and fractional integration, we conclude that it is enough to show that the following derivative series

$$ \begin{aligned} &\left({}^{C}_{0^+}\partial_t^{\beta} - \theta\ {}^{C}_{0^+}\partial_t^{\gamma} \right) \left[\sum_{k=1}^{+\infty} \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\, A_k(w)\, dw\, y_k(x) \right] \\ &= \sum_{k=1}^{+\infty} \left({}^{C}_{0^+}\partial_t^{\beta} - \theta\ {}^{C}_{0^+}\partial_t^{\gamma} \right) \left[\int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\, A_k(w)\, dw \right] y_k(x) \\ &= \sum_{k=1}^{+\infty} \left[-\lambda_k \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\, A_k(w)\, dw + A_k(t) \right] y_k(x) \end{aligned} \tag{94} $$

is uniformly convergent in any compact subset of $\Omega \times \mathbb{R}^+$. On the one hand, we have that

$$ \begin{aligned} &-\lambda_k \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\, A_k(w)\, dw \\ &= -\lambda_k \int_0^t w^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(w)\, A_k(t-w)\, dw \\ &= -\lambda_k \sum_{p=0}^{+\infty} \frac{(-\lambda_k)^p}{p!} \int_0^t w^{\beta p+\beta-1}\, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta, \beta-\gamma) \end{array} \middle|\, \theta\, w^{\beta-\gamma} \right] A_k(t-w)\, dw. \end{aligned} \tag{95} $$

On the other hand, taking into account the differentiation rule (24), we have that

$$ \begin{aligned} w^{\beta p+\beta-1}\, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta, \beta-\gamma) \end{array} \middle|\, \theta\, w^{\beta-\gamma} \right] &= w^{(\beta p+\beta+1)-2}\, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ ((\beta p+\beta+1)-1, \beta-\gamma) \end{array} \middle|\, \theta\, w^{\beta-\gamma} \right] \\ &= \frac{d}{dw} \left[w^{\beta p+\beta}\, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle|\, \theta\, w^{\beta-\gamma} \right] \right]. \end{aligned} \tag{96} $$

Therefore, from (96), we conclude that we can rewrite the inner integral in the right hand-side of (94) using the classical formula of integration by parts. Hence, we get

$$ -\lambda_k \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\, A_k(w)\, dw = -\lambda_k \int_0^t w^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(w)\, A_k(t-w)\, dw $$

$$= -\lambda_k \sum_{p=0}^{+\infty} \frac{(-\lambda_k)^p}{p!} \left[w^{\beta p+\beta} \, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, w^{\beta-\gamma} \right] A_k(t-w) \Bigg|_{w=0}^{w=t} \right.$$

$$\left. - \int_0^t w^{\beta p+\beta} \, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, w^{\beta-\gamma} \right] \frac{d}{dw} A_k(t-w)\, dw \right]$$

$$= -\lambda_k \sum_{p=0}^{+\infty} \frac{(-\lambda_k)^p}{p!} \left[t^{\beta p+\beta} \, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, t^{\beta-\gamma} \right] A_k(0) - 0 \right]$$

$$+ \lambda_k \sum_{p=0}^{+\infty} \frac{(-\lambda_k)^p}{p!} \int_0^t w^{\beta p+\beta} \, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, w^{\beta-\gamma} \right] \frac{d}{dw} A_k(t-w)\, dw.$$

Therefore, series (94) takes the form

$$\sum_{k=1}^{+\infty} \sum_{p=0}^{+\infty} \frac{(-\lambda_k)^{p+1}}{p!} t^{\beta p+\beta} \, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, t^{\beta-\gamma} \right] A_k(0)\, y_k(x)$$

$$- \sum_{k=1}^{+\infty} \sum_{p=0}^{+\infty} \frac{(-\lambda_k)^{p+1}}{p!} \int_0^t w^{\beta p+\beta} \, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, w^{\beta-\gamma} \right] \frac{d}{dw} A_k(t-w)\, dw\, y_k(x)$$

$$+ \sum_{k=1}^{+\infty} A_k(t)\, y_k(x). \tag{97}$$

Taking into account Lemma 3.2, estimate (34) for ${}_1\Psi_1$, the Cauchy-Schwarz inequality for series, and assumption (88), we have the following estimate for the first series of (97)

$$\left| \sum_{k=k_0}^{+\infty} \sum_{p=0}^{+\infty} \frac{(-\lambda_k)^{p+1}}{p!} t^{\beta p+\beta} \, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, t^{\beta-\gamma} \right] A_k(0)\, y_k(x) \right|$$

$$\leq \sum_{k=k_0}^{+\infty} \sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+1}}{p!} t^{\beta p+\beta} \left| {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, t^{\beta-\gamma} \right] \right| |A_k(0)|\ |y_k(x)|$$

$$\leq \sum_{k=k_0}^{+\infty} \sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+1}}{p!} t^{\beta p+\beta}\, t^{-\beta p-\beta}\, \mathcal{M}(\beta,\gamma,p,\theta,\beta p+\beta+1)\ |A_k(0)|\ M_0 \sqrt{|\lambda_k|}$$

$$= M_0 \sum_{k=k_0}^{+\infty} \sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{3}{2}}}{p!} \mathcal{M}(\beta,\gamma,p,\theta,\beta p+\beta+1)\ |A_k(0)|$$

$$\leq M_0 \left(\sum_{k=k_0}^{+\infty} \left(\sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{3}{2}}}{p!} \mathcal{M}(\beta,\gamma,p,\theta,\beta p+\beta+1) \right)^2 \right)^{\frac{1}{2}} \left(\sum_{k=k_0}^{+\infty} |A_k(0)|^2 \right)^{\frac{1}{2}} < +\infty, \tag{98}$$

whenever $k \geq k_0$. For the second series in (97), taking into account Lemma 3.2, estimate (34) for ${}_1\Psi_1$, the Cauchy-Schwarz inequality for series, and assumptions (88) and (89), we have that

$$\left| - \sum_{k=k_0}^{+\infty} \sum_{p=0}^{+\infty} \left[\frac{(-\lambda_k)^{p+1}}{p!} \int_0^t w^{\beta p+\beta} \, {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, w^{\beta-\gamma} \right] \frac{d}{dw} A_k(t-w)\, dw + A_k(t) \right] y_k(x) \right|$$

$$\leq \sum_{k=k_0}^{+\infty} \sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+1}}{p!} \int_0^t w^{\beta p+\beta} \left| {}_1\Psi_1 \left[\begin{array}{c} (p+1,1) \\ (\beta p+\beta+1, \beta-\gamma) \end{array} \middle| \, \theta\, w^{\beta-\gamma} \right] \right| \left| \frac{d}{dw} A_k(t-w) \right| dw\ |y_k(x)|$$

$$\leq \sum_{k=k_0}^{+\infty} \sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+1}}{p!} \int_0^t w^{\beta p+\beta}\, w^{-\beta p-\beta}\, \mathcal{M}(\beta,\gamma,p,\theta,\beta p+\beta+1) \left| \frac{d}{dw} A_k(t-w) \right| dw\, M_0 \sqrt{|\lambda_k|}$$

$$\leq M_0 \sum_{k=k_0}^{+\infty} \sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{3}{2}}}{p!} \left(\int_0^t \left(\mathcal{M}(\beta,\gamma,p,\theta,\beta p+\beta+1)\right)^2 \, dw \right)^{\frac{1}{2}} \left(\int_0^t \left| \frac{d}{dw} A_k(t-w) \right|^2 dw \right)^{\frac{1}{2}}$$

$$\leq M_0 \sum_{k=k_0}^{+\infty} \sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{3}{2}}}{p!} \mathcal{M}(\beta,\gamma,p,\theta,\beta p+\beta+1) \sqrt{t} \left\| \frac{d}{dw} A_k(t-w) \right\|_{L^2(0,T)}$$

$$\leq M_0 \sqrt{t} \left(\sum_{k=k_0}^{+\infty} \left(\sum_{p=0}^{+\infty} \frac{|\lambda_k|^{p+\frac{3}{2}}}{p!} \mathcal{M}(\beta,\gamma,p,\theta,\beta p+\beta+1) \right)^2 \right)^{\frac{1}{2}} \left(\sum_{k=k_0}^{+\infty} \left\| \frac{d}{dw} A_k(t-w) \right\|^2_{L^2(0,T)} \right)^{\frac{1}{2}} \quad < +\infty, \tag{99}$$

whenever $k \geq k_0$ and in any $(0,T)$, $T > 0$. For the third series in (97) we have

$$\left| \sum_{k=1}^{+\infty} A_k(t) \, y_k(x) \right| = |h(x,t)| \qquad < +\infty, \tag{100}$$

because by hypothesis $h \in L^2(\Omega \times (0,T))$, $T > 0$, and it is given in the form of the convergent series (87). Taking into account the conclusions obtained in (98), (99), and (100), we get our result by the Weierstrass criterion for uniform convergence.

■

Remark 4.4 *Due to Remark 2.10, the series representation (90) in Theorem 4.3 for the solution of the non-homogeneous time-space-fractional telegraph equation (65) coincides with the correspondent one presented in [6] when we consider $r(x) = \mu(x) = 1$, $\theta = -2\lambda$, $\beta = \alpha$, and $\gamma = \delta$ in Theorem 4.3, and $c = 1$ and $\beta = 2$ in Theorem 1 in [6].*

5 Examples

In this section we present some examples that illustrate some of our results.

Example 5.1 *Let us consider the following time-fractional Sturm-Liouville problem*

$$ {}_{0^+}^{\;C}\partial_t^\beta f(x,t) - \theta \; {}_{0^+}^{\;C}\partial_t^\gamma f(x,t) = \frac{\partial}{\partial x} \left(x^2 \frac{\partial f}{\partial x}(x,t) \right), \tag{101}$$

where $x \in [1,e]$ and $t \in \mathbb{R}^+$, and subject to the boundary and initial conditions

$$f(1,t) = f(e,t) = 0, \qquad t \in \mathbb{R}^+ \tag{102}$$

$$f(x,0) = g_0(x), \qquad \frac{\partial f}{\partial t}(x,0) = g_1(x) \qquad g_0, g_1 \in C_B([1,e]). \tag{103}$$

Problem (101)-(103) is a particular case of the problem studied in Theorem 4.1 with $n = 1$, $\alpha = 1$, $\mu(x) = x^2$, $\nu(x) = 0$, and $r(x) = 1$. For the classical Sturm-Liouville problem with $\mu(x) = x^2$, $\nu(x) = 0$, and $r(x) = 1$, the eigenvalues are of the form $\lambda_k = -\left(k^2\pi^2 + \frac{1}{4}\right)$ and the corresponding orthonormal eigenfunctions are

$$y_k(x) = \sqrt{\frac{2}{x}} \sin(k\pi \ln x), \qquad k = 1, 2, \ldots.$$

Theorem 4.1 implies that the solution of problem (101)-(103) is given by the series

$$f(x,t) = \sqrt{\frac{2}{x}} \sum_{k=1}^{+\infty} \left(\left\langle \sqrt{\frac{2}{x}} \sin(k\pi \ln x), g_0 \right\rangle u_k(t) + \left\langle \sqrt{\frac{2}{x}} \sin(k\pi \ln x), g_1 \right\rangle v_k(t) \right) \sin(k\pi \ln x),$$

where u and v are given by (35) and (36) with $\lambda = -\left(k^2\pi^2 + \frac{1}{4}\right)$.

Example 5.2 *Let us consider the following time-fractional Sturm-Liouville problem*

$$ {}_{0^+}^{\;C}\partial_t^{\beta} f(x,t) - \theta \; {}_{0^+}^{\;C}\partial_t^{\gamma} f(x,t) = x\,\frac{\partial}{\partial x}\left(x\,\frac{\partial f}{\partial x}(x,t)\right), \tag{104} $$

where $x \in [1,e]$ and $t \in \mathbb{R}^+$, and subject to the boundary and initial conditions

$$ f(1,t) = f(e,t) = 0, \qquad t \in \mathbb{R}^+ \tag{105} $$

$$ f(x,0) = g_0(x), \qquad \frac{\partial f}{\partial t}(x,0) = g_1(x) \qquad g_0, g_1 \in C_B\left([1,e]\right). \tag{106} $$

Problem (104)-(106) is a particular case of the problem studied in Theorem 4.1 with $n = 1$, $\alpha = 1$, $\mu(x) = x$, $\nu(x) = 0$, and $r(x) = \frac{1}{x}$. For the classical Sturm-Liouville problem with $\mu(x) = x$, $\nu(x) = 0$, and $r(x) = \frac{1}{x}$, eigenvalues are of the form $\lambda_k = -k^2\pi^2$ and the corresponding orthonormal eigenfunctions are

$$ y_k(x) = \sqrt{\frac{4k^2\pi^2+1}{2k^2\pi^2(e-1)}}\,\sin(k\pi\ln x), \qquad k = 1,2,\ldots. $$

Theorem 4.1 implies that the solution of problem (104)-(106) is given by the series

$$ f(x,t) = \sum_{k=1}^{+\infty} \left(\langle \sin(k\pi\ln x), g_0\rangle\; u_k(t) + \langle \sin(k\pi\ln x), g_1\rangle\; v_k(t)\right) \frac{4k^2\pi^2+1}{2k^2\pi^2(e-1)}\,\sin(k\pi\ln x), $$

where u and v are given by (35) and (36) with $\lambda = -k^2\pi^2$.

Let us now consider the following non-homogeneous equation

$$ {}_{0^+}^{\;C}\partial_t^{\beta} f(x,t) - \theta \; {}_{0^+}^{\;C}\partial_t^{\gamma} f(x,t) = x\,\frac{\partial}{\partial x}\left(x\,\frac{\partial f}{\partial x}(x,t)\right) + \sum_{l=1}^{m} A_l\,\sin(l\,\pi\ln x), \tag{107} $$

where $A_l \in \mathbb{R}$, $m \in \mathbb{N}$, and subject to conditions (105)-(106). By Theorem 4.3, with $n = 1$, its solution is given by

$$ \begin{aligned} f(x,t) = {} & \sum_{k=1}^{+\infty} \left(\langle \sin(k\pi\ln x),\, g_0\rangle\; u_k(t) + \langle \sin(k\pi\ln x),\, g_1\rangle\; v_k(t)\right) \frac{4k^2\pi^2+1}{2k^2\pi^2(e-1)}\,\sin(k\pi\ln x) \\ & + \sum_{l=1}^{m} A_l\,\sin(l\pi\ln x) \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\,dw, \end{aligned} \tag{108} $$

where $u_k(t)$ and $v_k(t)$ are given by (35) and (36), respectively, with $\lambda = -k^2\pi^2$, and $G_{\beta,\gamma;\theta,-\lambda_k}(z)$ is given by (38). Applying

$$ \int_0^z t^{b-1}\, E_{(a_1,a_2),b}\left(w_1 t^{a_1},\, w_2 t^{a_2}\right)\,dt = t^b\, E_{(a_1,a_2),b+1}\left(w_1 t^{a_1},\, w_2 t^{a_2}\right), \qquad a_1, a_2, b \in \mathbb{R}^+, $$

to the integral part of (108) we get

$$ \begin{aligned} \int_0^t (t-w)^{\beta-1}\, G_{\beta,\gamma;\theta,-\lambda_k}(t-w)\,dw &= \int_0^t (t-w)^{\beta-1}\, E_{(\beta-\gamma,\,\beta),\,\beta}\left(\theta\,(t-w)^{\beta-\gamma},\, -\lambda\,(t-w)^{\beta}\right)\,dw \\ &= t^{\beta}\, E_{(\beta-\gamma,\,\beta),\,\beta+1}\left(\theta\, t^{\beta-\gamma},\, -\lambda\, t^{\beta}\right). \end{aligned} \tag{109} $$

Combining (108) with (109) we conclude that

$$ \begin{aligned} f(x,t) = {} & \sum_{k=1}^{+\infty} \left(\langle \sin(k\pi\ln x),\, g_0\rangle\; u_k(t) + \langle \sin(k\pi\ln x),\, g_1\rangle\; v_k(t)\right) \frac{4k^2\pi^2+1}{2k^2\pi^2(e-1)}\,\sin(k\pi\ln x) \\ & + \sum_{l=1}^{m} A_l\,\sin(l\pi\ln x)\, t^{\beta}\, E_{(\beta-\gamma,\,\beta),\,\beta+1}\left(\theta\, t^{\beta-\gamma},\, -\lambda\, t^{\beta}\right), \end{aligned} $$

is a solution to equation (107) with conditions (105)-(106).

Example 5.3 *Let us consider the following time-fractional telegraph equation*

$$ {}^C_{0^+}\partial_t^\beta f(x,t) - \theta \ {}^C_{0^+}\partial_t^\gamma f(x,t) = \Delta f(x,t), \tag{110}$$

where $x \in \Omega = [0,\pi]^n$ *and* $t \in \mathbb{R}^+$*, and subject to the boundary conditions*

$$ f(x,t)|_{x_i=0} = 0 = f(x,t)|_{x_i=\pi}, \qquad t \in \mathbb{R}^+, \quad i=1,\dots,n, \tag{111}$$

and the initial conditions

$$ f(x,0) = g_0(x), \qquad \frac{\partial f}{\partial t}(x,0) = g_1(x) \qquad g_0, g_1 \in C_B(\Omega). \tag{112}$$

We observe that (110)-(112) is a particular case of the problem studied in Theorem 4.1 with $\alpha = (1,\dots,1)$, $\mu(x) \equiv 1$, $\nu(x) \equiv 0$*, and* $r(x) = 1$*. Moreover, for the classical Sturm-Liouville equation in* $\mathbb{R}^n$ *the orthonormal eigenfunctions are given by*

$$ y_\lambda(x) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} \sin(k_i\, x_i), $$

where $\lambda = \sum_{i=1}^{n} k_i^2$ *and* $k_i \in \mathbb{N}$*. Considering a reordering of the values of* λ *we can obtain a new sequence* λ_k*, with* $k \in \mathbb{N}$*, that constitutes the infinite set of eigenvalues. Theorem 4.1 implies that the solution of problem (110)-(112) is given by the series*

$$ f(x,t) = \left(\frac{2}{\pi}\right)^{n} \sum_{k=1}^{+\infty} \left(\left\langle \prod_{i=1}^{n} \sin(k_i\, x_i), g_0 \right\rangle u_k(t) + \left\langle \prod_{i=1}^{n} \sin(k_i\, x_i), g_1 \right\rangle v_k(t) \right) \prod_{i=1}^{n} \sin(k_i\, x_i), $$

where u *and* v *are given by (35) and (36) with* $\lambda = \lambda_k$*.*

6 Conclusion

In this paper, we considered the homogeneous and non-homogeneous fractional Sturm-Liouville telegraph equation in $\mathbb{R}^n \times \mathbb{R}^+$ with appropriate boundary conditions, where the time-fractional derivatives are in the Caputo sense and the space-fractional derivatives are expressed in terms of a fractional Sturm-Liouville operator. We obtained a series representation for the solution via the method of separation of variables and we studied the conditions that guarantee the convergence of the series solution. Moreover, it is shown that the obtained series representation is equivalent to the obtained in [6] when we reduce the fractional Sturm-Liouville operator to the classical Laplace operator.

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