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Dynamics on certain sets of stochastic matrices

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Abstract We study iteration of polynomials on symmetric stochastic matrices. In particular, we focus on a certain one-parameter family of quadratic maps which exhibits chaotic behavior for a wide range of the parameters. The well-known dynamical behavior of the quadratic family on the interval, and its dependence on the parameter, is reproduced on the spectrum of the stochastic matrices. For certain subclasses of stochastic matrices the referred dynamical behavior is also obtained in the matrix entries. Since a stochastic matrix characterizes a Markov chain, we obtain a discrete dynamical system on the space of reversible Markov chains. Therefore, depending on the parameter, there are initial conditions for which the corresponding reversible Markov chains will lead under iteration to a fixed point, to a periodic point, or to an aperiodic point. Moreover, there are sensitivity to initial conditions and the coexistence of infinite repulsive periodic orbits, both features of chaos.

Keywords Matrix dynamics · Stochastic matrices · Iterated interval maps · Reversible Markov chains

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1 Introduction and preliminaries

Iteration on $M_n(\mathbb{R})$, the set of $n \times n$ real matrices, can be viewed as a natural generalization of iteration on $\mathbb{R}^n$ or $\mathbb{C}^n$, concerning the algebraic structure of matrix algebras. In [7] we have analyzed the iteration of the quadratic polynomial $z \to z^2 + c$ on $M_2(\mathbb{R})$; see also [8]. Note that this example itself contains, as a subcase, the complex iteration and the related well-known Mandelbrot set, with all its dynamical, analytical and combinatorial complexity, see [6] for details.

In the following we define the assumptions needed in order to obtain the iteration on the set of symmetric stochastic matrices. The iteration is made under a quadratic map (a one-parameter family), conjugated to the logistic map, for which the one-dimensional version exhibits complex behavior, in particular, the coexistence of an infinite number of periodic points and the sensitivity to initial conditions or, in other words, exhibits chaotic behavior, see [4].

Moreover, the symmetric stochastic matrices determine reversible Markov chains, which are used in many applications and also have great interest from a theoretical point of view.

The *stochastic* matrices are matrices $X = (x_{ij})_{i,j=1}^n$ with non-negative entries, i.e., $x_{ij} \geq 0$ for which $\sum_{j=1}^n x_{ij} = 1$. In this case, the vector $u := (1, 1, \dots, 1) \in \mathbb{R}^n$ is a right Perron eigenvector of X with Perron eigenvalue equal to 1 (spectral radius $\rho(X) = 1$).

Now, consider the polynomial map $g_\lambda(x) := 1 - \lambda x(1 - x)$, which is a modified logistic map so that $x = 1$ is a fixed point. This fact is essential in order to the iteration of a stochastic matrix still be a stochastic matrix. We denote by G_λ the induced matrix map $G_\lambda(X) := \mathbf{1} - \lambda X(\mathbf{1} - X)$, where $\mathbf{1}$ denotes the identity matrix, with $\lambda \in [0, 4]$.

In the present paper we consider the restriction of the matrix set to the set of $n \times n$ positive definite symmetric stochastic matrices (necessarily doubly-stochastic since they are symmetric and stochastic), which we denote by Ω_n. Therefore, a matrix X in Ω_n has non-negative entries, the Perron eigenvalue 1 and spectrum contained in $]0; 1]$.

To deal with Markov chains we would like to have $G_\lambda(X) \in \Omega_n$ whenever $X \in \Omega_n$. However, this is false, in general. Assuming $X \in \Omega_n$, then $G_\lambda(X)$ is necessarily positive definite, symmetric and will have u as Perron eigenvector associated with the Perron eigenvalue 1 (see Lemma 1 below). However, $G_\lambda(X)$ may have negative entries. The alternative is to consider the minimal invariant subset of Ω_n which, under iteration of G_λ, maintains the property of non-negativeness of the entries, so that the iterations are stochastic. This is not a new situation, since it is similar to the case of interval maps dynamics, when the invariant set, instead of the whole interval, is a Cantor set, a self-similar fractal subset of the interval, see for example [3].

The paper is organized as follows. In Sect. 2 we give preliminary general results on Ω_n, relating the periodic points of the dynamics on Ω_n with those on one-dimensional dynamics. We also study two particular cases, for $n = 2$ and $n = 3$. Finally, in Sect. 3 we focus on the dynamical behavior on the Markov chain set, considering a reducible case and a primitive case.

2 Discrete dynamics on the set of positive definite matrices

2.1 General results

Consider $X \in M_n(\mathbb{R})$. Let $sp(X)$ denote the spectrum of X and X^T denote the transpose matrix of X. The symmetric *positive definite* matrices are those which satisfy $X = Y^2$, for some matrix $Y \in M_n(\mathbb{R})$; a symmetric matrix is positive definite if and only if all its eigenvalues are positive.

We say that $X, Y \in M_n(\mathbb{R})$ are *equivalent* if and only if they have the same spectrum, counting multiplicities. Since every symmetric matrix is diagonalizable, two matrices X, Y are equivalent if and only if there is an invertible P such that $Y = PXP^{-1}$.

Consider the modified logistic map

$$\begin{aligned} G_\lambda : \Omega_n &\longrightarrow M_n(\mathbb{R}) \\ X &\mapsto \mathbf{1} - \lambda X(\mathbf{1} - X), \end{aligned}$$

where $\mathbf{1}$ denotes the identity matrix, with the real parameter $\lambda \in [0, 4]$.

Lemma 1 *Let $X \in \Omega_n$, then $G_\lambda(X)$ is symmetric, positive definite and $u := (1, \dots, 1) \in \mathbb{R}^n$ is the Perron eigenvector associated with the Perron eigenvalue* 1.

Proof We first show that $G_\lambda(X)$ is symmetric; as $X^T = X$, then

$$\begin{aligned} \big(G_\lambda(X)\big)^T &= \mathbf{1} - \lambda\big(\mathbf{1} - X^T\big)X^T \\ &= \mathbf{1} - \lambda X^T\big(\mathbf{1} - X^T\big) \\ &= \mathbf{1} - \lambda X(\mathbf{1} - X) = G_\lambda(X). \end{aligned}$$

Now, since X is diagonalizable, there is an invertible matrix P such that $D = P^{-1}XP$ is diagonal and $G_\lambda(X) = G_\lambda(PDP^{-1}) = PG_\lambda(D)P^{-1}$. Since $D = \mathrm{diag}(x_1, \dots, x_n)$ is diagonal and $x_i \in [0, 1]$, $i = 1, \dots, n$, we have $G_\lambda(D) = \mathrm{diag}(g_\lambda(x_1), \dots, g_\lambda(x_n))$. Therefore the spectrum of $G_\lambda(X)$ is contained in $[0, 1]$, and $G_\lambda(X)$ is positive definite. Finally,

$$G_\lambda(X)u = u - \lambda X(\mathbf{1} - X)u = u - \lambda X(u - Xu) = u,$$

since u is the Perron eigenvector associated with the Perron eigenvalue 1 of X. □

Notice that, in order for $G_\lambda(X) \in \Omega_n$, the only necessary condition that might fail is that some of the entries of $G_\lambda(X)$ might be negative. Now, let

$$\Lambda_n(\lambda) := \{X \in \Omega_n : G_\lambda^k(X) \in \Omega_n, \text{ for all } k \in \mathbb{N}\}.$$

This is an invariant set, that is $G_\lambda(\Lambda_n(\lambda)) = \Lambda_n(\lambda)$, and it is the analogue of the invariant Cantor set for the iteration on the interval, see [3].

In the following we analyze how the matrix dynamics in some aspects can be reduced to the one-dimensional dynamics on the spectrum. For every $X \in \Omega_n$ there is an invertible matrix P such that $D = P^{-1}XP$ is diagonal; in particular, the diagonal entries, which correspond to the spectrum of X, belong to the interval $[0, 1]$. We thus have

$$G_\lambda(X) = G_\lambda(PDP^{-1}) = PG_\lambda(D)P^{-1}.$$

Since $D = \text{diag}(x_1, \dots, x_n)$ is diagonal and $x_i \in [0, 1]$, $i = 1, \dots, n$, we have $G_\lambda(D) = \text{diag}(g_\lambda(x_1), \dots, g_\lambda(x_n))$. Moreover,

$$\begin{aligned} G_\lambda^k(X) &= G_\lambda^k(PDP^{-1}) = PG_\lambda^k(D)P^{-1} \\ &= P\,\text{diag}(g_\lambda^k(x_1), \dots, g_\lambda^k(x_n))P^{-1}. \end{aligned}$$

This last expression allows us to use the results of iteration on interval maps in order to obtain results on the matrix dynamics.

Proposition 1 *The matrix $X \in \Omega_n$ is a periodic point, with respect to G_λ, if and only if the $sp(X)$ is a set of periodic points with respect to g_λ. Moreover, the period of X is the minimum common multiple of the periods of the eigenvalues of X.*

Proof Let $X \in \Omega_n$, which is diagonalizable with a certain P such that $P^{-1}XP = \text{diag}(sp(X))$. If $sp(X)$ is a set of periodic points of g_λ, then there is a positive integer m (minimum common multiple of the periods) for which

$$\text{diag}(g_\lambda^m(x_1), \dots, g_\lambda^m(x_n)) = \text{diag}(x_1, \dots, x_n),$$

therefore $G_\lambda^m(X) = P\,\text{diag}(x_1, \dots, x_n)P^{-1} = X$. Conversely, if there is a matrix $X \in \Omega_n$ such that $G_\lambda^m(X) = X$, for some positive integer m, then this implies that $\text{diag}(g_\lambda^m(x_1), \dots, g_\lambda^m(x_n)) = \text{diag}(x_1, \dots, x_n)$ and that the spectrum of X consists of periodic points with respect to g_λ. □

2.2 Some results in low dimension

2.2.1 *The $n = 2$ case*

In this situation we have $G_\lambda(\Omega_2) = \Omega_2$. A symmetric stochastic matrix $X \in M_2(\mathbb{R})$ is

$$X = \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix}, \quad x \in [0, 1].$$

The eigenvalues of X are 1 and $2x - 1$. Therefore, to ensure that X is positive definite, we must consider $x \in]1/2, 1]$. On the other hand, we have

$$G_\lambda(X) = \begin{pmatrix} 1+\lambda-3x\lambda+2x^2\lambda & (-1+3x-2x^2)\lambda \\ (-1+3x-2x^2)\lambda & 1+\lambda-3x\lambda+2x^2\lambda \end{pmatrix}.$$

Since $\lambda \in [0, 4]$, for $x \in]1/2, 1]$, the matrix $G_\lambda^k(X)$ has non-negative entries for every $k \in \mathbb{N}$. Thus the equality $G_\lambda(\Omega_2) = \Omega_2$ follows, that is, $\Omega_2 = \Lambda_2(\lambda)$, for every $\lambda \in [0, 4]$.

For each cycle $\{a_1, \dots, a_k\}$ of g_λ, with period k, we have the orbit

$$\begin{pmatrix} \frac{a_1+1}{2} & \frac{1-a_1}{2} \\ \frac{1-a_1}{2} & \frac{a_1+1}{2} \end{pmatrix}, \quad \dots, \quad \begin{pmatrix} \frac{a_k+1}{2} & \frac{1-a_k}{2} \\ \frac{1-a_k}{2} & \frac{a_k+1}{2} \end{pmatrix}$$

of G_λ, with the same period. Note that, after diagonalization, we have

$$\begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 1, \dots, k;$$

nevertheless, this matrix is not stochastic, since $a_i \neq 1$ for all $i = 1, \dots, k$, otherwise we would have the trivial case of the fixed point. When $n = 2$, there is no $Y \in \Omega_2$ equivalent to a given $X \in \Omega_2$, that is, there is no nontrivial $P \in GL_n$ such that $PXP^{-1} \in \Omega_2$. Moreover, there is a one-to-one correspondence between orbits in Ω_2, under G_λ, and orbits in $[0, 1]$, under g_λ. Therefore the matrix dynamics in Ω_2 is essentially equivalent to the one-dimensional dynamics.

2.2.2 *The $n = 3$ case*

In this case, it is not true that $\Omega_3 = \Lambda_3(\lambda)$, for every λ. For example, consider the matrix $X \in \Omega_3$,

$$X = \begin{pmatrix} 0.615903 & 0.114583 & 0.269514 \\ 0.114583 & 0.615903 & 0.269514 \\ 0.269514 & 0.269514 & 0.460972 \end{pmatrix}.$$

The eigenvalues of X are 1, 0.50132 and 0.191458 (roundoff to the 7th digit). For $\lambda = 3.9$ the second iteration $G_\lambda^2(X) = G_\lambda(G_\lambda(X))$ has two negative entries (although $G_\lambda(X)$ does not have it):

$$G_\lambda^2(X) = \begin{pmatrix} 0.79695 & -0.107963 & 0.311013 \\ -0.107963 & 0.79695 & 0.311013 \\ 0.311013 & 0.311013 & 0.377974 \end{pmatrix},$$

and therefore $G_\lambda^2(X) \notin \Omega_3$. In this case we have that X, given above, does not belong to $\Lambda_3(\lambda)$, for $\lambda = 3.9$.

For $n = 3$, a symmetric stochastic matrix is given by

$$X = \begin{pmatrix} x & y & 1-x-y \\ y & z & 1-y-z \\ 1-x-y & 1-y-z & x+2y+z-1 \end{pmatrix},$$

with $x, y, z \in [0, 1]$. In this case, both the explicit positivity condition on x, y, z and the condition for the non-negativity of the entries are more complicated than the ones for the $n = 2$ case. Let us analyze the subfamily that is characterized by $z = x$. In this case, $G_\lambda(X)$ also belongs to this subfamily and we obtain an invariant subset of $\Lambda_3(\lambda)$. This leads to an interesting theoretical question of what are the minimal invariant subsets of $\Lambda_n(\lambda)$, given n. The set $\Lambda_n(\lambda)$ itself is not minimal as we see here for $n = 3$. As we mentioned, let us now consider the family of matrices

$$X = \begin{pmatrix} x & y & 1-x-y \\ y & x & 1-x-y \\ 1-x-y & 1-x-y & 2y+2x-1 \end{pmatrix}, \quad (1)$$

with $x, y \in [0, 1]$.

We must impose additional conditions on x, y to guarantee that $X \in \Omega_3$. The eigenvalues of X are 1, $x - y$ and $3x + 3y - 2$. Therefore, to ensure that X is positive definite, we must consider $x > y$ and $x > 2/3 - y$. Let $\widetilde{\Lambda}(\lambda)$ be the subset of $\Lambda_3(\lambda)$ of matrices in the form (1). In Fig. 1 we show the invariant subset $\widetilde{\Lambda}(\lambda)$, for $\lambda = 3.9$, which illustrates how different can be the sets Ω_3 and $\Lambda_3(\lambda)$. In Fig. 2 we show a detail of Fig. 1. The example given in the beginning of this section was taken from this picture, searching for pairs (x, y) for which the color is close to white (in the example given, we have chosen $x = 0.615903$ and $y = 0.114583$). Note that it is much more difficult to give, with the same method, a pair (x, y) for which $G_\lambda^k(X)$ has non-negative entries for every positive integer k. This difficulty arises because the invariant set $\widetilde{\Lambda}(\lambda)$ is a rset with nontrivial fractal structure.

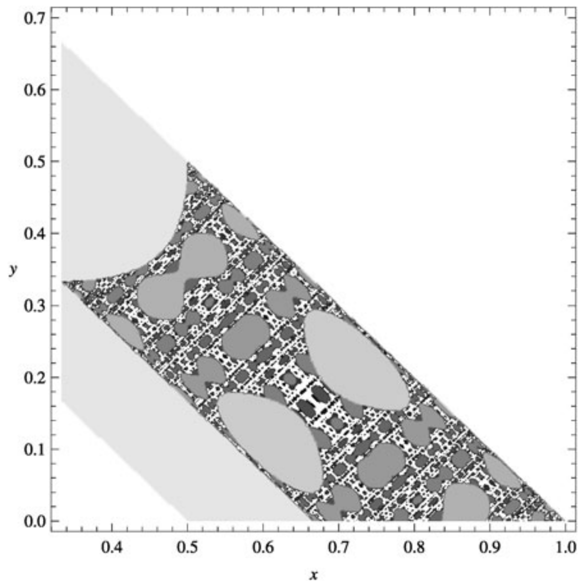


Fig. 1 Graph of the invariant set $\widetilde{\Lambda}(\lambda)$ for $\lambda = 3.9$

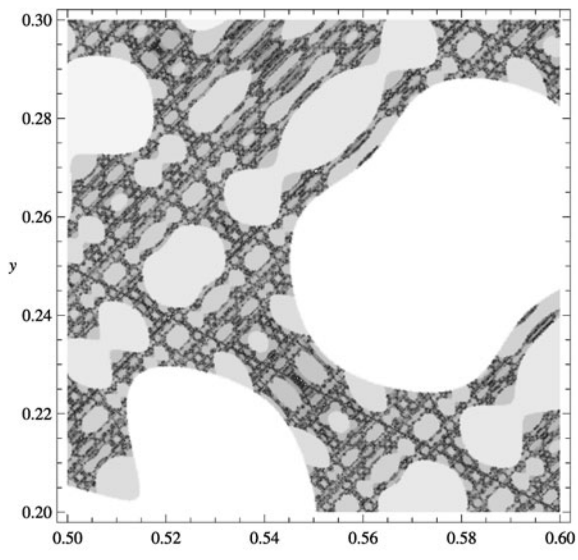


Fig. 2 Detail of the invariant set $\widetilde{\Lambda}(\lambda)$ for $\lambda = 3.9$

A more efficient way to give $X \in \widetilde{\Lambda}(\lambda)$, that is, X in the form (1) for which $G_\lambda^k(X)$ has non-negative entries for every positive integer k, is to give the periodic points of G_λ. Nevertheless, some care is needed, since even in this case we must assure the non-negativity of the entries. Let $a = x - y$ and $b = 3x + 3y - 2$ be the eigenvalues of X in terms of the entries x, y. Deter-

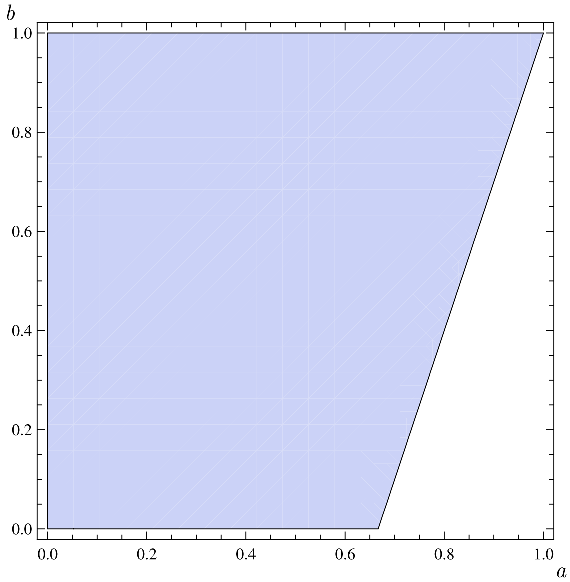


Fig. 3 Domain Δ in the plane (a, b)

mining x, y, we can explicitly write

$$X = \frac{1}{6}\begin{pmatrix} 3a+b+2 & -3a+b+2 & 2-2b \\ -3a+b+2 & 3a+b+2 & 2-2b \\ 2-2b & 2-2b & 2+4b \end{pmatrix}.$$

The conditions on the eigenvalues a, b, in order for $x, y \in [0, 1]$, are $\frac{1}{6}(3a + b + 2) \geq 0$ and $\frac{1}{6}(-3a + b + 2) \geq 0$. To ensure that X is positive definite we have also $0 < a, b \leq 1$, therefore there is a domain, in the plane (a, b), of possible eigenvalues of a matrix in Ω_3, see Fig. 3. Let Δ be this domain, which is the region defined by $\Delta = \{(a, b) \in \mathbb{R}^2 : 0 < a \leq 1,\ 0 < b,\ 3a - 2 \leq b \leq 1\}$. Note that $(a, b) \in \Delta$ does not imply that $(b, a) \in \Delta$.

Let $\{a_1, \ldots, a_k\}$ be a cycle of period k, with respect to g_λ, that is, $g_\lambda(a_i) = a_{i+1}$, $i = 1, 2, \ldots, k-1$, and $g_\lambda(a_k) = a_1$. To obtain a periodic matrix orbit for G_λ it is necessary to choose a pair of points from $\{a_1, \ldots, a_k\}$. In the considered set, this pair (a_i, a_j) will determine a periodic orbit for G_λ if and only if $(a_{i+r}, a_{j+r}) \in \Delta$ for every positive integer r (reminding that $g_\lambda(a_k) = a_1$). Since $(a, a) \in \Delta$, for which $a \in [0, 1]$, a possible choice is the pair (a_i, a_i); in this case, we say that the matrix orbit is *in phase*. Otherwise it is *out of phase*. The pair (a_i, a_j) gives the matrix

$$\frac{1}{6}\begin{pmatrix} 3a_i+a_j+2 & -3a_i+a_j+2 & 2-2a_j \\ -3a_i+a_j+2 & 3a_i+a_j+2 & 2-2a_j \\ 2-2a_j & 2-2a_j & 2+4a_j \end{pmatrix}$$

with $i, j = 1, \ldots, k$.

We can implement an algorithm to obtain all the periodic points of G_λ. Let $\lambda \in [0, 4]$. Each cycle, with respect to g_λ, can be determined using numerical routines or using symbolic dynamics and combinatorial arguments, see [5] and [2]. For the unidimensional orbit $\{a_1, \ldots, a_k\}$ of the least period k, we have a periodic matrix orbit in phase, generated by the pair (a_1, a_1). To obtain the other matrix orbits, the out of phase orbits, we only need to check if $(a_{1+r}, a_{i+r}) \in \Delta$, with $r = 1, \ldots, k-1$, for each pair (a_1, a_i), with $i = 2, \ldots, k$.

Example 1 Consider

$$X = \frac{1}{6}\begin{pmatrix} 3a_i+a_j+2 & -3a_i+a_j+2 & 2-2a_j \\ -3a_i+a_j+2 & 3a_i+a_j+2 & 2-2a_j \\ 2-2a_j & 2-2a_j & 2+4a_j \end{pmatrix}.$$

Let $\lambda = 4$. The two cycles of period 3, with respect to g_λ, are:

$$\{a_1, a_2, a_3\} = \{0.0301537, 0.883022, 0.586824\} \quad \text{and}$$
$$\{a_4, a_5, a_6\} = \{0.0495156, 0.811745, 0.38874\}$$

(obviously we exclude the fixed points and present approximate values). Remark that not all the pairs (a_i, a_j) generated by the elements of the cycles of g_λ, give a matrix orbit. In the table below, the symbol $\times$ represents the pairs (a_i, a_j) that generate periodic orbits for G_λ. The symbol $\varnothing$ represents the other cases.

	a_1	a_2	a_3	a_4	a_5	a_6
a_1	$\times$	$\varnothing$	$\varnothing$	$\times$	$\varnothing$	$\varnothing$
a_2	$\varnothing$	$\times$	$\varnothing$	$\varnothing$	$\times$	$\varnothing$
a_3	$\varnothing$	$\varnothing$	$\times$	$\varnothing$	$\varnothing$	$\times$
a_4	$\times$	$\times$	$\varnothing$	$\times$	$\varnothing$	$\varnothing$
a_5	$\varnothing$	$\times$	$\times$	$\varnothing$	$\times$	$\varnothing$
a_6	$\times$	$\varnothing$	$\times$	$\varnothing$	$\varnothing$	$\times$

This means that there exist two distinct matrix orbits of G_λ of period three that are in phase:

$(a_1, a_1) \longleftrightarrow$

$$\begin{pmatrix} 0.922015 & 0.0389926 & 0.0389926 \\ 0.0389926 & 0.922015 & 0.0389926 \\ 0.0389926 & 0.0389926 & 0.922015 \end{pmatrix},$$

(a_4, a_4)

$$\longleftrightarrow \begin{pmatrix} 0.874497 & 0.0627517 & 0.0627517 \\ 0.0627517 & 0.874497 & 0.0627517 \\ 0.0627517 & 0.0627517 & 0.874497 \end{pmatrix},$$

and three distinct matrix orbits of period three that are out of phase:

(a_1, a_4)

$$\longleftrightarrow \begin{pmatrix} 0.910135 & 0.027113 & 0.0627517 \\ 0.027113 & 0.910135 & 0.0627517 \\ 0.0627517 & 0.0627517 & 0.874497 \end{pmatrix},$$

(a_4, a_1)

$$\longleftrightarrow \begin{pmatrix} 0.886376 & 0.074631 & 0.0389926 \\ 0.0746313 & 0.886376 & 0.0389926 \\ 0.0389926 & 0.0389926 & 0.922015 \end{pmatrix},$$

(a_4, a_2)

$$\longleftrightarrow \begin{pmatrix} 0.83701 & 0.0252649 & 0.137725 \\ 0.0252649 & 0.83701 & 0.137725 \\ 0.137725 & 0.137725 & 0.724549 \end{pmatrix}.$$

3 Discrete dynamics on the Markov chain set

As we discussed in the introduction, each stochastic matrix determines a Markov chain. Therefore, the dynamics on $\Lambda_n(\lambda)$ generated by G_λ corresponds to a certain dynamical system on the set of reversible Markov chains (reversible since we deal with symmetric matrices). Since the set $\Lambda_n(\lambda)$ is too large, we will study families of matrices belonging to $\Lambda_n(\lambda)$, for a certain positive integer n, which are invariant under iteration of G_λ with potential interest for Markov chains analysis. We stress that many different Markov chains can be obtained, although the same iterative map G_λ is considered.

Let us explain the general idea. Let $\lambda \in [0, 4]$. We start with a particular Markov chain characterized by a stochastic matrix $X \in \Lambda_n(\lambda)$. This Markov chain can be seen as a model of a certain system in a regular regime. The periodic structural changes in the system are characterized by iteration of the map G_λ. After a singular regime (corresponding to an iteration under G_λ), in the next regular regime, the system is modeled again by a Markov chain characterized by the stochastic matrix $G_\lambda(X)$. Therefore, we obtain an evolutionary process for Markov chains and we use some useful techniques from iterated maps on the interval, that allow us to analyze this evolution.

3.1 A reducible case

An $n \times n$ matrix X is *reducible* if there is a permutation matrix P such that

$$PXP^{-1} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \tag{2}$$

where A, B, C are non necessarily square matrices, see (2). A reducible matrix X has the property that the powers of X, X^k have entries which are equal to 0 for every positive integer k (it must have at least $n-1$ zero entries).

Consider the following reducible matrix:

$$Y = \begin{pmatrix} y_0 & 0 & z_0 & 0 \\ 0 & y_0 & 0 & z_0 \\ z_0 & 0 & y_0 & 0 \\ 0 & z_0 & 0 & y_0 \end{pmatrix}.$$

In order for Y to be stochastic, we must have $0 \le y_0, z_0 \le 1$ and $y_0 + z_0 = 1$. The eigenvalues of Y are $y_0 - z_0$ and $y_0 + z_0 = 1$, both with multiplicity 2. Therefore if $y_0 > z_0$, we have that Y is positive definite and in that case $Y \in \Omega_4$. The kth iteration of Y will have the form

$$G_\lambda^k(Y) = \begin{pmatrix} y_k & 0 & z_k & 0 \\ 0 & y_k & 0 & z_k \\ z_k & 0 & y_k & 0 \\ 0 & z_k & 0 & y_k \end{pmatrix},$$

with

$$\begin{aligned} y_{k+1} &= 1 - \lambda y_k + \lambda y_k^2 + \lambda z_k^2 \quad \text{and} \\ z_{k+1} &= (2y_k - 1)\lambda z_k. \end{aligned} \tag{3}$$

We conclude that G_λ preserves the initial matrix form and the eigenvalues of $G_\lambda(Y)$ are $y_k - z_k$ and $y_k + z_k = 1$. Therefore, we have in fact a one-dimensional system with

$$\begin{aligned} y_{k+1} &= 1 - \lambda y_k + \lambda y_k^2 + \lambda(1 - y_k)^2 \\ &= 1 + \lambda - 3\lambda y_k + 2\lambda y_k^2. \end{aligned} \tag{4}$$

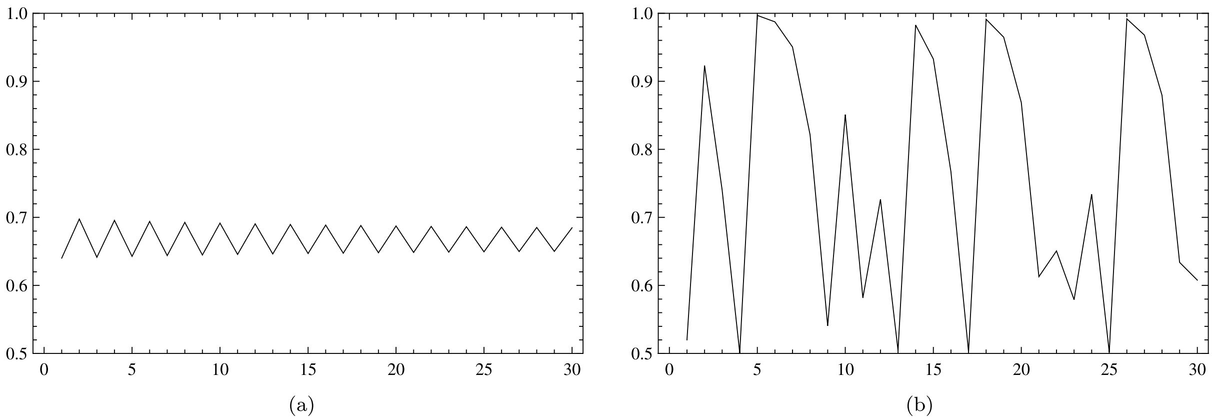


Fig. 4 Evolution of the transition probability y_k of maintaining the same state, with the initial condition $y_0 = 0.7$. The probability of transition to a different state is, for each iteration k, $z_k = 1 - y_k$: (**a**) with $\lambda = 3$; (**b**) with $\lambda = 4$

Now we are in position to analyze the dynamical behavior of G_λ directly from the matrix entries and, in particular, to analyze if any entry will eventually be negative. The map $x \mapsto h_\lambda(x) := 1 + \lambda - 3\lambda x + 2\lambda x^2$, restricted to $x \in [1/2, 1]$ (in order to satisfy the previous conditions, given in Sect. 2.2.1), remains in $[1/2, 1]$. This means that, for every $\lambda \in [0; 4]$, all the entries of $G_\lambda^k(Y)$ are non-negative, and that, for every k, the set of considered matrices is invariant under G_λ, in particular $y_k \in [1/2, 1]$.

We can see that we have two Markov chains, with underlying vertices $\{1, 3\}$ and $\{2, 4\}$, each one associated with a 2-full-shift, coupled by the recursion (3). For a given k we obtain a Markov chain for which the probability of maintaining the same state is y_k and to change the state is $z_k = 1 - y_k$. In the next iteration (generation), the probability of maintaining the same state is $y_{k+1} = h_\lambda(y_k) = 1 + \lambda - 3\lambda y_k + 2\lambda y_k^2$. In Fig. 4 we exemplify how this probability evolves, starting from $y_0 = 0.7$ and considering different values of λ.

It is interesting to note that the map h_λ, with $\lambda \in [0, 4]$, reproduces the different features of the quadratic maps dynamics, as we can see in the bifurcation diagram in Fig. 5.

If $\lambda = 2$ (or $\lambda < 3$), every initial condition in $[1/2, 1[$ is attracted, under h_λ, to the fixed point. If $\lambda = 3.56995\ldots$, we have the Feigenbaum point, the beginning of "chaos." If $\lambda > 3.56995\ldots$, we have positive topological entropy and an infinity of repulsive periodic points, see [9].

A natural generalization is considered in the next theorem.

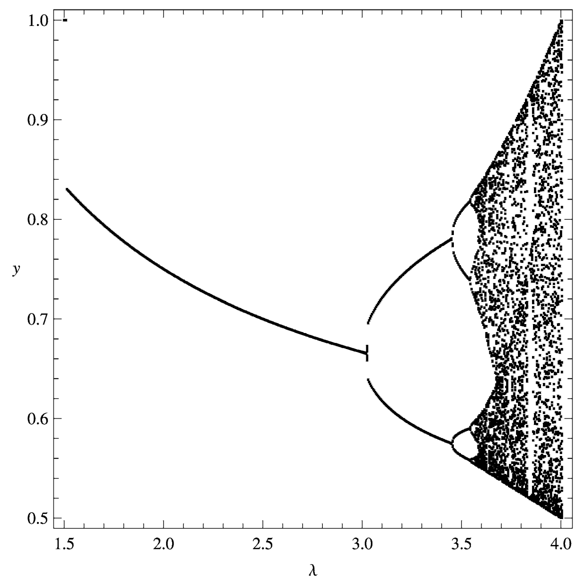


Fig. 5 Bifurcation diagram of the map h_λ

Theorem 1 *Let $Y \in \Omega_{2n+2}$, $n \geq 1$, be a stochastic matrix of the form*

$$Y = \begin{pmatrix} y & 0 & z & 0 & z & \cdots & 0 \\ 0 & y & 0 & z & 0 & \cdots & z \\ z & 0 & y & 0 & z & \cdots & 0 \\ 0 & z & 0 & y & 0 & \cdots & z \\ z & 0 & z & 0 & y & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & z & 0 & z & 0 & \cdots & y \end{pmatrix}. \tag{5}$$

Then

$$Y_k := G_\lambda^k(Y) = \begin{pmatrix} y_k & 0 & z_k & 0 & z_k & \cdots & 0 \\ 0 & y_k & 0 & z_k & 0 & \cdots & z_k \\ z_k & 0 & y_k & 0 & z_k & \cdots & 0 \\ 0 & z_k & 0 & y_k & 0 & \cdots & z_k \\ z_k & 0 & z_k & 0 & y_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & z_k & 0 & z_k & 0 & \cdots & y_k \end{pmatrix},$$

with

$$\begin{aligned} y_{k+1} &= 1 - \lambda y_k + \lambda y_k^2 + \lambda n z_k^2, \\ z_{k+1} &= (2y_k - 1)\lambda z_k + \lambda(n-1) z_k^2 \end{aligned} \tag{6}$$

and

$$y_0 = y, \qquad z_0 = z, \qquad y_k + n z_k = 1. \tag{7}$$

In particular, $G_\lambda^k(Y) \in \Omega_{2n+2}$ for every positive integer k, with $\lambda \in [0,4]$.

Proof If Y is in the form (5) and stochastic, then $y = 1 - nz$ and $z \in [0, 1/n]$. The eigenvalues of a matrix in the form (5) are $y + nz$ and $y - z$. As Y is positive definite, we have $y > z$ and $G_\lambda(Y_k)$ belongs to Ω_{2n+2}. Moreover, by direct computation of $G_\lambda(Y_k)$, we obtain the claimed relation for y_{k+1} and the matrix Y_{k+1} also in the form (5). Therefore, $y_{k+1} + n z_{k+1} = 1$. By induction, the result follows for every positive integer k. □

3.2 A primitive case

A matrix X is primitive if there is a positive integer k such that every entry of X^k is positive. If X is a non-negative primitive matrix then, by the Perron Frobenius theorem, one of its eigenvalues is positive (a simple root of the characteristic equation of X) and greater (in absolute value) than all the other eigenvalues, see [1].

Let

$$Y = \begin{pmatrix} x_0 & y_0 & y_0 \\ y_0 & x_0 & y_0 \\ y_0 & y_0 & x_0 \end{pmatrix}.$$

This matrix is primitive iff $y_0 \neq 0$. In order for Y to be stochastic, we must have $0 \leq y_0,\ x_0 \leq 1$ and $2y_0 + x_0 = 1$. The eigenvalues of Y are 1 and $1 - 3y_0$ with multiplicity 2. In this case, we have to consider $y_0 < 1/3$ to ensure that Y is positive definite so that $Y \in \Omega_3$. Now, it can be easily seen that $G_\lambda^k(Y)$ has the form

$$G_\lambda^k(Y) = \begin{pmatrix} x_k & y_k & y_k \\ y_k & x_k & y_k \\ y_k & y_k & x_k \end{pmatrix},$$

and consequently, we have $x_k = 1 - 2y_k$. Moreover, a simple calculation shows that $y_k = \lambda y_{k-1}(1 - 3y_{k-1})$. This means that in this case the dynamics of the matrix entries can be given explicitly (and also using the spectrum). Note that the quadratic $f_{\lambda,3}(y) = \lambda y(1-3y)$ maps the interval $[0, 1/3]$ into itself and reproduces the dynamics of the logistic equation on the unit interval. Furthermore, this shows that all the entries of $G_\lambda^k(Y)$ are non-negative for every $y_0 \in [0, 1/3[$ and every k.

As a generalization of this case, we can consider a family of $n \times n$ matrices Y for which the entries outside the diagonal are equal to y and in the diagonal are equal to $x = 1 - (n-1)y$. Let $f_{\lambda,n}(y) = \lambda y(1 - ny)$. The map $f_{\lambda,n}(y)$ sends the interval $[0, 1/n]$ into itself and reproduces the dynamics of the logistic equation on the unit interval. Now, we have the following theorem.

Theorem 2 *Let $Y \in \Omega_n$ be a stochastic matrix of the form*

$$Y = \begin{pmatrix} x & y & \cdots & y \\ y & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & y \\ y & \cdots & y & x \end{pmatrix}. \tag{8}$$

Then

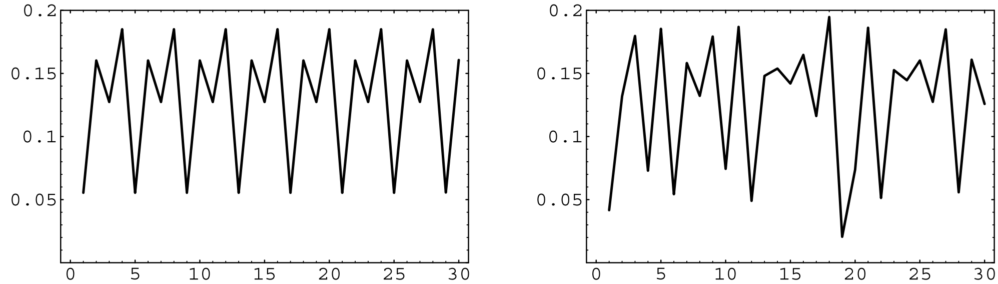
$$Y_k := G_\lambda^k(Y) = \begin{pmatrix} x_k & y_k & \cdots & y_k \\ y_k & x_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_k \\ y_k & \cdots & y_k & x_k \end{pmatrix},$$

with

$$x_{k+1} = 1 - (n-1) y_{k+1}, \qquad y_{k+1} = \lambda y_k (1 - n y_k),$$

and

$$y_0 = y, \qquad x_0 = x.$$



(a) (b)

Fig. 6 Evolution of the transition probability y_k of changing the state, with $\lambda = 4$. The probability of transition to the same state is for each iteration k, $x_k = 1 - (n-1)y_k$: (**a**) with $y_0 = 0.1850217\ldots$ (period 4); (**b**) with $y_0 = 0.1889865\ldots$ (aperiodic)

In particular, $G_\lambda^k(Y) \in \Omega_n$ *for every positive integer* k*, with* $\lambda \in [0, 4]$.

Proof If Y is in the form (8) and belongs to Ω_n, then $x = 1 - (n-1)y$ and $y \in [0, 1/n]$. The eigenvalues of a matrix in the form (8) are $x + (n-1)y$ and $x - y$. As Y is positive definite, we have $x > y$ and $G_\lambda(Y_k)$ belongs to Ω_n. Moreover, by direct computation of $G_\lambda(Y_k)$, we obtain the claimed relation for y_{k+1} and a matrix Y_{k+1} also in the form (8). Therefore, $x_{k+1} + (n-1)y_{k+1} = 1$. The result follows by induction, for every positive integer k. □

Example 2 Let us consider $n = 5$ and $\lambda = 4$. In Fig. 6 we can see the iterations of the Markov chains for different initial conditions $y_0 = 0.1850217\ldots$ (period 4), $y_0 = 0.1889865\ldots$ (aperiodic).

4 Conclusions

In this paper we have analyzed the dynamical behavior of iterated matrices under quadratic maps. Choosing an appropriate one-parameter family of quadratic maps it was possible to preserve the stochasticity of a matrix. Therefore, we have obtained a discrete dynamical system on certain invariant subsets of $n \times n$ matrices. These invariant sets depend on the size and the internal structure (relations between certain entries, fixed zero entries, etc.) of the initial matrix. In the case of a general 3×3 stochastic matrix, the invariant subset reveals a fractal structure resulting from discarding the matrices which have eventually negative entries. Different classes of $n \times n$ stochastic matrices, with certain given relations between the entries, such as those in Sects. 3.1 and 3.2, have as invariant sets domains in $\mathbb{R}^2$ (more generally, $\mathbb{R}^n$). Since the behavior of quadratic maps is well known and studied, we are now able to study and analyze in detail the dynamical behavior of reversible Markov chains, arising from the symmetric stochastic matrices, which evolve under the iteration of the quadratic map G_λ. This study can be made regarding the choice of the initial conditions (an initial Markov chain) and the dependence on the parameter $\lambda \in [0, 4]$. For instance, if $\lambda \leq 2$ for every initial condition, the corresponding orbit is attracted to the fixed point. If $\lambda \leq 3.56995\ldots$ (Feigenbaum point), the occurrence of periodic points is limited to those with period 2^k, for certain positive integer k. If $\lambda > 3.56995\ldots$, then there are infinite repulsive periodic points with periods conditioned by the Sharkovsky theorem. This is reflected directly in the dynamics of the stochastic matrices and the evolving Markov chains. As an example we have considered in Sect. 3.2 a family of $n \times n$ stochastic matrices, characterized by two variables: x (associated with the probability of staying in a certain state) and y (associated with probability of changing the state). This family is preserved by the map G_λ and the dynamics is, regarding the matrix entries, determined by a one-dimensional map $f_{\lambda,n}$, where n is the matrix size. Starting with $\lambda = 0$ and increasing its value, we follow the route to chaos by period doubling (and more generally by bifurcation phenomena). We obtain, therefore, successive oscillating Markov chains, first a fixed point, then period two, period four, and so on. In this setting and choosing the appropriate parameter, we obtained orbits of Markov chains which are aperiodic (as in Example 2).

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