Complex boosts: A Hermitian Clifford algebra approach†

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Abstract

The aim of this paper is to study complex boosts in complex Minkowski space-time that preserves the Hermitian norm. Starting from the spin group $\mathrm{Spin}^+(2n, 2m, \mathbb{R})$ in the real Minkowski space $\mathbb{R}^{2n,2m}$ we construct a Clifford realization of the pseudo-unitary group $\mathrm{U}(n, m)$ using the space-time Witt basis in the framework of Hermitian Clifford algebra. Restricting to the case of one complex time direction we derive a general formula for a complex boost in an arbitrary complex direction and its $KAK-$decomposition, generalizing the well-known formula of a real boost in an arbitrary real direction. In the end we derive the complex Einstein velocity addition law for complex relativistic velocities, by the projective model of hyperbolic $n-$space.

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1 Introduction

Lorentz boosts are linear transformations of space-time that preserve the space-time interval between any two events in Minkowski space. They are very important in many fields of mathematics and physics when relativistic effects come into play. In the real case, Lorentz boosts are elements of the Lorentz group $\mathrm{SO}(3, 1)$, which are rotation-free and preserve the indefinite norm $||\underline{x}||^2 - t^2$, with $\underline{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

The generalization of real boosts to complex boosts requires the study of the unitary group $\mathrm{U}(n, 1)$. This is just the group of isometries of the $(n + 1)-$dimensional complex space $\mathbb{C}^{n+1}$ which preserves the Hermitian indefinite norm $||\underline{z}||^2 - ||T||^2$, with $\underline{z} \in \mathbb{C}^n$ and $T \in \mathbb{C}$. One of the first papers studying

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the real group structure of the complex Lorentz group in four dimensions was the paper of Barut [1]. Real and complex boosts in arbitrary pseudo-Euclidean spaces were discussed in [19], where the law of composition of generalized velocities (subluminal and superluminal velocities) was found. Real Lorentz transformation groups in arbitrary pseudo-Euclidean spaces where also presented in Eq.(8.14e) of Ref. [24] using the language of gyrogroups, established by A. Ungar [25, 26]. Using this new formalism A. Ungar studied abstract real and complex Lorentz transformations and its associated gyrogroups in a series of papers [21, 22, 23, 18]. In these papers it is shown the strong connection between boosts and gyrogroups in the real and complex cases, through the study of the automorphisms of the unit ball (real or complex).

In the real Minkowski-space time $\mathbb{R}^{n,1}$ a boost in an arbitrary direction $\underline{\omega} \in S^{n-1}$ can be described using the universal real Clifford algebra $\mathbb{R}_{n,1}$ by the spin element $s_{\underline{\omega}} = \cosh(\alpha/2) + \underline{\omega}\epsilon \sinh(\alpha/2)$, where $\alpha \in \mathbb{R}$ and ϵ is the vector that spans the time axis. In [11] it was shown that there is a bijection between hyperbolic rotations generated by s_ω and relativistic velocity additions (non-standard velocities, coordinate velocities, and proper velocities), giving rise to three different models of hyperbolic geometry (Poincaré, Klein, and Hyperbola models). Thus, not only the infinitesimal generators of the Lorentz group are important, but also the formula of a boost in an arbitrary direction is of foremost importance for the construction of concrete examples of gyrogroups and the study of the relativistic velocities. The formula of the complex boost in an arbitrary complex direction constructed in this paper allows the derivation of the complex Einstein relativistic velocity addition by the projection of its spin action on Minkowski space to the complex unit ball. It turns out that this transformation belongs to the automorphism group of the complex unit ball considered by Rudin in [16]. Our work has also several applications in harmonic analysis, quantum phase-space analysis, coherent states and wavelets (c.f. [13, 14, 15, 10]). For instance, in [10] the author used the automorphisms of the unit ball to construct a family of spherical continuous wavelet transforms on the unit sphere in $\mathbb{R}^n$. Thus, the results of this paper are of interest for people working in physics and also mathematics.

In the last years Hermitian Clifford analysis has emerged as a refinement of Clifford analysis but also as an independent theory. While Clifford analysis focuses essentially on the study of the null-solutions of the Dirac operator on $\mathbb{R}^n$, called monogenic functions, Hermitian Clifford analysis focuses on the study of Hermitian monogenic functions taking values in a complex Clifford algebra or in a complex spinor space, which are null solutions of two complex mutually adjoint Dirac operators. In the real case, the Dirac operator is invariant under the orthogonal group $\mathrm{SO}(n)$ which is double covered by the group $\mathrm{Spin}(n)$, while in the complex case, the two Hermitian Dirac operators are invariant under the unitary group $\mathrm{U}(n)$. A vast literature on such function theories is available, see e.g. [4, 6, 12, 17, 2, 3, 5].

Clifford analysis has also been investigated in real Minkowski space-times $\mathbb{R}^{n,1}$ or $\mathbb{R}^{n,m}$, $m > 1$. In [9] it was developed a function theory for Clifford algebra valued null solutions for the Dirac operator on the hyperbolic unit ball, the so-called hyperbolic monogenics. In this case the invariance group is the proper real Lorentz group $\mathrm{Spin}^+(n,1)$ (see [9] and the vast literature therein). Our results can lead to the construction of a function theory for Hermitian hyperbolic monogenic functions on the complex projective model, generalizing the results in the real case (see e.g. [9]).

It is well-known that a very large class of Lie groups can be described as spin groups (see [8],[7]). Therefore, Clifford algebras or geometric algebras are a very powerful mathematical tool for the study of Lie groups. In this paper we construct a Clifford realization of the pseudo-unitary group $\mathrm{U}(n,m)$ as a subgroup of the real orthogonal group $\mathrm{Spin}^+(2n,2m,\mathbb{R})$. The paper is organized as follows. In Section 2 we define the space-time Witt basis for working in the Hermitian space $H_{n,m}$ and we establish all the algebraic relations and properties needed for our constructions. In Section 3 we study the spin group $\mathrm{Spin}^+(2n,2m,\mathbb{R})$ in the real Minkowski space $\mathbb{R}^{2n,2m}$ and we construct a Clifford realization of the pseudo-unitary group $\mathrm{U}(n,m)$ using the space-time Witt basis in the framework of Hermitian Clifford algebra. We compute all the complex infinitesimal transformations (holomorphic, anti-holomorphic and non-holomorphic transformations) in the Hermitian space $H_{n,m}$. Hereafter, in Section 4 we construct the holomorphic, anti-holomorphic and non-holomorphic complex boosts in an arbitrary complex direction for the case of one complex time direction. Each of these boosts turn out to be the composition of two specific real boosts. We show also the Cartan or $KAK-$decomposition of such complex boosts. Finally, in Section 5 we will derive the complex Einstein velocity addition for complex relativistic velocities by the projective model of hyperbolic $n-$space, which belongs to the automorphism group of the complex unit ball in $\mathbb{C}^n$.

2 The pseudo-Hermitian space $H_{n,m}$

We will denote by $H_{n,m}$ the standard pseudo-Hermitian space of type (n,m) which corresponds to the standard complex space $\mathbb{C}^{n+m}$, of complex dimension $p=n+m$, endowed with the non-degenerate sesquilinear Hermitian form, called the standard scalar product, defined by

$$\langle \underline{z}, \underline{w}\rangle = \sum_{j=1}^{n} z_j\overline{w_j} - \sum_{j=n+1}^{n+m} z_j\overline{w_j} \quad \text{for all } \underline{z},\underline{w}\in H_{n,m}. \tag{2.1}$$

We consider that $H_{n,m}$ is identified with $(\mathbb{R}^{2p}, J)$, where $\mathbb{R}^{2p=2n+2m}$ is the real vector space subordinate to $H_{n,m}$ and J is the $\mathbb{R}-$linear mapping fixing the complex structure. Since we want to incorporate complex space and complex time in this abstract setting we will consider $\underline{z}$=$(z_1,\ldots,z_n,t_1,\ldots,t_m)$ a vector in $\mathbb{C}^{n+m}$ with $z_j=x_j+iy_j\in\mathbb{C}, j=1,\ldots,n$ and $t_r=u_r+iv_r\in\mathbb{C}, r=1,\ldots,m$. Then, $\underline{z}$ can be identified with the vector $(x_1,\ldots,x_n, y_1,\ldots,y_n,u_1,\ldots,u_m,v_1,\ldots,v_m)\in\mathbb{R}^{2p}$. The vector space $\mathbb{R}^{2p=2n+2m}$ turns out to be a pseudo-Euclidean space of signature $(2n,2m)$.

Let us consider $\{e_j,\xi_r, j=1,\ldots 2n, r=1,\ldots,2m\}$ an orthonormal basis of the real Minkowski space-time $\mathbb{R}^{2n,2m}$, endowed with a non-degenerate real quadratic form of signature $(2n,2m)$, and let $\mathbb{R}_{2n,2m}$ be the associated real Universal Clifford algebra. The non-commutative multiplication in $\mathbb{R}_{2n,2m}$ is governed by the rules

$$e_je_k+e_ke_j=-2\delta_{jk},\quad \xi_r\xi_s+\xi_s\xi_r=2\delta_{rs},\quad \mathrm{e}_j\xi_r+\xi_r\mathrm{e}_j=0, \tag{2.2}$$

for $j,k=1,\ldots,n$, and $r,s=1,\ldots,m$. In particular, $e_j^2=-1, j=1,\ldots,n$ and $\xi_r^2=1, r=1,\ldots,m$. With these elements we construct the space-time Witt basis

$$\{\mathfrak{f}_j,\mathfrak{f}_j^\dagger, j=1,\ldots n\}\cup\{\mathfrak{h}_r,\mathfrak{h}_r^\dagger, r=1,\ldots m\}$$

where

$$\mathfrak{f}_j=\frac{e_j-ie_{n+j}}{2},\qquad \mathfrak{f}_j^\dagger=-\frac{e_j+ie_{n+j}}{2},\qquad j=1,\ldots,n \tag{2.3}$$

and

$$\mathfrak{h}_r=\frac{\xi_r-i\xi_{m+r}}{2},\qquad \mathfrak{h}_r^\dagger=-\frac{\xi_r+i\xi_{m+r}}{2},\qquad r=1,\ldots,m. \tag{2.4}$$

Here, the symbol † stands for the Hermitian conjugation, which is the composition of the usual conjugation on the Clifford algebra $\mathbb{R}_{2n,2m}$ defined by

$$a\mapsto\overline{a},\quad \overline{ab}=\overline{b}\overline{a},\quad \overline{a+b}=\overline{a}+\overline{b},\quad \overline{e_j}=-e_j,\quad \overline{\xi_r}=-\xi_r,\quad \overline{1}=1$$

and the complex conjugation $A \mapsto A^c$ for $A \in \mathbb{C}_{2p}$, where $\mathbb{C}_{2p}$ denotes the complexification of the Clifford algebra $\mathbb{R}_{2n,2m}$. The elements of the space-time Witt basis satisfy the following Grassmannian and duality identities:

$$\mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = 0 \qquad (2.5) \qquad\qquad \mathfrak{h}_r\mathfrak{h}_s^\dagger + \mathfrak{h}_r^\dagger\mathfrak{h}_s = -\delta_{rs} \qquad (2.10)$$
$$\mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger = 0 \qquad (2.6) \qquad\qquad \mathfrak{f}_j\mathfrak{h}_r + \mathfrak{h}_r\mathfrak{f}_j = 0 \qquad (2.11)$$
$$\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk} \qquad (2.7) \qquad\qquad \mathfrak{f}_j^\dagger\mathfrak{h}_r + \mathfrak{h}_r\mathfrak{f}_j^\dagger = 0 \qquad (2.12)$$
$$\mathfrak{h}_r\mathfrak{h}_s + \mathfrak{h}_r\mathfrak{h}_s = 0 \qquad (2.8) \qquad\qquad \mathfrak{f}_j\mathfrak{h}_r^\dagger + \mathfrak{h}_r^\dagger\mathfrak{f}_j = 0 \qquad (2.13)$$
$$\mathfrak{h}_r^\dagger\mathfrak{h}_s^\dagger + \mathfrak{h}_s^\dagger\mathfrak{h}_r^\dagger = 0 \qquad (2.9) \qquad\qquad \mathfrak{f}_j^\dagger\mathfrak{h}_r^\dagger + \mathfrak{h}_r^\dagger\mathfrak{f}_j^\dagger = 0 \qquad (2.14)$$

for $j,k = 1,\ldots,n$ and $r,s = 1,\ldots,m$. In particular, $\mathfrak{f}_j^2 = \left(f_j^\dagger\right)^2 = 0, j = 1,\ldots n$ and $\mathfrak{h}_r^2 = \left(\mathfrak{h}_r^\dagger\right)^2 = 0, r = 1,\ldots m$, i.e. these elements are isotropic. From (2.3) and (2.4) we obtain

$$e_j = \mathfrak{f}_j - \mathfrak{f}_j^\dagger, \quad \text{and} \quad e_{n+j} = i(\mathfrak{f}_j + \mathfrak{f}_j^\dagger), \quad j = 1,\ldots,n. \qquad (2.15)$$

$$\xi_r = \mathfrak{h}_r - \mathfrak{h}_r^\dagger, \quad \text{and} \quad \xi_{m+r} = i(\mathfrak{h}_r + \mathfrak{h}_r^\dagger), \quad r = 1,\ldots,m. \qquad (2.16)$$

Thus, every $\underline{X} \in \mathbb{R}^{2n,2m}$ is written in the Witt basis as

$$\underline{X} = \sum_{j=1}^{n}(x_je_j + y_je_{n+j}) + \sum_{r=1}^{m}(u_r\xi_r + v_r\xi_{m+r}) \qquad (2.17)$$
$$= \sum_{j=1}^{n}(z_j\mathfrak{f}_j - \overline{z_j}\mathfrak{f}_j^\dagger) + \sum_{r=1}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) \qquad (2.18)$$

where $\overline{z_j}$ and $\overline{t_r}$ are the conjugate variables of z_j and t_r, respectively. Defining the Hermitian vector variable

$$Z = \sum_{j=1}^{n} z_j\mathfrak{f}_j + \sum_{r=1}^{m} t_r\mathfrak{h}_r \qquad (2.19)$$

and its Hermitian vector conjugate variable

$$Z^\dagger = \sum_{j=1}^{n} \overline{z_j}\mathfrak{f}_j^\dagger + \sum_{r=1}^{m} \overline{t_r}\mathfrak{h}_r^\dagger \qquad (2.20)$$

then $\underline{X}$ is identified with a Clifford vector by $\underline{X} = Z - Z^\dagger$. Since $||\underline{X}|| = \sum_{j=1}^{n}(|x_j|^2 + |y_j|^2) - \sum_{r=1}^{m}(|u_r|^2 + |v_r|^2)$ and $Z^2 = (Z^\dagger)^2 = 0$ we have

$$\underline{X}^2 = -||\underline{X}||^2 = (Z - Z^\dagger)^2 = -(ZZ^\dagger + Z^\dagger Z). \qquad (2.21)$$

Using the Witt basis elements we can define two complex Grassmann algebras (see [5]):

$$\mathbb{C}\Lambda_{n,m} = \mathrm{Alg}_{\mathbb{C}}\{\mathfrak{f}_1,\ldots,\mathfrak{f}_n,\mathfrak{h}_1,\ldots,\mathfrak{h}_m\} \quad \text{and} \quad \mathbb{C}\Lambda_{n,m}^\dagger = \mathrm{Alg}_{\mathbb{C}}\{\mathfrak{f}_1^\dagger,\ldots,\mathfrak{f}_n^\dagger,\mathfrak{h}_1^\dagger,\ldots,\mathfrak{h}_m^\dagger\}.$$

The projection of the Clifford vector $Z - Z^\dagger$ onto these complex algebras can be made by introducing the primitive (anti-)idempotent element

$$I = \mathfrak{f}_1\mathfrak{f}_1^\dagger \cdots \mathfrak{f}_n\mathfrak{f}_n^\dagger \mathfrak{h}_1\mathfrak{h}_1^\dagger \cdots \mathfrak{h}_m\mathfrak{h}_m^\dagger \tag{2.22}$$

which satisfies $I^\dagger = I$, $I^2 = (-1)^m I$, and the conversion relations

$$e_j I = i e_{n+j} I = -\mathfrak{f}_j^\dagger I, \qquad \mathfrak{f}_j I = 0, \quad j = 1, \ldots, n; \tag{2.23}$$

$$I e_j = -i I e_{n+j} = I \mathfrak{f}_j, \qquad I \mathfrak{f}_j^\dagger = 0, \quad j = 1, \ldots, n; \tag{2.24}$$

$$\xi_r I = i \xi_{m+r} I = -\mathfrak{h}_r^\dagger I, \qquad \mathfrak{h}_r I = 0, \quad r = 1, \ldots, m; \tag{2.25}$$

$$I \xi_r = -i I \xi_{m+r} = I \mathfrak{h}_r, \qquad I \mathfrak{h}_r^\dagger = 0, \quad r = 1, \ldots, m. \tag{2.26}$$

Therefore, $I(Z - Z^\dagger) = IZ$, and $(Z - Z^\dagger)I = -Z^\dagger I$ i.e., I projects the Clifford vector $Z - Z^\dagger$ onto $\mathbb{C}\Lambda_{n,m}$ or $\mathbb{C}\Lambda_{n,m}^\dagger$ if the multiplication is performed on the left or on the right respectively.

Given two Hermitian vector variables Z_1, Z_2 we can define the dot and wedge product by

$$Z_1 \cdot Z_2 = \frac{1}{2}(Z_1 Z_2 + Z_2 Z_1) \quad \text{and} \quad Z_1 \wedge Z_2 = \frac{1}{2}(Z_1 Z_2 - Z_2 Z_1). \tag{2.27}$$

The following lemmas generalize Lemmas 1 and 2 presented in [5].

Lemma 2.1 *For each $j, k = 1, \ldots, n$ and $r, s = 1, \ldots, m$ we have*

$$\mathfrak{f}_j \cdot \mathfrak{f}_k = \mathfrak{f}_j^\dagger \cdot \mathfrak{f}_k^\dagger = 0 \tag{2.28}$$

$$\mathfrak{f}_j \cdot \mathfrak{f}_k^\dagger = \mathfrak{f}_k^\dagger \cdot \mathfrak{f}_j = \frac{1}{2}\delta_{jk} \tag{2.29}$$

$$\mathfrak{h}_r \cdot \mathfrak{h}_s = \mathfrak{h}_r^\dagger \cdot \mathfrak{h}_s^\dagger = 0 \tag{2.30}$$

$$\mathfrak{h}_r \cdot \mathfrak{h}_s^\dagger = \mathfrak{h}_s^\dagger \cdot \mathfrak{h}_r = -\frac{1}{2}\delta_{rs} \tag{2.31}$$

$$\mathfrak{f}_j \cdot \mathfrak{h}_r = \mathfrak{f}_j^\dagger \cdot \mathfrak{h}_r = \mathfrak{f}_j \cdot \mathfrak{h}_r^\dagger = \mathfrak{f}_j^\dagger \cdot \mathfrak{h}_r = 0. \tag{2.32}$$

Lemma 2.2 *For each $j, k = 1, \ldots, n, j \neq k$, and $r, s = 1, \ldots, m, r \neq s$ we have*

$$\mathfrak{f}_j \wedge \mathfrak{f}_k = -\mathfrak{f}_k \wedge \mathfrak{f}_j = \mathfrak{f}_j\mathfrak{f}_k = \frac{1}{4}(e_j e_k - i e_j e_{n+k} - i e_{n+j} e_k - e_{n+j} e_{n+k}) \tag{2.33}$$

$$\mathfrak{f}_j^\dagger \wedge \mathfrak{f}_k^\dagger = -\mathfrak{f}_k^\dagger \wedge \mathfrak{f}_j^\dagger = \mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger = \frac{1}{4}(e_j e_k + i e_j e_{n+k} + i e_{n+j} e_k - e_{n+j} e_{n+k}) \tag{2.34}$$

$$\mathfrak{f}_j \wedge \mathfrak{f}_k^\dagger = -\mathfrak{f}_k^\dagger \wedge \mathfrak{f}_j = \mathfrak{f}_j\mathfrak{f}_k^\dagger = -\frac{1}{4}(e_j e_k + i e_j e_{n+k} - i e_{n+j} e_k + e_{n+j} e_{n+k}) \tag{2.35}$$

$$\mathfrak{h}_r \wedge \mathfrak{h}_s = -\mathfrak{h}_s \wedge \mathfrak{h}_r = \mathfrak{h}_r\mathfrak{h}_s = \frac{1}{4}(\xi_r \xi_s - i \xi_r \xi_{m+s} - i \xi_{m+r} \xi_s - \xi_{m+r} \xi_{m+s}) \tag{2.36}$$

$$\mathfrak{h}_r^\dagger \wedge \mathfrak{h}_s^\dagger = -\mathfrak{h}_s^\dagger \wedge \mathfrak{h}_r^\dagger = \mathfrak{h}_r^\dagger\mathfrak{h}_s^\dagger = \frac{1}{4}(\xi_r \xi_s + i \xi_r \xi_{m+s} + i \xi_{m+r} \xi_s - \xi_{m+r} \xi_{m+s}) \tag{2.37}$$

$$\mathfrak{h}_r \wedge \mathfrak{h}_s^\dagger = -\mathfrak{h}_s^\dagger \wedge \mathfrak{h}_r = \mathfrak{h}_r\mathfrak{h}_s^\dagger = -\frac{1}{4}(\xi_r \xi_s + i \xi_r \xi_{m+s} - i \xi_{m+r} \xi_s + \xi_{m+r} \xi_{m+s}). \tag{2.38}$$

Furthermore, for each $j = 1, \ldots, n$, and $r = 1, \ldots, m$ we have

$$\mathfrak{f}_j \wedge \mathfrak{f}_j = \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j^\dagger = 0 \tag{2.39}$$

$$\mathfrak{f}_j \wedge \mathfrak{f}_j^\dagger = -\mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j = -\frac{i}{2} e_j e_{n+j} \tag{2.40}$$

$$\mathfrak{h}_r \wedge \mathfrak{h}_r = \mathfrak{h}_r^\dagger \wedge \mathfrak{h}_r^\dagger = 0 \tag{2.41}$$

$$\mathfrak{h}_r \wedge \mathfrak{h}_r^\dagger = -\mathfrak{h}_r^\dagger \wedge \mathfrak{h}_r = -\frac{i}{2} \xi_r \xi_{m+r} \tag{2.42}$$

$$\mathfrak{f}_j \wedge \mathfrak{h}_r = -\mathfrak{h}_r \wedge \mathfrak{f}_j = \mathfrak{f}_j \mathfrak{h}_r = \frac{1}{4}(e_j \xi_r - i e_j \xi_{m+r} - i e_{n+j} \xi_r - e_{n+j} \xi_{m+r}) \tag{2.43}$$

$$\mathfrak{f}_j^\dagger \wedge \mathfrak{h}_r = -\mathfrak{h}_r \wedge \mathfrak{f}_j^\dagger = \mathfrak{f}_j^\dagger \mathfrak{h}_r = -\frac{1}{4}(e_j \xi_r - i e_j \xi_{m+r} + i e_{n+j} \xi_r + e_{n+j} \xi_{m+r}) \tag{2.44}$$

$$\mathfrak{f}_j \wedge \mathfrak{h}_r^\dagger = -\mathfrak{h}_r^\dagger \wedge \mathfrak{f}_j = \mathfrak{f}_j \mathfrak{h}_r^\dagger = -\frac{1}{4}(e_j \xi_r + i e_j \xi_{m+r} - i e_{n+j} \xi_r + e_{n+j} \xi_{m+r}) \tag{2.45}$$

$$\mathfrak{f}_j^\dagger \wedge \mathfrak{h}_r^\dagger = -\mathfrak{h}_r^\dagger \wedge \mathfrak{f}_j^\dagger = \mathfrak{f}_j^\dagger \mathfrak{h}_r^\dagger = \frac{1}{4}(e_j \xi_r + i e_j \xi_{m+r} + i e_{n+j} \xi_r - e_{n+j} \xi_{m+r}). \tag{2.46}$$

3 The pseudo-unitary group $\mathrm{U}(n, m)$

The pseudo-unitary group $\mathrm{U}(n, m)$ is the group of holomorphic transformations preserving the Hermitian form (2.1). It is well-known that $\mathrm{U}(n, m) = \mathrm{SO}^+(2n, 2m, \mathbb{R}) \cap \mathrm{Sp}(2(n+m), \mathbb{R})$ i.e., $\mathrm{U}(n, m)$ is both a real subgroup of the pseudo-orthogonal group $\mathrm{SO}^+(2n, 2m, \mathbb{R})$ and of the sympletic group $\mathrm{Sp}(2(n+m), \mathbb{R})$.

In this section we will consider the group $\mathrm{Spin}^+(2n, 2m, \mathbb{R})$, the double covering group of $\mathrm{SO}^+(2n, 2m, \mathbb{R})$ to construct a representation of the unitary group $\mathrm{U}(n, m)$. The group $\mathrm{Spin}^+(2n, 2m, \mathbb{R})$ can be described by

$$\mathrm{Spin}^+(2n, 2m, \mathbb{R}) = \{s \in \Gamma^+(2n, 2m, \mathbb{R}) : s\bar{s} = \bar{s}s = 1\},$$

where $\Gamma^+(2n, 2m, \mathbb{R})$ is the even Clifford group in $\mathbb{R}^{2n,2m}$. Usually, the Lie algebra $spin^+(2n, 2m, \mathbb{R})$ is the real algebra spanned by the bivectors

$$e_i e_j, \qquad i, j = 1, \ldots, 2n, \; i < j \tag{3.1}$$

$$\xi_r \xi_s, \qquad r, s = 1, \ldots, 2m, \; r < s \tag{3.2}$$

$$e_i \xi_r, \qquad i = 1, \ldots, 2n, \; r = 1, \ldots, 2m \tag{3.3}$$

generating space rotations, time rotations, and space-time rotations, or boosts, in $\mathbb{R}^{2n,2m}$. It is easy to see that the dimension of $spin^+(2n, 2m, \mathbb{R})$ is $n(2n-1) + m(2m-1) + 4nm = (n+m)(2(n+m)-1)$. When we want to exploit complex symmetries of spaces of even real dimension it is more appropriate to split the vector basis of $\mathbb{R}^{2n,2m}$ into

$$\{e_j, e_{n+j}, j = 1, \ldots, n\} \cup \{\xi_r, \xi_{m+r}, r = 1, \ldots, m\}$$

in order to identify real and imaginary axes. Therefore, we can write another basis for the Lie algebra of $\mathrm{Spin}^+(2n, 2m, \mathbb{R})$, more suited for our purposes.

Lemma 3.1 *The Lie algebra spin$^+(2n,2m,\mathbb{R})$ can be generated by the (real) bivectors*

$$S^1_j = e_je_{n+j} = 2i\,\mathfrak{f}_j \wedge \mathfrak{f}^\dagger_j,\ j=1,\dots,n \tag{3.4}$$
$$S^2_{j,k} = e_je_k + e_{n+j}e_{n+k} = -2(\mathfrak{f}_j \wedge \mathfrak{f}^\dagger_k - \mathfrak{f}_k \wedge \mathfrak{f}^\dagger_j),\ j,k=1,\dots,n,\ j\neq k \tag{3.5}$$
$$S^3_{j,k} = e_je_{n+k} - e_{n+j}e_k = 2i(\mathfrak{f}_j \wedge \mathfrak{f}^\dagger_k + \mathfrak{f}_k \wedge \mathfrak{f}^\dagger_j),\ j,k=1,\dots,n,\ j\neq k \tag{3.6}$$
$$S^4_{j,k} = e_je_k - e_{n+j}e_{n+k} = 2(\mathfrak{f}_j \wedge \mathfrak{f}_k + \mathfrak{f}^\dagger_j \wedge \mathfrak{f}^\dagger_k),\ j,k=1,\dots,n,\ j\neq k \tag{3.7}$$
$$S^5_{j,k} = e_je_{n+k} + e_{n+j}e_k = 2i(\mathfrak{f}_j \wedge \mathfrak{f}_k - \mathfrak{f}^\dagger_j \wedge \mathfrak{f}^\dagger_k),\ j,k=1,\dots,n,\ j\neq k \tag{3.8}$$

$$T^1_r = \xi_r\xi_{m+r} = 2i\,\mathfrak{h}_r \wedge \mathfrak{h}^\dagger_r,\ r=1,\dots,m \tag{3.9}$$
$$T^2_{r,s} = \xi_r\xi_s + \xi_{m+r}\xi_{m+s} = -2(\mathfrak{h}_r \wedge \mathfrak{h}^\dagger_s - \mathfrak{h}_s \wedge \mathfrak{h}^\dagger_r), r,s=1,\dots,m,\ r\neq s \tag{3.10}$$
$$T^3_{r,s} = \xi_r\xi_{m+s} - \xi_{m+r}\xi_s = 2i(\mathfrak{h}_r \wedge \mathfrak{h}^\dagger_s + \mathfrak{h}_s \wedge \mathfrak{h}^\dagger_r),\ r,s=1,\dots,m,\ r\neq s \tag{3.11}$$
$$T^4_{r,s} = \xi_r\xi_s - \xi_{m+r}\xi_{m+s} = 2(\mathfrak{h}_r \wedge \mathfrak{h}_s + \mathfrak{h}^\dagger_r \wedge \mathfrak{h}^\dagger_s),\ r,s=1,\dots,m,\ r\neq s \tag{3.12}$$
$$T^5_{r,s} = \xi_r\xi_{m+s} + \xi_{m+r}\xi_s = 2i(\mathfrak{h}_r \wedge \mathfrak{h}_s - \mathfrak{h}^\dagger_r \wedge \mathfrak{h}^\dagger_s),\ r,s=1,\dots,m,\ r\neq s \tag{3.13}$$

$$B^1_{j,r} = e_j\xi_r + e_{n+j}\xi_{m+r} = -2(\mathfrak{f}^\dagger_j \wedge \mathfrak{h}_r + \mathfrak{f}_j \wedge \mathfrak{h}^\dagger_r),\ j=1,\dots,n,\ r=1,\dots,m \tag{3.14}$$
$$B^2_{j,r} = e_j\xi_{m+r} - e_{n+j}\xi_r = -2i(\mathfrak{f}^\dagger_j \wedge \mathfrak{h}_r - \mathfrak{f}_j \wedge \mathfrak{h}^\dagger_r), j=1,\dots,n,\ r=1,\dots,m \tag{3.15}$$
$$B^3_{j,r} = e_j\xi_r - e_{n+j}\xi_{m+r} = 2(\mathfrak{f}_j \wedge \mathfrak{h}_r + \mathfrak{f}^\dagger_j \wedge \mathfrak{h}^\dagger_r),\ j=1,\dots,n,\ r=1,\dots,m \tag{3.16}$$
$$B^4_{j,r} = e_j\xi_{m+r} + e_{n+j}\xi_r = 2i(\mathfrak{f}_j \wedge \mathfrak{h}_r - \mathfrak{f}^\dagger_j \wedge \mathfrak{h}^\dagger_r),\ j=1,\dots,n,\ r=1,\dots,m. \tag{3.17}$$

Proof: Since the proposed elements are linearly independent and the dimension equals $n+4\binom{n}{2}+m+4\binom{m}{2}+4mn=(n+m)(2(n+m)-1)$ they constitute a basis of *spin*$^+(2n,2m,\mathbb{R})$.

∎

Henceforward, we shall refer to elements (3.4)-(3.8) as complex space bivectors, elements (3.9)-(3.13) as complex time bivectors, and elements (3.14)-(3.17) as complex space-time bivectors since they will generate complex space rotations, complex time rotations, and complex-time rotations or complex boosts, respectively, as we will see in this section.

Lemma 3.2 *For $j,k,l,l'=1,\dots n$, $r,s,q,q'=1,\dots,m$, $\beta_1,\beta_2=2,3,4,5$, and $\gamma=1,2,3,4$ the following commutation rules hold:*

1. *Commutation relations between complex space bivectors:*

$$\begin{aligned}
[S^1_j, S^1_k] &= 0\\
[S^1_j, S^2_{l,k}] &= -2S^3_{j,k}\delta_{jl}, j\neq k\\
[S^1_j, S^3_{l,k}] &= 2S^2_{j,k}\delta_{jl}, j\neq k\\
[S^1_j, S^4_{l,k}] &= 2S^5_{j,k}\delta_{jl}, j\neq k\\
[S^1_j, S^5_{l,k}] &= -2S^4_{j,k}\delta_{jl}, j\neq k\\
[S^2_{j,k}, S^2_{l,k}] &= 2S^2_{j,l}(1-\delta_{jl})\\
[S^2_{j,k}, S^3_{j,k}] &= 4\left(S^1_k - S^1_j\right)\\
[S^2_{j,k}, S^3_{l,k}] &= -2S^3_{j,l}, j\neq l\\
[S^2_{j,k}, S^4_{l,k}] &= 2S^4_{j,l}(1-\delta_{jl})
\end{aligned}$$

$$\begin{aligned}
[S^2_{j,k}, S^5_{l,k}] &= 2S^5_{j,l}(1-\delta_{jl})\\
[S^3_{j,k}, S^3_{l,k}] &= 2S^2_{j,l}(1-\delta_{jl})\\
[S^3_{j,k}, S^4_{l,k}] &= -2S^5_{j,l}(1-\delta_{jl})\\
[S^3_{j,k}, S^5_{l,k}] &= 2S^4_{j,l}(1-\delta_{jl})\\
[S^4_{j,k}, S^4_{l,k}] &= 2S^2_{j,l}(1-\delta_{jl})\\
[S^4_{j,k}, S^5_{j,k}] &= 4\left(S^1_j + S^1_k\right)\\
[S^4_{j,k}, S^5_{l,k}] &= 2S^3_{j,l}, j\neq l\\
[S^5_{j,k}, S^5_{l,k}] &= 2S^2_{j,l}(1-\delta_{jl})\\
\left[S^{\beta_1}_{j,k}, S^{\beta_2}_{l,l'}\right] &= 0,\ j\neq l, k\neq l'
\end{aligned}$$

2. *Commutation relations between complex time bivectors:*

$$
\begin{aligned}
\left[T^1_r, T^1_s\right] &= 0 \\
\left[T^1_r, T^2_{q,s}\right] &= 2T^3_{r,s}\delta_{rq}, r \neq s \\
\left[T^1_r, T^3_{q,s}\right] &= -2T^2_{r,s}\delta_{rq}, r \neq s \\
\left[T^1_r, T^4_{q,s}\right] &= -2T^5_{r,s}\delta_{rq}, r \neq s \\
\left[T^1_r, T^5_{q,s}\right] &= 2T^4_{r,s}\delta_{rq}, r \neq s \\
\left[T^2_{r,s}, T^2_{q,s}\right] &= -2T^2_{r,q}(1-\delta_{rq}) \\
\left[T^2_{r,s}, T^3_{r,s}\right] &= 4\left(T^1_r - T^1_s\right) \\
\left[T^2_{r,s}, T^3_{q,s}\right] &= 2T^3_{r,q}, r \neq q \\
\left[T^2_{r,s}, T^4_{q,s}\right] &= -2T^4_{r,q}(1-\delta_{rq})
\end{aligned}
$$

$$
\begin{aligned}
\left[T^2_{r,s}, T^5_{q,s}\right] &= -2T^5_{r,q}(1-\delta_{rq}) \\
\left[T^3_{r,s}, T^3_{q,s}\right] &= -2T^3_{r,q}(1-\delta_{rq}) \\
\left[T^3_{r,s}, T^4_{q,s}\right] &= 2T^5_{r,q}(1-\delta_{rq}) \\
\left[T^3_{r,s}, T^5_{q,s}\right] &= -2T^4_{r,q}(1-\delta_{rq}) \\
\left[T^4_{r,s}, T^4_{q,s}\right] &= -2T^2_{r,q}(1-\delta_{rq}) \\
\left[T^4_{r,s}, T^5_{r,s}\right] &= -4\left(T^1_r + T^1_s\right) \\
\left[T^4_{r,s}, T^5_{q,s}\right] &= -2T^3_{r,q}, r \neq q \\
\left[T^5_{r,s}, T^5_{q,s}\right] &= -2T^2_{r,q}(1-\delta_{rq}) \\
\left[T^{\beta_1}_{r,s}, T^{\beta_2}_{q,q'}\right] &= 0,\, r \neq q, s \neq q'
\end{aligned}
$$

3. *Commutation relations between complex space-time bivectors:*

$$
\begin{aligned}
\left[B^\gamma_{j,r}, B^\gamma_{j,q}\right] &= 2T^2_{r,q}(1-\delta_{rq}),\, \forall\gamma \\
\left[B^1_{j,r}, B^1_{l,q}\right] &= -2S^2_{j,l}\delta_{rq}, j \neq l \\
\left[B^1_{j,r}, B^2_{j,r}\right] &= 4\left(S^1_j + T^1_r\right) \\
\left[B^1_{j,r}, B^2_{j,q}\right] &= -2T^3_{r,q}, r \neq q \\
\left[B^1_{j,r}, B^2_{l,q}\right] &= 2S^3_{j,l}\delta_{rq}, j \neq l
\end{aligned}
$$

$$
\begin{aligned}
\left[B^2_{j,r}, B^2_{l,q}\right] &= -2S^2_{j,l}\delta_{rq},\, j \neq l \\
\left[B^2_{j,r}, B^3_{j,q}\right] &= 2T^5_{r,q}(1-\delta_{rq}) \\
\left[B^2_{j,r}, B^3_{l,q}\right] &= 2S^5_{j,l}\delta_{rq},\, j \neq l \\
\left[B^2_{j,r}, B^4_{j,q}\right] &= -2T^4_{r,q}(1-\delta_{rq}) \\
\left[B^2_{j,r}, B^4_{l,q}\right] &= 2S^2_{j,l}\delta_{rq},\, j \neq l
\end{aligned}
$$

$$
\begin{aligned}
\left[B^1_{j,r}, B^3_{j,q}\right] &= 2T^4_{r,q}(1-\delta_{rq}) \\
\left[B^2_{j,r}, B^3_{l,q}\right] &= -2B^4_{j,l}\delta_{rq}, j \neq l \\
\left[B^1_{j,r}, B^4_{j,r}\right] &= -4S^1_j \\
\left[B^1_{j,r}, B^4_{j,q}\right] &= 2T^5_{r,q}, r \neq q \\
\left[B^1_{j,r}, B^4_{l,q}\right] &= -2S^5_{j,l}\delta_{rq}, j \neq l
\end{aligned}
$$

$$
\begin{aligned}
\left[B^3_{j,r}, B^3_{l,q}\right] &= -2S^2_{j,l}\delta_{rq},\, j \neq l \\
\left[B^3_{j,r}, B^4_{j,r}\right] &= -4\left(S^1_j - T^1_r\right) \\
\left[B^3_{j,r}, B^4_{j,q}\right] &= 2T^3_{r,q}, r \neq q \\
\left[B^3_{j,r}, B^4_{l,q}\right] &= -2S^3_{j,l}\delta_{rq},\, j \neq l \\
\left[B^4_{j,r}, B^4_{l,q}\right] &= -2S^3_{j,l}\delta_{rq},\, j \neq l
\end{aligned}
$$

4. *Commutation relations between complex space and time bivectors:*

$$
\begin{aligned}
\left[S^1_j, T^1_r\right] &= 0 \\
\left[S^1_j, T^{\beta_2}_{r,s}\right] &= 0
\end{aligned}
$$

$$
\begin{aligned}
\left[S^{\beta_1}_{j,k}, T^1_r\right] &= 0 \\
\left[S^{\beta_1}_{j,k}, T^{\beta_2}_{r,s}\right] &= 0
\end{aligned}
$$

5. *Commutation relations between complex space and space-time bivectors:*

$$
\begin{aligned}
\left[S_j^1, B_{l,r}^1\right] &= -2B_{j,r}^2\delta_{jl} & \left[S_{j,k}^3, B_{l,r}^3\right] &= 2B_{k,r}^4\delta_{jl} \\
\left[S_j^1, B_{l,r}^2\right] &= 2B_{j,r}^1\delta_{jl} & \left[S_{j,k}^3, B_{l,r}^4\right] &= -2B_{k,r}^3\delta_{jl} \\
\left[S_j^1, B_{l,r}^3\right] &= 2B_{j,r}^4\delta_{jl} & \left[S_{j,k}^4, B_{l,r}^1\right] &= 2B_{k,r}^4\delta_{jl} \\
\left[S_j^1, B_{l,r}^4\right] &= -2B_{j,r}^3\delta_{jl} & \left[S_{j,k}^4, B_{l,r}^2\right] &= 2B_{k,r}^4\delta_{jl} \\
\left[S_{j,k}^2, B_{l,r}^1\right] &= 2B_{k,r}^1\delta_{jl} & \left[S_{j,k}^4, B_{l,r}^3\right] &= 2B_{k,r}^1\delta_{jl} \\
\left[S_{j,k}^2, B_{l,r}^2\right] &= 2B_{k,r}^2\delta_{jl} & \left[S_{j,k}^4, B_{l,r}^4\right] &= 2B_{k,r}^2\delta_{jl} \\
\left[S_{j,k}^2, B_{l,r}^3\right] &= 2B_{k,r}^3\delta_{jl} & \left[S_{j,k}^5, B_{l,r}^1\right] &= 2B_{k,r}^4\delta_{jl} \\
\left[S_{j,k}^2, B_{l,r}^4\right] &= 2B_{k,r}^4\delta_{jl} & \left[S_{j,k}^5, B_{l,r}^2\right] &= -2B_{k,r}^1\delta_{jl} \\
\left[S_{j,k}^3, B_{l,r}^1\right] &= -2B_{k,r}^2\delta_{jl} & \left[S_{j,k}^5, B_{l,r}^3\right] &= -2B_{k,r}^4\delta_{jl} \\
\left[S_{j,k}^3, B_{l,r}^2\right] &= 2B_{k,r}^1\delta_{jl} & \left[S_{j,k}^5, B_{l,r}^4\right] &= 2B_{k,r}^1\delta_{jl}
\end{aligned}
$$

6. *Commutation relations between complex time and space-time bivectors:*

$$
\begin{aligned}
\left[T_r^1, B_{j,q}^1\right] &= -2B_{j,r}^2\delta_{rq} & \left[T_{r,s}^3, B_{j,q}^3\right] &= -2B_{j,s}^4\delta_{rq} \\
\left[T_r^1, B_{j,q}^2\right] &= 2B_{j,r}^1\delta_{rq} & \left[T_{r,s}^3, B_{j,q}^4\right] &= 2B_{j,s}^3\delta_{rq} \\
\left[T_r^1, B_{j,q}^3\right] &= -2B_{j,r}^4\delta_{rq} & \left[T_{r,s}^4, B_{j,q}^1\right] &= -2B_{j,s}^3\delta_{rq} \\
\left[T_r^1, B_{j,q}^4\right] &= 2B_{j,r}^3\delta_{rq} & \left[T_{r,s}^4, B_{j,q}^2\right] &= 2B_{j,s}^4\delta_{rq} \\
\left[T_{r,s}^2, B_{j,q}^1\right] &= -2B_{j,s}^1\delta_{rq} & \left[T_{r,s}^4, B_{j,q}^3\right] &= -2B_{j,s}^1\delta_{rq} \\
\left[T_{r,s}^2, B_{j,q}^2\right] &= -2B_{j,s}^2\delta_{rq} & \left[T_{r,s}^4, B_{j,q}^4\right] &= 2B_{j,s}^2\delta_{rq} \\
\left[T_{r,s}^2, B_{j,q}^3\right] &= -2B_{j,s}^1\delta_{rq} & \left[T_{r,s}^5, B_{j,q}^1\right] &= -2B_{j,s}^4\delta_{rq} \\
\left[T_{r,s}^2, B_{j,q}^4\right] &= -2B_{j,s}^4\delta_{rq} & \left[T_{r,s}^5, B_{j,q}^2\right] &= -2B_{j,s}^3\delta_{rq} \\
\left[T_{r,s}^3, B_{j,q}^1\right] &= -2B_{j,s}^2\delta_{rq} & \left[T_{r,s}^5, B_{j,q}^3\right] &= 2B_{j,s}^2\delta_{rq} \\
\left[T_{r,s}^3, B_{j,q}^2\right] &= 2B_{j,s}^1\delta_{rq} & \left[T_{r,s}^5, B_{j,q}^4\right] &= -2B_{j,s}^1\delta_{rq}.
\end{aligned}
$$

From these commutation relations, a subalgebra of $spin^+(2n, 2m, \mathbb{R})$ can be identified.

Lemma 3.3 *The elements*

$$\begin{aligned}
S_j^1 &= e_j e_{n+j} = 2i\, \mathfrak{f}_j \wedge \mathfrak{f}_j^\dagger, j = 1, \ldots, n \\
S_{j,k}^2 &= e_j e_k + e_{n+j} e_{n+k} = -2(\mathfrak{f}_j \wedge \mathfrak{f}_k^\dagger - \mathfrak{f}_k \wedge \mathfrak{f}_j^\dagger)\,, j,k = 1, \ldots, n,\ j \neq k \\
S_{j,k}^3 &= e_j e_{n+k} - e_{n+j} e_k = 2i(\mathfrak{f}_j \wedge \mathfrak{f}_k^\dagger + \mathfrak{f}_k \wedge \mathfrak{f}_j^\dagger)\ , j,k = 1, \ldots, n,\ j \neq k \\
T_r^1 &= \xi_r \xi_{m+r} = 2i\, \mathfrak{h}_r \wedge \mathfrak{h}_r^\dagger,\ r = 1, \ldots, m \\
T_{r,s}^2 &= \xi_r \xi_s + \xi_{m+r} \xi_{m+s} = -2(\mathfrak{h}_r \wedge \mathfrak{h}_s^\dagger - \mathfrak{h}_s \wedge \mathfrak{h}_r^\dagger),\ r,s = 1, \ldots, m,\ r \neq s \\
T_{r,s}^3 &= \xi_r \xi_{m+s} - \xi_{m+r} \xi_s = 2i(\mathfrak{h}_r \wedge \mathfrak{h}_s^\dagger + \mathfrak{h}_s \wedge \mathfrak{h}_r^\dagger)\ , r,s = 1, \ldots, m,\ r \neq s \\
B_{j,r}^1 &= e_j \xi_r + e_{n+j} \xi_{m+r} = -2(\mathfrak{f}_j^\dagger \wedge \mathfrak{h}_r + \mathfrak{f}_j \wedge \mathfrak{h}_r^\dagger)\ , j = 1, \ldots, n,\ r = 1, \ldots, m \\
B_{j,r}^2 &= e_j \xi_{m+r} - e_{n+j} \xi_r = -2i(\mathfrak{f}_j^\dagger \wedge \mathfrak{h}_r - \mathfrak{f}_j \wedge \mathfrak{h}_r^\dagger),\ j = 1, \ldots, n,\ r = 1, \ldots, m
\end{aligned}$$

constitute a Lie subalgebra of the Lie algebra $spin^+(2n, 2m, \mathbb{R})$.

Proof: Since the Lie bracket is closed under these elements they define a subalgebra of $spin^+(2n, 2m, \mathbb{R})$.

■

The Lie subalgebra defined in Lemma 3.3 defines a Lie group of dimension $(n+m)^2$ isomorphic to the unitary group $\mathrm{U}(n,m)$. Before we realize this let us compute the spin actions generated by the elements (3.4)-(3.17) of the Lie algebra $spin^+(2n, 2m, \mathbb{R})$. For $s \in \mathrm{Spin}^+(2n, 2m, \mathbb{R})$ its spin-1 representation is given by

$$h(s) : \underline{X} \mapsto s\underline{X}\overline{s}, \quad \underline{X} \in \mathbb{R}_{2n,2m},$$

which preserves the multi-structure of $\mathbb{R}_{2n,2m}$. The spin elements $s \in \mathrm{Spin}^+(2n, 2m, \mathbb{R})$ associated to (3.4)-(3.8) are obtained by exponentiation of the elements of the Lie algebra $spin^+(2n, 2m, \mathbb{R})$:

$$\begin{aligned}
s_j^1 &= e^{\frac{\theta}{2} e_j e_{n+j}},\ j = 1, \ldots, n && (3.18) \\
s_{j,k}^2 &= e^{\frac{\theta}{2}(e_j e_k + e_{n+j} e_{n+k})} = e^{\frac{\theta}{2} e_j e_k} e^{\frac{\theta}{2} e_{n+j} e_{n+k}},\ j,k = 1, \ldots, n, j \neq k && (3.19) \\
s_{j,k}^3 &= e^{\frac{\theta}{2}(e_j e_{n+k} - e_{n+j} e_k)} = e^{\frac{\theta}{2} e_j e_{n+k}} e^{-\frac{\theta}{2} e_{n+j} e_k},\ j,k = 1, \ldots, n, j \neq k && (3.20) \\
s_{j,k}^4 &= e^{\frac{\theta}{2}(e_j e_k - e_{n+j} e_{n+k})} = e^{\frac{\theta}{2} e_j e_k} e^{-\frac{\theta}{2} e_{n+j} e_{n+k}},\ j,k = 1, \ldots, n, j \neq k && (3.21) \\
s_{j,k}^5 &= e^{\frac{\theta}{2}(e_j e_{n+k} + e_{n+j} e_k)} = e^{\frac{\theta}{2} e_j e_{n+k}} e^{\frac{\theta}{2} e_{n+j} e_k},\ j,k = 1, \ldots, n, j \neq k && (3.22) \\
s_r^6 &= e^{\frac{\theta}{2} \xi_r \xi_{m+s}},\ r = 1, \ldots, m && (3.23) \\
s_{r,s}^7 &= e^{\frac{\theta}{2}(\xi_r \xi_s + \xi_{m+r} \xi_{m+s})} = e^{\frac{\theta}{2} \xi_r \xi_s} e^{\frac{\theta}{2} \xi_{m+r} \xi_{m+s}},\ r,s = 1, \ldots, m, r \neq s && (3.24) \\
s_{r,s}^8 &= e^{\frac{\theta}{2}(\xi_r \xi_{m+s} - \xi_{m+r} \xi_s)} = e^{\frac{\theta}{2} \xi_r \xi_{m+s}} e^{-\frac{\theta}{2} \xi_{m+r} \xi_s},\ r,s = 1, \ldots, m, r \neq s && (3.25) \\
s_{r,s}^9 &= e^{\frac{\theta}{2}(\xi_r \xi_s - \xi_{m+r} \xi_{m+s})} = e^{\frac{\theta}{2} \xi_r \xi_s} e^{-\frac{\theta}{2} \xi_{m+r} \xi_{m+s}},\ r,s = 1, \ldots, m, r \neq s && (3.26) \\
s_{r,s}^{10} &= e^{\frac{\theta}{2}(\xi_r \xi_{m+s} + \xi_{m+r} \xi_s)} = e^{\frac{\theta}{2} \xi_r \xi_{m+s}} e^{\frac{\theta}{2} \xi_{m+r} \xi_s},\ r,s = 1, \ldots, m, r \neq s && (3.27)
\end{aligned}$$

$$s_{j,r}^{11} = e^{\frac{\alpha}{2}(e_j\xi_r + e_{n+j}\xi_{m+r})} = e^{\frac{\alpha}{2}e_j\xi_r} e^{\frac{\alpha}{2}e_{n+j}\xi_{m+r}}, j = 1, \ldots, n, r = 1, \ldots, m \tag{3.28}$$
$$s_{j,r}^{12} = e^{\frac{\alpha}{2}(e_j\xi_{m+r} - e_{n+j}\xi_r)} = e^{\frac{\alpha}{2}e_j\xi_{m+r}} e^{-\frac{\alpha}{2}e_{n+j}\xi_r}, j = 1, \ldots, n, r = 1, \ldots, m \tag{3.29}$$
$$s_{j,r}^{13} = e^{\frac{\alpha}{2}(e_j\xi_r - e_{n+j}\xi_{m+r})} = e^{\frac{\alpha}{2}e_j\xi_r} e^{-\frac{\alpha}{2}e_{n+j}\xi_{m+r}}, j = 1, \ldots, n, r = 1, \ldots, m \tag{3.30}$$
$$s_{j,r}^{14} = e^{\frac{\alpha}{2}(e_j\xi_{m+r} + e_{n+j}\xi_r)} = e^{\frac{\alpha}{2}e_j\xi_{m+r}} e^{\frac{\alpha}{2}e_{n+j}\xi_r}, j = 1, \ldots, n,\ r = 1, \ldots, m. \tag{3.31}$$

In (3.19)-(3.22), and (3.24)-(3.31), the exponential law is valid since

$$\begin{aligned}
[e_je_k, e_{n+j}e_{n+k}] &= [e_je_{n+k}, e_{n+j}e_k] = 0,\ j,k = 1,\ldots,n,\ j \neq k \\
[\xi_r\xi_s, \xi_{m+r}\xi_{m+s}] &= [\xi_r\xi_{m+s}, \xi_{m+r}\xi_s] = 0,\ r,s = 1,\ldots,m,\ r \neq s, \\
[e_j\xi_{m+r}, e_{n+j}\xi_{m+r}] &= [e_j\xi_{m+r}, e_{n+j}\xi_r] = 0,\ j = 1,\ldots,n, r = 1,\ldots,m.
\end{aligned}$$

The elements (3.18)-(3.31) are a basis of $\mathrm{Spin}^+(2n, 2m, \mathbb{R})$. We choose the circular angle $\theta \in [0, 2\pi[$ for Euclidean space and time rotations whereas the hyperbolic angle $\alpha \in \mathbb{R}$ will be associated to hyperbolic rotations. Since for an arbitrary bivector B such that $B^2 = -1$ we have

$$e^{\frac{\theta}{2}B} = \cos\frac{\theta}{2} + B\sin\frac{\theta}{2} \tag{3.32}$$

and if $B^2 = +1$ we have

$$e^{\frac{\alpha}{2}B} = \cosh\frac{\alpha}{2} + B\sinh\frac{\alpha}{2}$$

we can easily compute the spin actions of the elements (3.18)-(3.31) on a given vector $\underline{X} \in \mathbb{R}^{2n,2m}$. In real coordinates they are given by

$$\begin{aligned}
s_j^1\underline{X}\overline{s_j^1} = & \sum_{t=1,t\neq j}^{n} (x_te_t + y_te_{n+t}) + (x_j\cos\theta - y_j\sin\theta)e_j + (x_j\sin\theta + y_j\cos\theta)e_{n+j} \\
& + \sum_{r=1}^{m}(u_r\xi_r + v_r\xi_{m+r})
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
s_{j,k}^2\underline{X}\overline{s_{j,k}^2} = & \sum_{t=1,t\neq j,k}^{n} (x_te_t + y_te_{n+t}) + (x_j\cos\theta - x_k\sin\theta)e_j + (y_j\cos\theta - y_k\sin\theta)e_{n+j} \\
& + (x_j\sin\theta + x_k\cos\theta)e_k + (y_j\sin\theta + y_k\cos\theta)e_{n+k} + \sum_{r=1}^{m}(u_r\xi_r + v_r\xi_{m+r})
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
s_{j,k}^3\underline{X}\overline{s_{j,k}^3} = & \sum_{t=1,t\neq j,k}^{n} (x_te_t + y_te_{n+t}) + (x_j\cos\theta - y_k\sin\theta)e_j + (y_j\cos\theta + x_k\sin\theta)e_{n+j} \\
& + (-y_j\sin\theta + x_k\cos\theta)e_k + (x_j\sin\theta + y_k\cos\theta)e_{n+k} + \sum_{r=1}^{m}(u_r\xi_r + v_r\xi_{m+r})
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
s_{j,k}^4\underline{X}\overline{s_{j,k}^4} = & \sum_{t=1,t\neq j,k}^{n} (x_te_t + y_te_{n+t}) + (x_j\cos\theta - x_k\sin\theta)e_j + (y_j\cos\theta + y_k\sin\theta)e_{n+j} \\
& + (x_j\sin\theta + x_k\cos\theta)e_k + (-y_j\sin\theta + y_k\cos\theta)e_{n+k} + \sum_{r=1}^{m}(u_r\xi_r + v_r\xi_{m+r})
\end{aligned} \tag{3.36}$$

$$s^5_{j,k}\underline{X}\overline{s^5_{j,k}} = \sum_{t=1,t\neq j,k}^{n}(x_te_t+y_te_{n+t})+(x_j\cos\theta-y_k\sin\theta)e_j+(y_j\cos\theta-x_k\sin\theta)e_{n+j}$$
$$+(y_j\sin\theta+x_k\cos\theta)e_k+(x_j\sin\theta+y_k\cos\theta)e_{n+k}+\sum_{r=1}^{m}(u_r\xi_r+v_r\xi_{m+r}) \tag{3.37}$$

$$s^6_{r,s}\underline{X}\overline{s^6_{r,s}} = \sum_{j=1}^{n}(x_je_j+y_je_{n+j})+\sum_{r=1,r\neq s}^{m}(u_r\xi_r+v_r\xi_{m+r})+(u_r\cos\theta+v_r\sin\theta)\xi_r$$
$$+(-u_r\sin\theta+v_r\cos\theta)\xi_{m+r} \tag{3.38}$$

$$s^7_{r,s}\underline{X}\overline{s^7_{r,s}} = \sum_{j=1}^{n}(x_je_j+y_je_{n+j})+\sum_{t=1,t\neq r}^{m}(u_t\xi_t+v_t\xi_{m+t})+(u_r\cos\theta+u_s\sin\theta)\xi_r$$
$$+(v_r\cos\theta+v_s\sin\theta)\xi_{m+r}+(-u_r\sin\theta+u_s\cos\theta)\xi_s+(-v_r\sin\theta+v_s\cos\theta)\xi_{m+s} \tag{3.39}$$

$$s^8_{r,s}\underline{X}\overline{s^8_{r,s}} = \sum_{j=1}^{n}(x_je_j+y_je_{n+j})+\sum_{t=1,t\neq r}^{m}(u_t\xi_t+v_t\xi_{m+t})+(u_r\cos\theta+v_s\sin\theta)\xi_r$$
$$+(v_r\cos\theta-u_s\sin\theta)\xi_{m+r}+(v_r\sin\theta+u_s\cos\theta)\xi_s+(-u_r\sin\theta+v_s\cos\theta)\xi_{m+s} \tag{3.40}$$

$$s^9_{r,s}\underline{X}\overline{s^9_{r,s}} = \sum_{j=1}^{n}(x_je_j+y_je_{n+j})+\sum_{t=1,t\neq r}^{m}(u_t\xi_t+v_t\xi_{m+t})+(u_r\cos\theta+u_s\sin\theta)\xi_r$$
$$+(v_r\cos\theta-v_s\sin\theta)\xi_{m+r}+(-u_r\sin\theta+u_s\cos\theta)\xi_s+(v_r\sin\theta+v_s\cos\theta)\xi_{m+s} \tag{3.41}$$

$$s^{10}_{r,s}\underline{X}\overline{s^{10}_{r,s}} = \sum_{j=1}^{n}(x_je_j+y_je_{n+j})+\sum_{t=1,t\neq r}^{m}(u_t\xi_t+v_t\xi_{m+t})+(u_r\cos\theta+v_s\sin\theta)\xi_r$$
$$+(v_r\cos\theta+u_s\sin\theta)\xi_{m+r}+(-v_r\sin\theta+u_s\cos\theta)\xi_s+(-u_r\sin\theta+v_s\cos\theta)\xi_{m+s} \tag{3.42}$$

$$s^{11}_{r,s}\underline{X}\overline{s^{11}_{r,s}} = \sum_{t=1,t\neq j}^{n}(x_te_t+y_te_{n+t})+\sum_{t=1,t\neq s}^{m}(u_t\xi_t+v_t\xi_{m+t})+(x_j\cosh\alpha+u_r\sinh\alpha)e_j$$
$$+(y_j\cosh\alpha+v_r\sinh\alpha)e_{n+j}+(x_j\sinh\alpha+u_r\cosh\alpha)\xi_r+(y_j\sinh\alpha+v_r\cosh\alpha)\xi_{m+r} \tag{3.43}$$

$$s^{12}_{r,s}\underline{X}\overline{s^{12}_{r,s}} = \sum_{t=1,t\neq j}^{n}(x_te_t+y_te_{n+t})+\sum_{t=1,t\neq r}^{m}(u_t\xi_t+v_t\xi_{m+t})+(x_j\cosh\alpha+v_r\sinh\alpha)e_j+$$
$$(y_j\cosh\alpha-u_r\sinh\alpha)e_{n+j}+(-y_j\sinh\alpha+u_r\cosh\alpha)\xi_r+(x_j\sinh\alpha+v_r\cosh\alpha)\xi_{m+r} \tag{3.44}$$

$$s^{13}_{r,s}\underline{X}\overline{s^{13}_{r,s}} = \sum_{t=1,t\neq j}^{n}(x_te_t+y_te_{n+t})+\sum_{t=1,t\neq r}^{m}(u_t\xi_t+v_t\xi_{m+t})+(x_j\cosh\alpha+u_r\sinh\alpha)e_j+$$
$$(y_j\cosh\alpha-v_r\sinh\alpha)e_{n+j}+(x_j\sinh\alpha+u_r\cosh\alpha)\xi_r+(-y_j\sinh\alpha+v_r\cosh\alpha)\xi_{m+r} \tag{3.45}$$

$$s^{14}_{r,s}\underline{X}\overline{s^{14}_{r,s}} = \sum_{t=1,t\neq j}^{n}(x_te_t+y_te_{n+t})+\sum_{t=1,t\neq r}^{m}(u_t\xi_t+v_t\xi_{m+t})+(x_j\cosh\alpha+v_r\sinh\alpha)e_j$$
$$+(y_j\cosh\alpha+u_r\sinh\alpha)e_{n+j}+(y_j\sinh\alpha+u_r\cosh\alpha)\xi_r+(x_j\sinh\alpha+v_r\cosh\alpha)\xi_{m+r} \tag{3.46}$$

Their complex form is obtained by passing to the Witt basis. Using (2.15) and (2.16) we obtain the following complex transformations:

$$s^1_j\underline{X}\overline{s^1_j} = \sum_{t=1,t\neq j}^{n}(z_t\mathfrak{f}_t-\overline{z_t}\mathfrak{f}_t)+e^{i\theta}z_j\mathfrak{f}_j-\overline{e^{i\theta}}\overline{z_j}\mathfrak{f}_j^\dagger+\sum_{r=1}^{m}(t_r\mathfrak{h}_r-\overline{t_r}\mathfrak{h}_r^\dagger) \tag{3.47}$$

$$s^2_{j,k}\underline{X}\overline{s^2_{j,k}} = \sum_{t=1,t\neq j,k}^{n}(z_t\mathfrak{f}_t - \overline{z_t}\mathfrak{f}_t^\dagger) + (z_j\cos\theta - z_k\sin\theta)\mathfrak{f}_j - \overline{(z_j\cos\theta - z_k\sin\theta)}\mathfrak{f}_j^\dagger$$
$$+(z_j\sin\theta + z_k\cos\theta)\mathfrak{f}_k - \overline{(z_j\sin\theta + z_k\cos\theta)}\mathfrak{f}_k^\dagger + \sum_{r=1}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) \tag{3.48}$$

$$s^3_{j,k}\underline{X}\overline{s^3_{j,k}} = \sum_{t=1,t\neq j,k}^{n}(z_t\mathfrak{f}_t - \overline{z_t}\mathfrak{f}_t^\dagger) + (z_j\cos\theta + iz_k\sin\theta)\mathfrak{f}_j - \overline{(z_j\cos\theta + iz_k\sin\theta)}\mathfrak{f}_j^\dagger$$
$$+(iz_j\sin\theta + z_k\cos\theta)\mathfrak{f}_k - \overline{(iz_j\sin\theta + z_k\cos\theta)}\mathfrak{f}_k^\dagger + \sum_{r=1}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) \tag{3.49}$$

$$s^4_{j,k}\underline{X}\overline{s^4_{j,k}} = \sum_{t=1,t\neq j,k}^{n}(z_t\mathfrak{f}_t - \overline{z_t}\mathfrak{f}_t^\dagger) + (z_j\cos\theta - \overline{z_k}\sin\theta)\mathfrak{f}_j - \overline{(z_j\cos\theta - \overline{z_k}\sin\theta)}\mathfrak{f}_j^\dagger$$
$$+(\overline{z_j}\sin\theta + z_k\cos\theta)\mathfrak{f}_k - \overline{(\overline{z_j}\sin\theta + z_k\cos\theta)}\mathfrak{f}_k^\dagger + \sum_{r=1}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) \tag{3.50}$$

$$s^5_{j,k}\underline{X}\overline{s^5_{j,k}} = \sum_{t=1,t\neq j,k}^{n}(z_t\mathfrak{f}_t - \overline{z_t}\mathfrak{f}_t^\dagger) + (z_j\cos\theta - i\overline{z_k}\sin\theta)\mathfrak{f}_j - \overline{(z_j\cos\theta - i\overline{z_k}\sin\theta)}\mathfrak{f}_j^\dagger$$
$$+(i\overline{z_j}\sin\theta + z_k\cos\theta)\mathfrak{f}_k - \overline{(i\overline{z_j}\sin\theta + z_k\cos\theta)}\mathfrak{f}_k^\dagger + \sum_{r=1}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) \tag{3.51}$$

$$s^6_{r,s}\underline{X}\overline{s^6_{r,s}} = \sum_{j=1}^{n}(z_j\mathfrak{f}_j - \overline{z_j}\mathfrak{f}_j^\dagger) + \sum_{t=1,t\neq r}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) + t_re^{-i\theta}\mathfrak{h}_r - \overline{t_r}e^{i\theta}\mathfrak{h}_r^\dagger \tag{3.52}$$

$$s^7_{r,s}\underline{X}\overline{s^7_{r,s}} = \sum_{j=1}^{n}(z_j\mathfrak{f}_j - \overline{z_j}\mathfrak{f}_j^\dagger) + \sum_{t=1,t\neq r,s}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) + (t_r\cos\theta + t_s\sin\theta)\mathfrak{h}_r$$
$$-\overline{(t_r\cos\theta + t_s\sin\theta)}\mathfrak{h}_r^\dagger + (-t_r\sin\theta + t_s\cos\theta)\mathfrak{h}_s - \overline{(-t_r\sin\theta + t_s\cos\theta)}\mathfrak{h}_s^\dagger \tag{3.53}$$

$$s^8_{r,s}\underline{X}\overline{s^8_{r,s}} = \sum_{j=1}^{n}(z_j\mathfrak{f}_j - \overline{z_j}\mathfrak{f}_j^\dagger) + \sum_{t=1,t\neq r,s}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) + (t_r\cos\theta - it_s\sin\theta)\mathfrak{h}_r$$
$$-\overline{(t_r\cos\theta - it_s\sin\theta)}\mathfrak{h}_r^\dagger + (-it_r\sin\theta + t_s\cos\theta)\mathfrak{h}_s - \overline{(-it_r\sin\theta + t_s\cos\theta)}\mathfrak{h}_s^\dagger \tag{3.54}$$

$$s^9_{r,s}\underline{X}\overline{s^9_{r,s}} = \sum_{j=1}^{n}(z_j\mathfrak{f}_j - \overline{z_j}\mathfrak{f}_j^\dagger) + \sum_{t=1,t\neq r,s}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) + (t_r\cos\theta + \overline{t_s}\sin\theta)\mathfrak{h}_r$$
$$-\overline{(t_r\cos\theta + \overline{t_s}\sin\theta)}\mathfrak{h}_r^\dagger + (-\overline{t_r}\sin\theta + t_s\cos\theta)\mathfrak{h}_s - \overline{(-\overline{t_r}\sin\theta + t_s\cos\theta)}\mathfrak{h}_s^\dagger \tag{3.55}$$

$$s^{10}_{r,s}\underline{X}\overline{s^{10}_{r,s}} = \sum_{j=1}^{n}(z_j\mathfrak{f}_j - \overline{z_j}\mathfrak{f}_j^\dagger) + \sum_{t=1,t\neq r,s}^{m}(t_r\mathfrak{h}_r - \overline{t_r}\mathfrak{h}_r^\dagger) + (t_r\cos\theta + i\overline{t_s}\sin\theta)\mathfrak{h}_r$$
$$-\overline{(t_r\cos\theta + i\overline{t_s}\sin\theta)}\mathfrak{h}_r^\dagger + (-i\overline{t_r}\sin\theta + t_s\cos\theta)\mathfrak{h}_s - \overline{(-i\overline{t_r}\sin\theta + t_s\cos\theta)}\mathfrak{h}_s^\dagger \tag{3.56}$$

$$s^{11}_{j,r}\underline{X}\overline{s^{11}_{j,r}} = \sum_{t=1,t\neq j}^{n}(z_t\mathfrak{f}_t - \overline{z_t}\mathfrak{f}_t^\dagger) + (z_j\cosh\alpha + t_r\sinh\alpha)\mathfrak{f}_j - \overline{(z_j\cosh\alpha + t_r\sinh\alpha)}\mathfrak{f}_j^\dagger$$
$$+\sum_{s=1,s\neq r}^{m}(t_s\mathfrak{h}_s - \overline{t_s}\mathfrak{h}_s^\dagger) + (z_j\sinh\alpha + t_r\cosh\alpha)\mathfrak{h}_r - \overline{(z_j\sinh\alpha + t_r\cosh\alpha)}\mathfrak{h}_r^\dagger \tag{3.57}$$

$$s^{12}_{j,r}\underline{X}\overline{s^{12}_{j,r}} = \sum_{t=1,t\neq j}^{n}(z_t\mathfrak{f}_t - \overline{z_t}\mathfrak{f}_t^\dagger) + (z_j\cosh\alpha - it_r\sinh\alpha)\mathfrak{f}_j - \overline{(z_j\cosh\alpha - it_r\sinh\alpha)}\mathfrak{f}_j^\dagger$$
$$+\sum_{s=1,s\neq r}^{m}(t_s\mathfrak{h}_s - \overline{t_s}\mathfrak{h}_s^\dagger) + (iz_j\sinh\alpha + t_r\cosh\alpha)\mathfrak{h}_r - \overline{(iz_j\sinh\alpha + t_r\cosh\alpha)}\mathfrak{h}_r^\dagger \tag{3.58}$$

$$(3.59)$$

$$s_{j,r}^{13}\underline{X}\overline{s_{j,r}^{13}} = \sum_{t=1,t\neq j}^{n}(z_t\mathfrak{f}_t - \overline{z_t}\mathfrak{f}_t^\dagger) + (z_j\cosh\alpha + \overline{t_r}\sinh\alpha)\mathfrak{f}_j - \overline{(z_j\cosh\alpha + \overline{t_r}\sinh\alpha)}\mathfrak{f}_j^\dagger$$
$$+ \sum_{s=1,s\neq r}^{m}(t_s\mathfrak{h}_s - \overline{t_s}\mathfrak{h}_s^\dagger) + (\overline{z_j}\sinh\alpha + t_r\cosh\alpha)\mathfrak{h}_r - \overline{(\overline{z_j}\sinh\alpha + t_r\cosh\alpha)}\mathfrak{h}_r^\dagger \quad (3.60)$$

$$s_{j,r}^{14}\underline{X}\overline{s_{j,r}^{14}} = \sum_{t=1,t\neq j}^{n}(z_t\mathfrak{f}_t - \overline{z_t}\mathfrak{f}_t^\dagger) + (z_j\cosh pha + i\overline{t_r}\sinh\alpha)\mathfrak{f}_j - \overline{(z_j\cosh\alpha + i\overline{t_r}\sinh\alpha)}\mathfrak{f}_j^\dagger$$
$$+ \sum_{s=1,s\neq r}^{m}(t_s\mathfrak{h}_s - \overline{t_s}\mathfrak{h}_s^\dagger) + (i\overline{z_j}\sinh\alpha + t_r\cosh\alpha)\mathfrak{h}_r - \overline{(i\overline{z_j}\sinh\alpha + t_r\cosh\alpha)}\mathfrak{h}_r^\dagger \quad (3.61)$$

The complex transformations (3.47)-(3.61) preserve the Hermitian norm and they can be divided into two classes: the holomorphic transformations (a group in itself) and the non-holomorphic transformations. Indeed, multiplying at left the spin actions (3.47)-(3.61) with the (anti-)primitive idempotent I (projection onto $\mathbb{C}\Lambda_{n,m}$), we immediately see that transformations (3.47)-(3.49),(3.52)-(3.54),(3.57),(3.58) are holomorphic transformations in the variables $z_j, j = 1,\ldots,n$, and $t_r, r = 1,\ldots,m$, belonging to the unitary group $\mathrm{U}(n,m)$, whereas the remaining transformations are non holomorphic. It is interesting to observe that the projection of the spin actions (3.47)-(3.49),(3.52)-(3.54),(3.57),(3.58) onto $\mathbb{C}\Lambda_{n,m}^\dagger$, obtained by the multiplication at right with the (anti-)primitive idempotent I, yield the anti-holomorphic transformations in the variables $z_j, j = 1,\ldots,n$, and $t_r, r = 1,\ldots,m$. Therefore, the Hermitian Clifford algebra approach encodes, in the same structure, holomorphic and anti-holomorphic transformations.

The spin elements (3.18)-(3.20),(3.23)-(3.25),(3.28),(3.29) that give rise to the holomorphic (and anti-holomorphic) transformations can be fully characterized by the primitive idempotent I by defining the Clifford group

$$\widetilde{\mathrm{U}}(n,m) = \{s \in \mathrm{Spin}^+(2n,2m,\mathbb{R}) | \, \exists\theta \geq 0 : Is = e^{i\theta}I\}. \quad (3.62)$$

Thus, it follows that the Clifford group $\widetilde{\mathrm{U}}(n,m)$ is isomorphic to the unitary group $\mathrm{U}(n,m)$. Removing the elements in $\widetilde{\mathrm{U}}(n,m)$ responsible by a global phase term we obtain a Clifford realization of the special unitary group $\mathrm{SU}(n,m)$ which is given by

$$\widetilde{\mathrm{SU}}(n,m) = \{s \in \widetilde{\mathrm{U}}(n,m) : Is = I\}. \quad (3.63)$$

Corollary 3.4 *The Lie algebra $\widetilde{su}(n,m)$ is generated by the real bivectors (3.5),(3.6), (3.10),(3.11), together with the elements*

$$S_j^{1*} = e_je_{n+j} - e_ne_{2n} = 2i\,\mathfrak{f}_j\wedge\mathfrak{f}_j^\dagger - 2i\mathfrak{f}_n\wedge\mathfrak{f}_n^\dagger, j = 1,\ldots,n-1$$
$$T_r^{1*} = \xi_r\xi_{m+r} - \xi_m\xi_{2m} = 2i\,\mathfrak{h}_r\wedge\mathfrak{h}_r^\dagger - 2i\mathfrak{h}_m\wedge\mathfrak{h}_m^\dagger, r = 1,\ldots,m-1.$$

Proof: First we see that all the real bivectors are linearly independent and their number equals $(n+m)^2-1$, which is exactly $\dim(su(n,m))= \dim(\mathrm{SU}(n,m))$. Furthermore, the spin elements associated to these bivectors satisfy the condition $Is = I$.

■

4 Complex boosts in an arbitrary complex direction

We have seen that elements (3.28)-(3.31) are the Lorentz boosts generators in $\mathbb{R}^{2n,2m}$, yielding complex Lorentz transformations. In this section we restrict ourselves to $\mathbb{R}^{2n,2}$, the case of one complex time dimension, and we compute the formula for a general complex boost in an arbitrary complex direction. In the real case it is well-know that a real boost in the real Minkowski space-time $\mathbb{R}^{n,1}$ is parameterized by a direction $\underline{\omega} \in S^{n-1}$ (S^{n-1} is the unit sphere in $\mathbb{R}^n$) and a hyperbolic angle $\alpha \in \mathbb{R}$, by the formula

$$s_{\underline{\omega},\alpha} = \cosh\left(\frac{\alpha}{2}\right) + \underline{\omega}\xi \sinh\left(\frac{\alpha}{2}\right) \tag{4.1}$$

where ξ is the Clifford basis element which spans the time axis and satisfies $\xi^2 = 1$. The spin element (4.1) corresponds to the exponentiation of the element $\underline{\omega}\xi$, which belongs to the Lie algebra $spin^+(n,1)$. Therefore, we have

$$s_{\omega,\alpha} = e^{\underline{\omega}\xi} = e^{(w_1 e_1 + \ldots + w_n e_n)\xi} = e^{w_1 e_1 \xi + \ldots + w_n e_n \xi} \tag{4.2}$$

i.e., the real boost $s_{\omega,\alpha}$ in an arbitrary direction $\omega \in S^{n-1}$ appears as the exponentiation of the real linear combination of the boosts $e_1\xi, \ldots, e_n\xi$ in $spin^+(n,1)$. For a real space-time vector $\underline{x} + t\xi$ where $\underline{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, the spin action induced by $s_{\underline{\omega},\alpha}$ is given by

$$\begin{aligned} s_{\underline{\omega},\alpha}(\underline{x} + t\xi)\overline{s_{\underline{\omega},\alpha}} &= \underline{x} + t\xi + ((\cosh(\alpha) - 1)\langle \underline{\omega}, \underline{x} \rangle + \sinh(\alpha)t)\underline{\omega} + \\ &\quad +((\cosh(\alpha) - 1)t + \sinh(\alpha)\langle \underline{\omega}, \underline{x} \rangle)\xi. \end{aligned} \tag{4.3}$$

In this section we will derive a similar formula for the complex case, by studying all possible linear combinations of the complex boost bivectors in the Lie algebra $spin^+(2n, 2, \mathbb{R})$, which are given by

$$(I) \qquad e_j\xi_1 + e_{n+j}\xi_2 \tag{4.4}$$

$$(II) \qquad e_j\xi_2 - e_{n+j}\xi_1 \tag{4.5}$$

$$(III) \qquad e_j\xi_1 - e_{n+j}\xi_2 \tag{4.6}$$

$$(IV) \qquad e_j\xi_2 + e_{n+j}\xi_1 \tag{4.7}$$

with $j = 1, \ldots, n$. As we have seen in Section 3, boosts (I) and (II) are the complex holomorphic boosts whereas boosts (III) and (IV) are the non-holomorphic complex boosts. To obtain a holomorphic generalization of formula (4.3) we need to investigate the linear combination of the real boosts (I) and (II).

Case 1: (I) + (II):
Let us consider the spin element associated to the linear combination of (I) and (II):

$$\begin{aligned} s &= e^{\frac{\alpha}{2}(\lambda_j(e_j\xi_1 + e_{n+j}\xi_2) + \lambda_{n+j}(e_j\xi_2 - e_{n+j}\xi_1))} \\ &= e^{\frac{\alpha}{2}((\lambda_j e_j - \lambda_{n+j}e_{n+j})\xi_1 + (\lambda_j e_{n+j} + \lambda_{n+j}e_j)\xi_2)} \\ &= \underbrace{e^{\frac{\alpha}{2}(\lambda_j e_j - \lambda_{n+j}e_{n+j})\xi_1}}_{s_1} \underbrace{e^{\frac{\alpha}{2}(\lambda_j e_{n+j} + \lambda_{n+j}e_j)\xi_2}}_{s_2} \end{aligned} \tag{4.8}$$

where $\lambda_j, \lambda_{n+j} \in \mathbb{R}$, for some $j = 1, \ldots, n$. The last equality is valid since

$$[(\lambda_j e_j - \lambda_{n+j}e_{n+j})\xi_1, (\lambda_j e_{n+j} + \lambda_{n+j}e_j)\xi_2] = 0$$

and thus, the exponential law holds. The elements s_1 and s_2 are defined by

$$s_1 = \cosh\left(\frac{\alpha}{2}\right) + (\lambda_j e_j - \lambda_{n+j} e_{n+j})\xi_1 \sinh\left(\frac{\alpha}{2}\right) \tag{4.9}$$

and

$$s_2 = \cosh\left(\frac{\alpha}{2}\right) + (\lambda_j e_{n+j} + \lambda_{n+j} e_j)\xi_2 \sinh\left(\frac{\alpha}{2}\right) \tag{4.10}$$

which belong to $\mathrm{Spin}^+(2n, 2, \mathbb{R})$ if and only if $\lambda_j^2 + \lambda_{n+j}^2 = 1$. This is the normalization condition that we have to impose. By straightforward computations, the spin actions induced by s_1 and s_2 in an arbitrary space-time vector $\underline{X} \in \mathbb{R}^{2n,2}$ are given in real coordinates by

$$\begin{aligned} s_1\underline{X}\overline{s_1} = & \sum_{j=1}^{n}(x_j e_j + y_j e_{n+j}) + ((\cosh\alpha - 1)(\lambda_j^2 x_j - \lambda_j\lambda_{n+j} y_j) + \lambda_j u_1 \sinh\alpha) e_j \\ & + ((\cosh\alpha - 1)(\lambda_{n+j}^2 y_j - \lambda_j \lambda_{n+j} x_j) - \lambda_{n+j} u_1 \sinh\alpha) e_{n+j} \\ & + (u_1 \cosh\alpha + \sinh\alpha(\lambda_j x_j - \lambda_{n+j} y_j))\xi_1 + u_2 \xi_2 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} s_2\underline{X}\overline{s_2} = & \sum_{j=1}^{n}(x_j e_j + y_j e_{n+j}) + ((\cosh\alpha - 1)(\lambda_{n+j}^2 x_j + \lambda_j \lambda_{n+j} y_j) + \\ & + \lambda_{n+j} u_2 \sinh\alpha) e_j + ((\cosh\alpha - 1)(\lambda_j^2 y_j + \lambda_j\lambda_{n+j} x_j) + \lambda_j u_2 \sinh\alpha) e_{n+j} \\ & + u_1 \xi_1 + (u_2 \cosh\alpha + \sinh\alpha(\lambda_{n+j} x_j + \lambda_j y_j))\xi_2. \end{aligned} \tag{4.12}$$

Finally, the composition of s_1 and s_2 gives us the spin action $sX\overline{s}$:

$$\begin{aligned} s_1(s_2\underline{X}\overline{s_2})\overline{s_1} = & \sum_{j=1}^{n}(x_j e_j + y_j e_{n+j}) + ((\cosh\alpha - 1) x_j + \sinh\alpha(\lambda_j u_1 + \lambda_{n+j} u_2)) e_j \\ & + ((\cosh\alpha - 1) y_j + \sinh\alpha(\lambda_j u_2 - \lambda_{n+j} u_1)) e_{n+j} \\ & + (u_1 \cosh\alpha + \sinh\alpha(\lambda_j x_j - \lambda_{n+j} y_j))\xi_1 \\ & + (u_2 \cosh\alpha + \sinh\alpha(\lambda_{n+j} x_j + \lambda_j y_j))\xi_2. \end{aligned} \tag{4.13}$$

Writing (4.13) in terms of the Witt basis using (2.15) and (2.16), we obtain the following complex transformation:

$$\begin{aligned} s\underline{X}\overline{s} = & Z - Z^\dagger + ((\cosh\alpha - 1) z_j w_j + T\sinh\alpha)\overline{w_j}\mathfrak{f}_j \\ & - \overline{((\cosh\alpha - 1) z_j w_j + T\sinh\alpha)} w_j \mathfrak{f}_j^\dagger + (T(\cosh\alpha - 1) + \sinh\alpha\, z_j w_j)\mathfrak{h}_1 \\ & - \overline{(T(\cosh\alpha - 1) + \sinh\alpha\, z_j w_j)}\mathfrak{h}_1^\dagger \end{aligned} \tag{4.14}$$

with $w_j = \lambda_j + i\lambda_{n+j}$ such that $\lambda_j^2 + \lambda_{n+j}^2 = 1$, and $T = u_1 + iu_2$. Since $\lambda_{n+j} \in \mathbb{R}$ is arbitrary we can replace λ_{n+j} by $-\lambda_{n+j}$ in (4.14). Considering also $\underline{\omega_j} = (0, \ldots, w_j, \ldots, 0) \in \mathbb{C}^n$, which satisfies $||\underline{\omega_j}||^2 = 1$, we finally obtain the spin action

$$\begin{aligned} s\underline{X}\overline{s} = & Z - Z^\dagger + ((\cosh\alpha - 1)\langle \underline{z}, \underline{\omega_j}\rangle + T\sinh\alpha) w_j \mathfrak{f}_j \\ & - \overline{((\cosh\alpha - 1)\langle \underline{z}, \underline{\omega_j}\rangle + T\sinh\alpha)}\overline{w_j}\mathfrak{f}_j^\dagger + (T(\cosh\alpha - 1) + \sinh\alpha\langle \underline{z}, \underline{\omega_j}\rangle)\mathfrak{h}_1 \\ & - \overline{(T(\cosh\alpha - 1) + \sinh\alpha\langle \underline{z}, \underline{\omega_j}\rangle)}\mathfrak{h}_1^\dagger, \end{aligned} \tag{4.15}$$

with $\underline{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, and $\langle \underline{z}, \underline{\omega_j} \rangle$ the usual Hermitian inner product on $\mathbb{C}^n$. Formula (4.15) corresponds to the action of a complex boost in the complex direction $\underline{\omega_j} = (0, \dots, w_j, \dots, 0) \in \mathbf{S}$, where $\mathbf{S}$ denotes the complex unit sphere in $\mathbb{C}^n$. Replacing λ_{n+j} by $-\lambda_{n+j}$ in (4.8) the spin element can be written in Hermitian form as

$$\begin{aligned} s_{\underline{\omega_j},\alpha} &= \left(\cosh\left(\frac{\alpha}{2}\right) + (w_j \mathfrak{f}_j - \overline{w}_j \mathfrak{f}_j^\dagger)(\mathfrak{h}_1 - \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right) \times \\ &\quad \times \left(\cosh\left(\frac{\alpha}{2}\right) - (w_j \mathfrak{f}_j + \overline{w}_j \mathfrak{f}_j^\dagger)(\mathfrak{h}_1 + \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right). \end{aligned} \tag{4.16}$$

To have a boost in an arbitrary direction of the complex sphere we can consider the linear combination of all boosts of types I and II or we can simply consider rotation arguments. Let $\underline{\omega} = (w_1, \dots, w_n) \in \mathbf{S}$ be an arbitrary direction in $\mathbb{C}^n$. Then it is always possible to find $s_* \in \widetilde{\mathrm{U}}(n, 0)$ such that $\omega = s_* \underline{w_j} \overline{s_*}$, where $\underline{\omega_j} = (0, \dots, w_j, \dots, 0) \in \mathbf{S}$. Therefore, by performing the action $s_* s_{\underline{\omega_j},\alpha} \overline{s_*}$ we will arrive at the formula of the complex boost $s_{\underline{\omega},\alpha}$ in an arbitrary direction $\underline{\omega} \in S$, which is given by

$$\begin{aligned} s_{\underline{\omega},\alpha} = s_* s_{\underline{\omega_j},\alpha} \overline{s_*} &= \left(\cosh\left(\frac{\alpha}{2}\right) + (\underline{\omega} - \underline{\omega}^\dagger)(\mathfrak{h}_1 - \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right) \times \\ &\quad \left(\cosh\left(\frac{\alpha}{2}\right) - (\underline{\omega} + \underline{\omega}^\dagger)(\mathfrak{h}_1 + \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right). \end{aligned} \tag{4.17}$$

Its action on $\underline{X} \in \mathbb{R}^{2n,2}$ is given by

$$\begin{aligned} s_{\underline{\omega},\alpha} \underline{X}\, \overline{s_{\underline{\omega},\alpha}} = Z - Z^\dagger &+ ((\cosh\alpha - 1)\langle \underline{z}, \underline{\omega}\rangle + T\sinh\alpha)\omega \\ &- \overline{((\cosh\alpha - 1)\langle \underline{z}, \underline{\omega}\rangle + T\sinh\alpha)}\omega^\dagger + (T\cosh\alpha + \sinh\alpha\langle \underline{z}, \underline{\omega}\rangle)\mathfrak{h}_1 \\ &- \overline{(T\cosh\alpha + \sinh\alpha\langle \underline{z}, \underline{\omega}\rangle)}\mathfrak{h}_1^\dagger. \end{aligned} \tag{4.18}$$

Multiplying at left the spin action (4.18) with the primitive (anti-)idempotent I, we obtain a holomorphic transformation in the variables $z_j, j = 1, \dots, n$, and T, whereas, the multiplication of I at right gives an anti-holomorphic transformation. We summarize our results in the next theorem.

Theorem 4.1 *The holomorphic and anti-holomorphic complex boost parameterized by a complex direction $\underline{\omega} \in \mathbf{S}$ and a hyperbolic angle $\alpha \in \mathbb{R}$ is given by*

$$\begin{aligned} s_{\underline{\omega},\alpha} &= \left(\cosh\left(\frac{\alpha}{2}\right) + (\underline{\omega} - \underline{\omega}^\dagger)(\mathfrak{h}_1 - \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right) \times \\ &\quad \times \left(\cosh\left(\frac{\alpha}{2}\right) - (\underline{\omega} + \underline{\omega}^\dagger)(\mathfrak{h}_1 + \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right) \end{aligned} \tag{4.19}$$

and it admits the KAK decomposition $s_{\underline{\omega},\alpha} = s s_{\underline{\omega_j},\alpha} \overline{s}$, where $s_{\underline{\omega_j},\alpha} \in \widetilde{U}(1,1) \cong U(1,1)$ is the group of complex boosts on a fixed complex direction $\underline{\omega_j} = (0, \dots, w_j, \dots, 0) \in \mathbf{S}$, and $s \in \widetilde{U}(n,0) \cong U(n)$ such that $\omega = s\underline{\omega_j}\overline{s}$. The holomorphic spin action of $s_{\underline{\omega},\alpha}$ in an arbitrary space-time vector $\underline{X} \in \mathbb{R}^{2n,2}$ is given by

$$\begin{aligned} I s_{\underline{\omega},\alpha} \underline{X}\, \overline{s_{\underline{\omega},\alpha}} &= I\,(Z + ((\cosh\alpha - 1)\langle \underline{z}, \underline{\omega}\rangle + T\sinh\alpha)\omega + \\ &\quad + (T(\cosh\alpha - 1) + \sinh\alpha\langle \underline{z}, \underline{\omega}\rangle)\mathfrak{h}_1) \end{aligned} \tag{4.20}$$

and the anti-holomorphic spin action is given by

$$\begin{aligned} s_{\underline{\omega},\alpha} \underline{X}\, \overline{s_{\underline{\omega},\alpha}} I &= -\Big(Z^\dagger + ((\cosh\alpha - 1)\overline{\langle \underline{z}, \underline{\omega}\rangle} + \overline{T}\sinh\alpha)\omega^\dagger \\ &\quad + (\overline{T}(\cosh\alpha - 1) + \sinh\alpha\overline{\langle \underline{z}, \underline{\omega}\rangle})\mathfrak{h}_1^\dagger\Big)\, I. \end{aligned} \tag{4.21}$$

Case 2: (III) + (IV):
In this case we study the linear combination of boosts (III) and (IV). Since these are non-holomorphic boosts we will obtain the non-holomorphic analogue of formula (4.18). The spin element associated to the linear combination of boost elements (III) and (IV) is given by exponentiation as

$$\begin{aligned} s &= e^{\frac{\alpha}{2}(\lambda_j(e_j\xi_1 - e_{n+j}\xi_2)+\lambda_{n+j}(e_j\xi_2+e_{n+j}\xi_1))} \\ &= e^{\frac{\alpha}{2}((\lambda_j e_j+\lambda_{n+j}e_{n+j})\xi_1+(-\lambda_j e_{n+j}+\lambda_{n+j}e_j)\xi_2)} \\ &= \underbrace{e^{\frac{\alpha}{2}(\lambda_j e_j+\lambda_{n+j}e_{n+j})\xi_1}}_{s_1}\underbrace{e^{\frac{\alpha}{2}(-\lambda_j e_{n+j}+\lambda_{n+j}e_j)\xi_2}}_{s_2}, \end{aligned} \tag{4.22}$$

where $\lambda_j, \lambda_{n+j} \in \mathbb{R}$. The last equality is valid since

$$[(\lambda_j e_j + \lambda_{n+j}e_{n+j})\xi_1, (-\lambda_j e_{n+j} + \lambda_{n+j}e_j)\xi_2] = 0$$

and, thus, the exponential law holds. By similar computations as in Case 1 we obtain the spin action

$$\begin{aligned} s\underline{X}\overline{s} &= Z - Z^\dagger + ((\cosh\alpha - 1)z_j\overline{w_j} + \overline{T}\sinh\alpha)w_j\mathfrak{f}_j \\ &\quad -\overline{((\cosh\alpha - 1)z_j\overline{w_j}+\overline{T}\sinh\alpha)}\overline{w_j}\mathfrak{f}_j^\dagger+(T(\cosh\alpha - 1)+\sinh\alpha\,\overline{z_j}w_j)\mathfrak{h}_1 \\ &\quad -\overline{(T(\cosh\alpha - 1) + \sinh\alpha\,\overline{z_j}w_j)}\mathfrak{h}_1^\dagger, \end{aligned} \tag{4.23}$$

with $w_j = \lambda_j + i\lambda_{n+j}$ such that $\lambda_j^2 + \lambda_{n+j}^2 = 1$. Let $\underline{\omega_j} = (0,\ldots,w_j,\ldots,0) \in \mathbf{S}$. Then we obtain the non-holomorphic complex Lorentz boost in the $\underline{\omega_j}$ direction:

$$\begin{aligned} s\underline{X}\overline{s} = Z - Z^\dagger &+ ((\cosh\alpha - 1)\langle\underline{z},\underline{\omega_j}\rangle + \overline{T}\sinh\alpha)w_j\mathfrak{f}_j \\ &-\overline{((\cosh\alpha-1)\langle\underline{z},\underline{\omega_j}\rangle+\overline{T}\sinh\alpha)}\overline{w_j}\mathfrak{f}_j^\dagger+(T(\cosh\alpha-1)+\sinh\alpha\overline{\langle\underline{z},\underline{\omega_j}\rangle})\mathfrak{h}_1 \\ &-\overline{(T(\cosh\alpha - 1) + \sinh\alpha\overline{\langle\underline{z},\underline{\omega_j}\rangle})}\mathfrak{h}_1^\dagger, \end{aligned} \tag{4.24}$$

with $\underline{z} = (z_1,\ldots,z_n) \in \mathbb{C}^n$. In this case, the non-holomorphic complex boost $s_{\underline{\omega_j},\alpha}$ in the direction $\underline{\omega_j}$ is written in the Witt basis as

$$\begin{aligned} s_{\underline{\omega_j},\alpha} &= \left(\cosh\left(\frac{\alpha}{2}\right) + (w_j\mathfrak{f}_j - \overline{w}_j\mathfrak{f}_j^\dagger)(\mathfrak{h}_1 - \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right) \times \\ &\quad\times\left(\cosh\left(\frac{\alpha}{2}\right) + (w_j\mathfrak{f}_j + \overline{w}_j\mathfrak{f}_j^\dagger)(\mathfrak{h}_1 + \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right). \end{aligned} \tag{4.25}$$

By the same rotation arguments as in the Case 1 we can obtain the formula for a non-holomorphic complex boost parameterized by an arbitrary direction $\underline{\omega} \in \mathbf{S}$, that we will describe in the next theorem.

Theorem 4.2 *The non-holomorphic complex boost parameterized by a complex direction $\underline{\omega} \in \mathbf{S}$ and a hyperbolic angle $\alpha \in \mathbb{R}$ is given by*

$$\begin{aligned} s_{\underline{\omega},\alpha} &= \left(\cosh\left(\frac{\alpha}{2}\right) + (\underline{\omega} - \underline{\omega}^\dagger)(\mathfrak{h}_1 - \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right) \times \\ &\quad\times\left(\cosh\left(\frac{\alpha}{2}\right) + (\underline{\omega} + \underline{\omega}^\dagger)(\mathfrak{h}_1 + \mathfrak{h}_1^\dagger)\sinh\left(\frac{\alpha}{2}\right)\right) \end{aligned} \tag{4.26}$$

and it admits the KAK decomposition $s_{\underline{\omega},\alpha} = ss_{\underline{\omega_j},\alpha}\overline{s}$, where $s_{\underline{\omega_j},\alpha} \in \widetilde{U}(1,1) \cong U(1,1)$ is the group of complex boosts on a fixed complex direction

$\underline{\omega_j} = (0, \ldots, w_j, \ldots, 0) \in \mathbf{S}$, *and* $s \in \widetilde{U}(n,0) \cong U(n)$ *such that* $\omega = s\underline{\omega_j}\overline{s}$. *The non-holomorphic spin action of* $s_{\underline{\omega},\alpha}$ *in an arbitrary space-time vector* $\underline{X} \in \mathbb{R}^{2n,2}$ *is given by*

$$\begin{aligned} Is_{\underline{\omega},\alpha}\underline{X}\overline{s_{\underline{\omega},\alpha}} &= I\big(Z + ((\cosh\alpha - 1)\langle \underline{z}, \underline{\omega}\rangle + \overline{T}\sinh\alpha)\omega \\ &\quad + (T(\cosh\alpha - 1) + \sinh\alpha\overline{\langle \underline{z}, \underline{\omega}\rangle})\mathfrak{h}_1\big) \end{aligned} \tag{4.27}$$

or

$$\begin{aligned} s_{\underline{\omega},\alpha}\underline{X}\overline{s_{\underline{\omega},\alpha}}I &= -\Big(Z^\dagger + ((\cosh\alpha - 1)\overline{\langle \underline{z}, \underline{\omega}\rangle} + T\sinh\alpha)\omega^\dagger \\ &\quad + (\overline{T}(\cosh\alpha - 1) + \sinh\alpha\langle \underline{z}, \underline{\omega}\rangle)\mathfrak{h}_1^\dagger\Big) I. \end{aligned} \tag{4.28}$$

Case 3: (I) + (III):
Finally, we study the spin element associated to the linear combination of elements (I) and (III):

$$\begin{aligned} s &= e^{\frac{\alpha}{2}(\lambda_j(e_j\xi_1 + e_{n+j}\xi_2) + \lambda_{n+j}(e_j\xi_2 - e_{n+j}\xi_1))} \\ &= e^{\frac{\alpha}{2}\lambda_j(e_j\xi_1 + e_{n+j}\xi_2)} e^{\frac{\alpha}{2}\lambda_{n+j}(e_j\xi_2 - e_{n+j}\xi_1)} \\ &= \underbrace{e^{\frac{\alpha}{2}\lambda_j e_j\xi_1}}_{s_1} \underbrace{e^{\frac{\alpha}{2}\lambda_j e_{n+j}\xi_2}}_{s_2} \underbrace{e^{\frac{\alpha}{2}\lambda_{n+j} e_j\xi_2}}_{s_3} \underbrace{e^{-\frac{\alpha}{2}\lambda_{n+j} e_{n+j}\xi_1}}_{s_4}. \end{aligned} \tag{4.29}$$

Thus, $s_1, s_2, s_3, s_4 \in \mathrm{Spin}^+(2n, 2, \mathbb{R})$ if and only if $\lambda_j^2 = \pm 1$ and $\lambda_{n+j}^2 = \pm 1$. Here, four sub-cases appear.

Sub-case 1: $\lambda_j = 1$ and $\lambda_{n+j} = 1$. We have

$$s = \underbrace{e^{\frac{\alpha}{2}e_j\xi_1} e^{\frac{\alpha}{2}e_{n+j}\xi_2}}_{s_{j,1}^{11}} \underbrace{e^{\frac{\alpha}{2}e_j\xi_2} e^{-\frac{\alpha}{2}e_{n+j}\xi_1}}_{s_{j,1}^{13}}$$

The spin action induced by s is just the composition of the spin actions induced by the spin elements $s_{j,1}^{11}$ and $s_{j,1}^{13}$ of Section 3. We obtain

$$\begin{aligned} s\underline{X}\overline{s} = s_{j,1}^{11}(s_{j,1}^{13}\underline{X}\overline{s_{j,1}^{13}})\overline{s_{j,1}^{11}} &= Z - Z^\dagger + \left((z_j + \overline{z_j})\sinh^2(\alpha) + \frac{T + \overline{T}}{2}\sinh(2\alpha)\right)\mathfrak{f}_j \\ &- \overline{\left((z_j + \overline{z_j})\sinh^2(\alpha) + \frac{T + \overline{T}}{2}\sinh(2\alpha)\right)}\mathfrak{f}_j^\dagger + \left(\frac{z_j + \overline{z_j}}{2}\sinh(2\alpha) + \right. \\ &\left. + (T + \overline{T})\sinh^2(\alpha)\right)\mathfrak{h}_1 - \overline{\left(\frac{z_j + \overline{z_j}}{2}\sinh(2\alpha) + (T + \overline{T})\sinh^2(\alpha)\right)}\mathfrak{h}_1^\dagger. \end{aligned} \tag{4.30}$$

Sub-case 2: $\lambda_j = -1$ and $\lambda_{n+j} = 1$. We have

$$\begin{aligned} s\underline{X}\overline{s} = \overline{s_{j,1}^{11}}(s_{j,1}^{13}\underline{X}\overline{s_{j,1}^{13}})s_{j,1}^{11} &= Z - Z^\dagger + \left((z_j - \overline{z_j})\sinh^2(\alpha) + \frac{\overline{T} - T}{2}\sinh(2\alpha)\right)\mathfrak{f}_j \\ &- \overline{\left((z_j - \overline{z_j})\sinh^2(\alpha) + \frac{\overline{T} - T}{2}\sinh(2\alpha)\right)}\mathfrak{f}_j^\dagger + \left(-\frac{z_j - \overline{z_j}}{2}\sinh(2\alpha)\right. \\ &\left. - (\overline{T} - T)\sinh^2(\alpha)\right)\mathfrak{h}_1 - \overline{\left(-\frac{z_j - \overline{z_j}}{2}\sinh(2\alpha) - (\overline{T} - T)\sinh^2(\alpha)\right)}\mathfrak{h}_1^\dagger. \end{aligned} \tag{4.31}$$

Sub-case 3: $\lambda_j = 1$ and $\lambda_{n+j} = -1$. In this case we obtain the inverse transformation of (4.31) replacing α by $-\alpha$.

Sub-case 4: $\lambda_j = -1$ and $\lambda_{n+j} = -1$. In this case we obtain the inverse transformation of (4.30) replacing α by $-\alpha$.

In conclusion, the linear combination of boosts elements (I) and (III) gives the composition of well-known spin actions. The same happens for the linear combinations of boost elements (I) and (IV), (II) and (III), and, (II) and (IV).

5 The complex Einstein's addition

In this section we show that the complex Einstein velocity addition can be realized by the projection of the holomorphic spin action (4.20) to the hypersurface defined by $T = 1$ in the projective model of the Hermitian space $H_{n,1}$. The Einstein velocity addition belongs to the group of automorphisms of the complex unit ball in $\mathbb{C}^n$ defined by Rudin in [16] (cf. [22] and [23]).

For $(\underline{z}, T) \in H_{n,1}$, with $\underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $T \in \mathbb{C}$ we consider the indefinite real quadratic form $Q(\underline{z}, T) = ||\underline{z}||^2 - |T|^2$ inherited by the Hermitean inner product (2.1). The associated complex null cone is defined by $Q(\underline{z}, T) = 0$. It is a cone with real signature $(2n, 2)$ and it stands for the complexification of the real cone of signature $(n, 1)$ on the Minkowski space $\mathbb{R}^{n,1}$. The time and space-like regions defined by

$$TLR = \{(\underline{z}, T) \in H_{n,1} : Q(\underline{z}, T) < 0\} \tag{5.1}$$

$$SLR = \{(\underline{z}, T) \in H_{n,1} : Q(\underline{z}, T) > 0\}. \tag{5.2}$$

Inside the TLR region a complex projective model can be defined using the manifold of rays given by

$$Rays(TLR) = \{\{\lambda(\underline{z}, T) : \lambda \in \mathbb{C}\}, Q(\underline{z}, T) < 0\}. \tag{5.3}$$

This projective model is known as complex hyperbolic $n-$space. The unitary group $\mathrm{U}(n, 1)$ acts transitively in $Rays(TLR)$. Considering $T = 1$ we obtain the embedding of the complex unit ball $B^n_{\mathbb{C}}$ in the complex hyperbolic space since $||\underline{z}||^2 < 1$. Using the same arguments as in [11] we finally show how to obtain the normalized complex Einstein velocity addition.

Theorem 5.1 *Let $T = 1$ and $\underline{a}, \underline{z} \in \mathbb{C}^n$ with $||\underline{z}|| < 1$, $||\underline{a}|| < 1$, $\cosh(\alpha) = \frac{1}{\sqrt{1-||\underline{a}||^2}}$, $\sinh(\alpha) = \frac{||\underline{a}||}{\sqrt{1-||\underline{a}||^2}}$ and $\underline{\omega} = \frac{\underline{a}}{||\underline{a}||}$. Then the restriction of the holomorphic spin action (4.20) to the complex unit ball $B^n_{\mathbb{C}}$ gives the complex Einstein's addition $\oplus_E$ defined by*

$$\underline{a} \oplus_E \underline{z} = \frac{1}{1 + \langle \underline{z}, \underline{a} \rangle} \left(\sqrt{1 - ||\underline{a}||^2}\, \underline{z} + \frac{1}{1 + \sqrt{1 - ||\underline{a}||^2}} \langle \underline{z}, \underline{a} \rangle \underline{a} + \underline{a} \right) \tag{5.4}$$

(cf. [25,p.282]).

Proof:

Considering in (4.20) the substitutions mentioned we obtain the complex time and space components:

$$T^* = \frac{1 + \langle \underline{z}, \underline{a} \rangle}{\sqrt{1 - ||\underline{a}||^2}}$$

and

$$\underline{z}^* = \frac{\sqrt{1 - ||\underline{a}||^2}\underline{z} + \frac{1}{1 - \sqrt{1 + ||\underline{a}||^2}} \langle \underline{z}, \underline{a} \rangle \underline{a} + \underline{a}}{\sqrt{1 - ||\underline{a}||^2}}.$$

Finally, multiplying $\underline{z}^*$ by $\lambda = \frac{1}{T^*}$ (corresponds to the restriction to $B^n_{\mathbb{C}}$) we obtain the complex space coordinate

$$\lambda \underline{z}^* = \frac{1}{1 + \langle \underline{z}, \underline{a} \rangle} \left(\sqrt{1 - ||\underline{a}||^2}\underline{z} + \frac{1}{1 + \sqrt{1 - ||\underline{a}||^2}} \langle \underline{z}, \underline{a} \rangle \underline{a} + \underline{a} \right)$$

which corresponds to the complex Einstein velocity addition.

■

In [22] it was shown that $(B^n_{\mathbb{C}}, \oplus_E)$ is a complex gyrogroup. The restriction of formula (5.4) to the real case gives the real Einstein velocity addition. By the same arguments we can restrict the anti-holomorphic and non-holomorphic spin actions (4.21), (4.27), and (4.28) onto the complex unit ball $B^n_{\mathbb{C}}$ by the complex projective model. In this way all complex Einstein velocity additions (holomorphic, anti-holomorphic and non-holomorphic) are found. A more detailed study of complex relativistic velocity additions will be made in a forthcoming paper. With this work we hope to have shown the importance of Hermitian Clifford algebra for a better understanding of Hermitian spaces and the automorphisms of the complex unit ball in $\mathbb{C}^n$.

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