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ψ-Hilfer fractional relaxation-oscillation equation

N. Vieira[†,⋆] M. Ferreira [‡] M. Manuela Rodrigues [†]

[†] Department of Mathematics, University of Aveiro & CIDMA, Portugal
[‡] School of Technology and Management, Polytechnic Institute of Leiria & CIDMA, Portugal

[⋆] `nloureirovieira@gmail.com`

Abstract: In this work, we solve the ψ-Hilfer fractional relaxation-oscillation equation with a force term, where the time-fractional derivatives are in the ψ-Hilfer sense. The solution of the equation is presented in terms of bivariate Mittag-Leffler functions. An asymptotic analysis of the solution of the associated homogeneous equation is performed.

keywords: Fractional relaxation-oscillation equation; ψ-Hilfer derivative; Bivariate Mittag-Leffler function.

MSC2010: 35R11; 26A33; 34C26; 35B05.

1 Introduction

The relaxation and oscillation processes are of great relevance in physics. From a mathematical point of view, they are modelled by linear differential equations of first and second orders in time. In [2] the fractional relaxation and oscillation equations with Caputo derivatives were studied separately. The simultaneous consideration of time-fractional derivatives or first and second orders leads to the so-called fractional relaxation-oscillation phenomena, that we study in this paper.

2 Preliminaries

In this section, we recall some basic definitions about ψ-Hilfer fractional derivatives, special functions, and the ψ-Laplace transform, that are necessary for this work.

Definition 1 *(cf. [4, Def. 4]) Let (a,b) be a finite or infinite interval on the real line $\mathbb{R}$ and $\alpha>0$. Also let ψ be a monotone increasing and positive function on (a,b), having a continuous derivative ψ' in (a,b). The left Riemann-Liouville fractional integral of a function f with respect to another function ψ on $[a,b]$ is given by*

$$\left(I_{a^+}^{\alpha,\psi} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(w)\, (\psi(t)-\psi(w))^{\alpha-1} f(w)\, dw, \qquad t>a. \tag{1}$$

Next, we give the definition of the so-called ψ-Hilfer fractional derivative of a function f with respect to another function.

Definition 2 *(cf. [4, Def. 7]) Let $\alpha>0$ and $m=[\alpha]+1$, where $[\alpha]$ denotes the integer part of α. Let also $I=[a,b]$ be a finite or infinite interval on the real line and $f,\psi\in C^m[a,b]$ two functions such that ψ is a positive monotone increasing function and $\psi'(t)\neq 0$, for all $t\in I$. The left ψ-Hilfer fractional derivative ${}^H D_{a^+}^{\alpha,\mu;\psi}$ of order α and type $\mu\in[0,1]$ is defined by*

$$\left({}^H D_{a^+}^{\alpha,\mu;\psi} f\right)(t) = I_{a^+}^{\mu(m-\alpha),\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^m I_{a^+}^{(1-\mu)(m-\alpha),\psi} f(t). \tag{2}$$

We observe that when $\mu = 0$ we recover the left Riemann-Liouville fractional derivative of a function with respect to ψ (see [4, Def. 5]) and when $\mu = 1$ we obtain the left Caputo fractional derivative of a function with respect to ψ (see [4, Def. 6]). In Section 5 of [4] is presented a list of several fractional integrals and fractional derivatives that can be obtained from (1) and (2), respectively, for different choices of μ and ψ. The solution of the ψ-fractional relaxation-oscillation equation is presented in terms of the bivariate Mittag-Leffler function which has the following double series representation:

$$E_{(a_1,a_2),b}\left(z_1, z_2\right) = \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} \frac{(l_1 + l_2)!}{l_1!\, l_2!} \frac{z_1^{l_1}\, z_2^{l_2}}{\Gamma\left(b + a_1 l_1 + a_2 l_2\right)}. \tag{3}$$

When $z_1 = -c_1 t^{a_1}$, $z_2 = -c_2 t^{a_2}$, with $a_1, a_2, c_1, c_2, b, t > 0$, we have the following asymptotic expansions near the origin and at the infinity:

$$E_{(a_1,a_2),b}\left(-c_1\, t^{a_1}, -c_2\, t^{a_2}\right) \sim \frac{1}{\Gamma\left(b\right)} - \frac{c_1\, t^{a_1}}{\Gamma\left(b + a_1\right)} - \frac{c_2\, t^{a_2}}{\Gamma\left(b + a_2\right)}, \quad t \to 0^+, \tag{4}$$

$$E_{(a_1,a_2),b}\left(-c_1\, t^{a_1}, -c_2\, t^{a_2}\right) \sim \frac{t^{-a_1}}{c_1\, \Gamma\left(b - a_1\right)}, \quad b \neq a_1,\ t \to +\infty. \tag{5}$$

The ψ-Laplace transform of a real valued function $f\left(t\right)$ with respect to ψ is defined by (see [3, Def. 13])

$$\mathcal{L}_{\psi}\left\{f\left(t\right)\right\}\left(\mathbf{s}\right) = \widetilde{f}_{\psi}\left(\mathbf{s}\right) = \int_0^{+\infty} e^{-\mathbf{s}\,\psi(t)}\, \psi'\left(t\right)\, f\left(t\right)\, dt, \quad \operatorname{Re}\left(\mathbf{s}\right) \in \mathbb{C},$$

where ψ is a non negative monotone increasing function in $\mathbb{R}_0^+$ and such that $\psi\left(0\right) = 0$. The ψ-Laplace transform may be written as the following composition operator involving the classical Laplace transform: $\mathcal{L}_{\psi} = \mathcal{L} \circ Q_{\psi^{-1}}$ where $\left(Q_{\psi^{-1}} f\right)\left(t\right) = f\left(\psi^{-1}\left(t\right)\right)$ (cf. [3, Thm. 4]). As a consequence of the previous relation, if f is a function whose classical Laplace transform is $\widetilde{f}$, the ψ-Laplace transform of $f\left(\psi\left(t\right)\right)$ is also $\widetilde{f}\left(\mathbf{s}\right)$ (see [3, Cor. 2]), that is,

$$\mathcal{L}\left\{f\left(t\right)\right\}\left(\mathbf{s}\right) = \widetilde{f}\left(\mathbf{s}\right) \quad \Rightarrow \quad \mathcal{L}_{\psi}\left\{f\left(\psi\left(t\right)\right)\right\}\left(\mathbf{s}\right) = \widetilde{f}\left(\mathbf{s}\right).$$

We observe that the definition of the ψ-Laplace can be adapted fon any interval $[a, +\infty[\subseteq \mathbb{R}_0^+$ with ψ satisfying $\psi\left(a\right) = 0$. This is important in our work in order to the ψ-Hilfer derivative encompasses the largest number of fractional derivatives. When the ψ-Laplace transform is applied to the ψ-Hilfer derivative we obtain (see [3, Thm. 6])

$$\mathcal{L}_{\psi}\left\{ {}^{H}D_{a^+}^{\alpha,\mu;\psi} f\left(t\right)\right\}\left(\mathbf{s}\right) = \mathbf{s}^{\alpha}\, \widetilde{f}_{\psi}\left(\mathbf{s}\right) - \sum_{j=0}^{m-1} \mathbf{s}^{m-\mu(m-\alpha)-1-j} \left(I_{t,a^+}^{(1-\mu)(m-\alpha)-j;\psi} f\right)\left(a^+\right), \tag{6}$$

where $m = [\alpha] + 1$ and the initial-value terms $\left(I_{a^+}^{(1-\mu)(m-\alpha)-j,\psi} f\right)\left(a^+\right)$ are evaluated in the limit $t \to a^+$. The ψ-Laplace convolution operator of two functions is defined by (see [3, Def. 15])

$$\left(f *_{\psi} g\right)\left(t\right) = \int_0^t f\left(\psi^{-1}\left(\psi\left(t\right) - \psi\left(w\right)\right)\right)\, \psi'\left(w\right)\, g\left(w\right)\, dw, \quad t \in \mathbb{R}^+, \tag{7}$$

and the correspondent Convolution Theorem is (see [3, Thm. 8])

$$\mathcal{L}_{\psi}\left\{\left(f *_{\psi} g\right)\left(t\right)\right\}\left(\mathbf{s}\right) = \mathcal{L}_{\psi}\left\{f\right\}\left(\mathbf{s}\right)\, \mathcal{L}_{\psi}\left\{g\right\}\left(\mathbf{s}\right). \tag{8}$$

Moreover, from relation (17.6) in [1] we have that

$$\mathcal{L}_{\psi}\left\{\psi\left(t\right)^{\alpha-\gamma} \sum_{p=0}^{+\infty} \left(-a\, \psi\left(t\right)^{\alpha-\beta}\right)^p E_{\alpha,\alpha+(\alpha+\beta)p-\gamma+1}^{p+1}\left(-b\, \psi\left(t\right)^{\alpha}\right)\right\}\left(\mathbf{s}\right) = \frac{\mathbf{s}^{\gamma-1}}{\mathbf{s}^{\alpha} + a\mathbf{s}^{\beta} + b}, \tag{9}$$

where $\operatorname{Re}\left(\alpha\right), \operatorname{Re}\left(\beta\right), \operatorname{Re}\left(\gamma\right) \in \mathbb{R}^+$, $\left|\frac{a\mathbf{s}^{\beta}}{\mathbf{s}^{\alpha+\beta}}\right| < 1$, and provided that the series in (9) is convergent.

3 ψ-Hilfer fractional relaxation-oscillation equation

In this section, we solve the ψ-Hilfer fractional forced damped oscillator modelled by the following fractional differential equation

$$c_2\,{}^{H}\!D_{a^+}^{\alpha_2,\mu_2;\psi}u\left(t\right)+c_1\,{}^{H}\!D_{a^+}^{\alpha_1,\mu_1;\psi}u\left(t\right)+d^2\,u\left(t\right)\,=q\left(t\right), \tag{10}$$

and subject to the following initial conditions

$$\left(I_{t,a^+}^{(1-\mu_1)(1-\alpha_1);\psi}u\right)\left(a^+\right)=\kappa_1,\;\;\left(I_{t,a^+}^{(1-\mu_2)(2-\alpha_2);\psi}u\right)\left(a^+\right)=\kappa_2,\;\;\frac{d}{dt}\left[\left(I_{t,a^+}^{(1-\mu_2)(2-\alpha_2);\psi}u\right)\right]\left(a^+\right)=\kappa_3, \tag{11}$$

which are evaluated in the limit $t\to a^+$. Moreover, $c_2,c_1,d,\kappa_1,\kappa_2,\kappa_3\in\mathbb{R}$, $c_2\neq 0$, $t\in I$, with $I=[a,b]$ being a finite or infinite interval on $\mathbb{R}^+$, the partial time-fractional derivatives of orders $\alpha_1\in]0,1]$ and $\alpha_2\in]1,2]$, and types $\mu_1,\mu_2\in[0,1]$, respectively, are the ψ-Hilfer derivatives given by (2), q belongs to $L_1\left(I\right)$ (when $q\left(t\right)=0$ the solution of equation (10) corresponds to an unforced damped oscillator). We look for solutions u of our problem in the space $C^2\left(a,b\right)$.

When $\psi\left(t\right)=t$, with $t\in\mathbb{R}^+$, $\mu_1=\mu_2=1$, and $c_2=0$ or $c_1=0$ in equation (10), we obtain, respectively, the time-fractional relaxation/oscillation equations with Caputo fractional derivatives. These two equations were studied separately in [2]. Moreover, equation (10) is a particular case of the time-fractional telegraph equation with ψ-Hilfer derivatives studied in [5].

Now, we solve our relaxation-oscillation problem. Applying the ψ-Laplace transform to (10) and taking into account (11), we get

$$\left(c_2\,\mathbf{s}^{\alpha_2}+c_1\,\mathbf{s}^{\alpha_1}+d^2\right)\,\widetilde{u}_\psi\left(\mathbf{s}\right)-c_1\,\kappa_1\,\mathbf{s}^{-\mu_1(1-\alpha_1)}-c_2\,\kappa_2\,\mathbf{s}^{1-\mu_2(2-\alpha_2)}-c_2\,\kappa_3\,\mathbf{s}^{-\mu_2(2-\alpha_2)}=\widetilde{q}_\psi\left(\mathbf{s}\right). \tag{12}$$

Solving the above equation in order to $\widetilde{u}_\psi$, we obtain:

$$\widetilde{u}_\psi\left(\mathbf{s}\right)=\frac{\frac{c_1\,\kappa_1}{c_2}\,\mathbf{s}^{-\mu_1(1-\alpha_1)}+\kappa_2\,\mathbf{s}^{1-\mu_2(2-\alpha_2)}+\kappa_3\,\mathbf{s}^{-\mu_2(2-\alpha_2)}}{\mathbf{s}^{\alpha_2}+\frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}+\frac{d^2}{c_2}}+\frac{1}{c_2}\,\widetilde{q}_\psi\left(\mathbf{s}\right)\,\frac{1}{\mathbf{s}^{\alpha_2}+\frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}+\frac{d^2}{c_2}}. \tag{13}$$

Inverting the ψ-Laplace transform and taking into account (9), we have

$$\begin{aligned}
u\left(t\right)=&\ \frac{c_1\,\kappa_1}{c_2}\,\psi\left(t\right)^{\alpha_2-1+\mu_1(1-\alpha_1)}\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi\left(t\right)^{\alpha_2-\alpha_1}\right)^p E_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_1(1-\alpha_1)}^{p+1}\left(-\frac{d^2}{c_2}\,\psi\left(t\right)^{\alpha_2}\right)\\
&+\kappa_2\,\psi\left(t\right)^{\alpha_2-2+\mu_2(2-\alpha_2)}\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi\left(t\right)^{\alpha_2-\alpha_1}\right)^p E_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p-1+\mu_2(2-\alpha_2)}^{p+1}\left(-\frac{d^2}{c_2}\,\psi\left(t\right)^{\alpha_2}\right)\\
&+\kappa_3\,\psi\left(t\right)^{\alpha_2-1+\mu_2(2-\alpha_2)}\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi\left(t\right)^{\alpha_2-\alpha_1}\right)^p E_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_2(2-\alpha_2)}^{p+1}\left(-\frac{d^2}{c_2}\,\psi\left(t\right)^{\alpha_2}\right)\\
&+\frac{1}{c_2}\,q\left(\psi\left(t\right)\right)*_\psi\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi\left(t\right)^{\alpha_2-\alpha_1}\right)^p\psi\left(t\right)^{\alpha_2-1}\,E_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p}^{p+1}\left(-\frac{d^2}{c_2}\,\psi\left(t\right)^{\alpha_2}\right),
\end{aligned} \tag{14}$$

where the ψ-convolution is given by (7). From the definition of the bivariate Mittag-Leffler function (see (3)) we can rewrite (14) as

$$\begin{aligned}
u\left(t\right)=&\ \frac{c_1\,\kappa_1}{c_2}\,\psi\left(t\right)^{\alpha_2-1+\mu_1(1-\alpha_1)}\,E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2+\mu_1(1-\alpha_1)}\left(-\frac{d^2}{c_2}\,\psi\left(t\right)^{\alpha_2},-\frac{c_1}{c_2}\,\psi\left(t\right)^{\alpha_2-\alpha_1}\right)\\
&+\kappa_2\,\psi\left(t\right)^{\alpha_2-2+\mu_2(2-\alpha_2)}\,E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2-1+\mu_2(2-\alpha_2)}\left(-\frac{d^2}{c_2}\,\psi\left(t\right)^{\alpha_2},-\frac{c_1}{c_2}\,\psi\left(t\right)^{\alpha_2-\alpha_1}\right)
\end{aligned} \tag{15}$$

$$
\begin{aligned}
&+ \kappa_3\, \psi(t)^{\alpha_2-1+\mu_2(2-\alpha_2)}\, E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2+\mu_2(2-\alpha_2)}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2}, -\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right) \\
&+ \frac{1}{c_2}\, q(\psi(t)) *_\psi \left[\psi(t)^{\alpha_2-1}\, E_{(\alpha_2,\alpha_2-\alpha_1),\alpha_2}\left(-\frac{d^2}{c_2}\,\psi(t)^{\alpha_2}, -\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)\right]. \qquad (16)
\end{aligned}
$$

Therefore, the solution involves series of three-parameter Mittag-Leffler functions of one variable or just bivariate Mittag-Leffler functions.

For the unforced case, the solution can be written as $u_h(t) = u_1(t) + u_2(t) + u_3(t)$, where u_1, u_2, u_3 corresponds to the first three terms in (16). These constitute a set of fundamental solutions of the homogeneous equation. Let us study the behaviour of u_h when $t \to a^+$ and $t \to +\infty$. From (4) we have the following asymptotic behaviour near the starting point $t = a$

$$
u_h(t) \sim \frac{\kappa_2}{\Gamma(\alpha_2 + \mu_2(2-\alpha_2) - 1)}\, \psi(t)^{\alpha_2+\mu_2(2-\alpha_2)-2}, \quad t \to a^+.
$$

Moreover, from (5) we have the following asymptotic behaviour for large values of t

$$
u_h(t) \sim \frac{\kappa_1}{\Gamma(\alpha_1 + \mu_1(1-\alpha_1))}\, \psi(t)^{\mu_1(1-\alpha_1)-1}, \quad t \to +\infty,
$$

whenever $\mu_2(2-\alpha_2) - \mu_1(1-\alpha_1) < 0$, for $\mu_1, \mu_2 \in [0,1]$, $\alpha_1 \in]0,1]$, and $\alpha_2 \in]1,2]$.

4 Conclusions

In this work, we solved the ψ-Hilfer fractional relaxation-oscillation equation and we showed that the solution can be expressed in terms of bivariate Mittag-Leffler functions. We studied the asymptotic behaviour of the solution of the associated homogeneous equation. This is important to understand and classify the relaxation-oscillation phenomena. Our results generalise those presented in Section 3 of [2].

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