Perfect DFT Sequences Transformed Into Orthogonal Sequences

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Abstract—**In this paper, we present a method and a code generator to transform perfect DFT (discrete Fourier transform) sequences into orthogonal sequences and into bipolar codes, with good correlation properties. These new sequences or codes produce a low peak-to-average power ratio (PAPR) and a low error probability in a code-division multiple access (CDMA) system. All these codes can also be useful in a pre-coded version of an orthogonal frequency-division multiplexing (OFDM) system**.

Keywords-component; CDMA, DFT, OFDM, and Perfect DFT Sequences.

I. INTRODUCTION

3GPP-LTE (3rd Generation Partnership Project – Long Term Evolution) is a step toward the 4th generation (4G) of radio technologies designed to increase the capacity and speed of mobile telephone networks. LTE uses OFDM (orthogonal frequency-division multiplexing) for the downlink. In the uplink, LTE uses a pre-coded version of OFDM called single carrier frequency-division multiple access (SC-FDMA) [1]. This is a solution to compensate for a drawback with normal OFDM, which has a very high peak-to-average power ratio (PAPR). SC-FDMA solves this problem by grouping together the resource blocks in a way that reduces the need for linearity, and thus the power consumption, in the power amplifier.

The standardization effort of 3GPP LTE is done in parallel with the IEEE 802.16 (WiMax) Broadband Wireless Access standard. Both systems can have a good performance if a Generalized DFT (GDFT) [2] is used. For example, a GDFT framework can be used to improve the pre-coded signal that is applied before the IDFT (inverse discrete Fourier transform) used in the uplink 3GPP LTE transmitter. As it is well known, the IDFT of an OFDM signal can be considered as a multi-carrier code-division multiple access (MC-CDMA) signal.

The generation of GDFT sub-sets is based on a perfect periodic autocorrelation that is equal to a periodic cross-correlation of two unimodular sequences. The product of these two sequences is a specific sequence defined as a periodic unimodular complex sequence, with period N, identified as the rth power of the first primitive Nth roots of unity raised to the nth power [2]. Usually, a brute force search method is used to obtain optimum GDFT sets of orthogonal sequences [3]. However, in this paper we prefer to present an analytical method to obtain many sets of perfect DFT (discrete Fourier transform) sequences with a low PAPR and a maximum absolute value of periodic cross-correlation (MaxCC) close to the well known lower bound $\sqrt{N}$. Moreover, our new perfect DFT sequences can be used in the construction of many orthogonal sequences or in new bipolar codes.

Akansu [2] believes that the next generation multicarrier communication systems might benefit with the GDFT family of sequences. For the same reasons, we also believe that our new perfect DFT sequences will be a good choice for the next generation systems. Perfect DFT sequences are, in theory, immune to multi-path interference (MPI) and can be easily transformed into orthogonal sequences or into bipolar codes. Besides, the low or null cross-correlations of our codes should limit the multiple access interference (MAI) in asynchronous or synchronous transmission systems. We have also confirmed that our new sequences have a low PAPR.

Other perfect sequences with a low MaxCC have been proposed in the literature. One of the largest sets of perfect sequences with the lowest MaxCC ($\sqrt{N}$) is the Chu polyphase set [4], which has N–1 perfect sequences. Other well known sets, with a number of perfect sequences smaller or equal to N, are the Generalized Chirp-Like Polyphase Sequences, Frank Perfect Sequences, Frank-Zadoff-Chu (or FZC) Perfect Sequences, and Generalized Chu polyphase sequences [5].

In section II, a method to generate N+1 orthogonal sequences derived from perfect DFT sequences is described. Our perfect DFT sequences and orthogonal sequences have good cross-correlation properties and can be easily implemented with an electronic generator. Numerical and analytical results are presented in section III. We also present a new family of bipolar codes based on mutually orthogonal complementary sequences that can be a good alternative solution to the Gold codes. Finally, main conclusions are gathered in section IV.

II. PERFECT DFT SEQUENCES TRANSFORMED INTO ORTHOGONAL SEQUENCES

A. Some Definitions

Let $x(n)$, with $n = 0,1,2...,N-1$, be one of the N points of a periodic sequence x. Its DFT and IDFT [6] is given respectively by:

$$DFT\left[x(n)\right] = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} , \quad (1)$$

$$IDFT\left[X(k)\right] = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} . \quad (2)$$

For convenience of notation, W_N is defined as $W_N = \exp(-j2\pi/N)$, where $j = \sqrt{-1}$.

A sequence m_r will defined as $m_r = \chi(u \oplus T^r v)$, with $-1 \le r \le N-1$, where r is an integer. Let T^r denote the operator which shifts sequences cyclically to the left by r places, i.e.:

$$T^r x = \left(x(r), x(r+1), ..., x(N-1), x(0), ..., x(r-1)\right) . \quad (3)$$

The $\oplus$ operator is used to denote modulo-2 addition (or the EXCLUSIVE-OR operation). For convenience of notation, $T^0 x = x$ and $m_{-1} = \chi(u \oplus T^{-1}v) = \chi(u)$. The sequences u and v are m-sequences with length N. The function $\chi(.)$ is defined by $\chi(0) = +1$ and $\chi(1) = -1$. If x denotes an arbitrary $\{0,1\}$-valued sequence, then $\chi(x)$ indicates the corresponding $\{+1,-1\}$-valued sequence, and the i-th element of $\chi(x)$ is just $\chi[x(i)]$. It is important to remember that, for an m-sequence v, there is a unique integer k, distinct from both integers r and s, with $0 \le r,s,k \le N-1$, that verifies $T^r v \oplus T^s v = T^k v$. Note also that $\chi(u \oplus T^k v) = \chi(u)\chi(T^k v)$. Another useful property of $\{+1,-1\}$-valued m-sequences is:

$$\left|N \times IDFT\left[\chi(v)\right]\right|^2 = DFT\left[\theta_{\chi(v)\chi(v)}\right] = (N+1) - N\delta(n) . \quad (4)$$

Using the well known DFT and IDFT transforms, the periodic cross-correlation $\theta_{xy}(n) = \sum_{k=0}^{N-1} x(k) y^*\left[\mathrm{mod}(k+n,N)\right]$ between two periodic sequences x and y can be alternatively expressed as [7]:

$$\theta_{xy}(n) = IDFT\left\{DFT[x] DFT^*[y]\right\} . \quad (5)$$

The MaxCC between two sequences x and y is defined by $\max\left\{\left|\theta_{xy}\right|\right\}$ or $\max\left\{\left|\theta_{xy}(n)\right|\right\}$, with $0 \le n \le N-1$. The in-phase periodic cross-correlation is $\theta_{xy}(0)$.

B. Property

Orthogonal Sequences Derived from Perfect DFT Sequences: N+1 orthogonal codes derived from N-length perfect DFT sequences are constructed when a new element (chip) with value 1 is appended (in the same position) to each perfect DFT sequence of the set:

$$\varphi = \left\{\sqrt{N} \times IDFT[m_{-1}], \sqrt{N} \times IDFT[m_0], ..., \sqrt{N} \times IDFT[m_{N-1}]\right\} . (6)$$

With the appended chip, these N+1 different sequences are orthogonal and have a length equal to N+1.

Using the mathematical properties (2), (4), and (5), it is possible to verify that two different sequences y and z of φ set have a $\max\left\{\left|\theta_{yz}\right|\right\} = \sqrt{N+1}$ [8] and the exact $\theta_{yz}(0)$ value, considering all N+1 sequences of φ set, is equal to -1.

This last result can also be obtained using a useful property of an m-sequence [7]: *the chip sum in any bipolar m-sequence is equal to* -1. With $n = 0$, $\theta_{yz}(0) = \frac{N}{N}\sum_{t=0}^{N-1} \chi(T^k v) W_N^0 = \sum^{N} \chi(v) = -1$, when y and z are two different DFT sequences defined in (6).

Now, we can use the same method that transforms a Gold code into an orthogonal Gold code. By appending a chip with the value 1 to all N-length perfect DFT sequences of the φ set, we obtain orthogonal sequences with a length equal to N+1.

It is expectable that the correlation properties of our new orthogonal sequences will not be significantly affected by the appended chip, when N is much higher than 1.

C. M-ary OPPAC/PPAC Code Generator

In Fig. 1, we propose an OPPAC (Orthogonal Perfect Periodic Autocorrelation) generator based on the previous property. The M-ary codes, in Fig. 1, are OPPAC codes implemented with a finite number of discrete amplitude levels. One useful application of our M-ary OPPAC and PPAC generator is the LTE uplink. However, many others can be found.

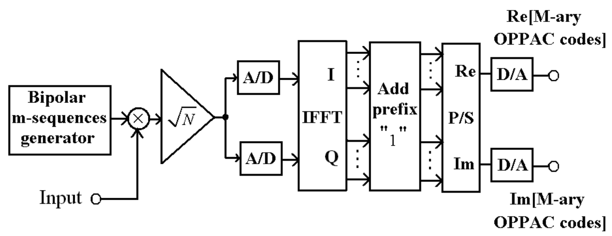

Figure 1: Generator of real and imaginary parts of M-ary OPPAC codes.

Any unimodular code can be transformed by an IDFT into a PPAC complex code (alternative designation of a perfect DFT sequence). Nevertheless, it is possible to generate PPAC codes with a low MaxCC when the input of the IDFT is a unimodular sequence. For this reason, the block "Bipolar *m*-sequences generator" of Fig. 1 is used to create the N+1 bipolar sequences (*m*-sequences). The Fig. 1 "Input" is the information carried by the generator and should be equal to a "1" or "-1" constant sequence. The product of these two sequences are amplified to the levels $\pm\sqrt{N}$. Two analog-to-digital converters (A/D) are included in Fig. 1. The outputs of these are the digital inputs of an IFFT (inverse fast Fourier transform) block. The block "Add prefix "1" " is used at the IFFT outputs to transform the PPAC codes into orthogonal codes. The block "P/S" is the parallel-to-serial converter. The real and imaginary parts of the complex M-ary OPPAC codes are obtained at the outputs of two digital-to-analog converters (D/A). The bit resolution of the four converters (A/D and D/A) should be selected according to the value of M. The higher is this value, the closer are the generated M-ary OPPAC codes to the ideal OPPAC codes. Another advantage is that the generator of M-ary OPPAC codes can be converted into a generator of M-ary PPAC codes by removing the block "Add prefix "1" ".

III. Some Numerical And Analytical Results

In order to evaluate the performance of a future digital implementation of our generator, the M-ary PPAC codes and M-ary OPPAC codes have been tested by simulation in synchronous and asynchronous CDMA transmission scenarios with periodic and aperiodic correlation functions (with one bit carried per sequence). In the presence of MPI, we have found that the OPPAC codes can be slightly more robust than other orthogonal codes, such as Walsh-Hadamard codes or orthogonal Gold codes.

A. PAPR results

An OFDM signal with N=32 subcarriers, and a 64-QAM (Quadrature Amplitude Modulation) constellation of symbols, has a PAPR ≈ 15.5 with the complementary-cumulative-distribution-function CCDF $\approx 1/1000$. Another OFDM PAPR reference value [1] is PAPR ≈ 4.4 with CCDF $\approx 1/10$. Lower PAPRs will be found with our OPPAC codes.

In the design of perfect sequence, the difficulty is to achieve both low cross-correlation and low PAPR [2]. Note that unimodular sequences have PAPR=1. When N = 31, the PAPR average of (6) is close to 2.15. With N=127 or N=511, we have also found many perfect DFT sequences with a PAPR lower than 2.7. We have additionally verified that when N=31+1, our OPPAC codes have a null in-phase periodic cross-correlation, a MaxCC close to $3.2\times\sqrt{N}$, and a PAPR approximately equal to 2.1. For these reasons, we say that many OPPAC codes or perfect DFT sequences, defined with (6), can be useful for communication systems based on OFDM or CDMA.

B. Aperiodic Cross-Correlation

A periodic cross-correlation function can be rewritten as an aperiodic cross-correlation function $C_{x,y}(l)$ [7]. For example, a periodic cross-correlation function between two sequences, x and y, is defined as:

$$\theta_{x,y}(l) = C_{x,y}(l) + C_{x,y}(l-N) \tag{7}$$

where $0 \le l < N$. When $x = y$, (7) can be transformed into

$$N + \left[\sum_{l=1}^{N-1} \left|\theta_x(l)\right|\right] \le N + 2\left[\sum_{l=1}^{N-1} \left|C_x(l)\right|\right] + \left|C_x(-N)\right|, \tag{8}$$

where $0 < l < N$. This result is obtained using also the properties $C_x(-l) = \left[C_x(l)\right]^*$, $\left|C_x(-N)\right| = 0$, and $C_x(0) = \theta_x(0) = N$ [7]. Then, when $0 < l < N$, it is possible to obtain a lower bound for the average of the aperiodic autocorrelation modulus:

$$\left[<\left|C_x(l)\right|>\right] \ge \frac{\left[<\left|\theta_x(l)\right|>\right]}{2}. \tag{9}$$

This lower bound can be minimized if $\left|\theta_x(l)\right| = 0$ for $0 < l < N$ .The sequences that satisfy this condition are perfect sequences.

When $x \ne y$, (7) can be transformed into the expression (with $1 - N \le l \le N - 1$):

$$\left[<\left|C_{x,y}(l)\right|>\right] \ge \frac{\left[<\left|\theta_{x,y}(l)\right|>\right]}{2}. \tag{10}$$

This lower bound can also be minimized if $\left|\theta_{x,y}(l)\right| = 0$, when $1 - N \le l \le N - 1$ (for any cyclic shift). Sequences that satisfy this condition are mutually orthogonal complementary (MOC) sequences [8]. The real and imaginary parts of our perfect DFT sequences, defined in (6), are MOC sequences.

C. CDMA Simulations With Different Bipolar Codes

The perfect DFT sequences of φ set can be transformed into bipolar codes using a signal function sgn{}. New bipolar codes, in (11), are constructed with the sum of two MOC sequences based on Gold codes (or φ set):

$$\text{sgn}\left\{\text{Re}\left[IDFT(Gold)\right] + \text{Im}\left[IDFT(Gold)\right]\right\}. \tag{11}$$

Alternatively, the union of the bipolar codes $\text{sgn}\left\{\text{Re}\left[IDFT(Gold)\right]\right\}$ and $\text{sgn}\left\{\text{Im}\left[IDFT(Gold)\right]\right\}$ can also be an optimum solution. It is expectable that, when N is high, these bipolar codes (with PAPR=1) are almost orthogonal and quasi perfect. For this reason, we have used (12) to find out if our new bipolar codes are better than the well known bipolar Gold codes [9]-[11].

In an asynchronous DS-CDMA (Direct Sequence Code Division Multiple Access) system with BPSK (Binary Phase Shift Keying) modulation, the average error probability P_e can be approximated by [12]:

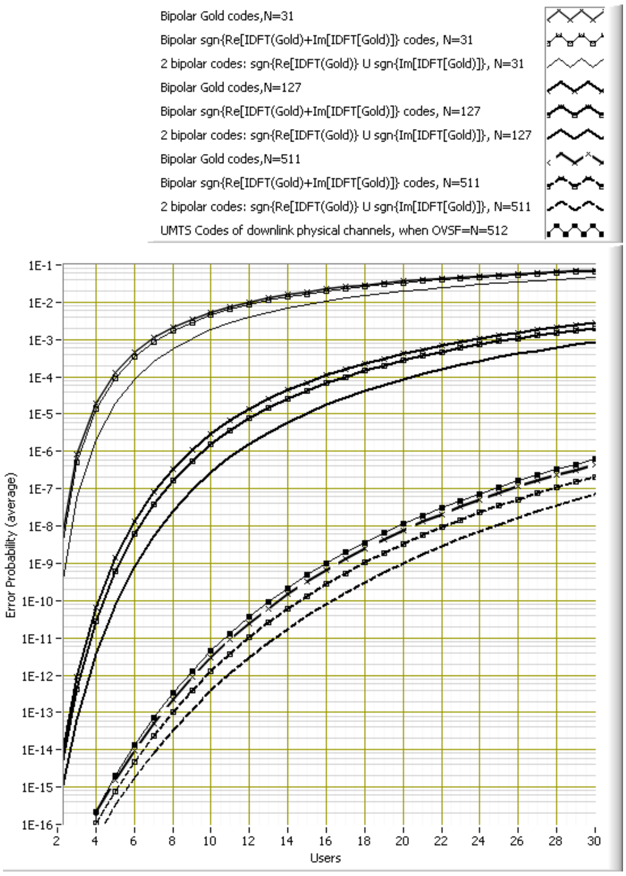

Figure 2: Error probability (average) *versus* number of Users.

$$P_e \approx 1 - \phi\left[\left(\frac{N_0}{2E_b} + \frac{1}{6N^3}\sum_{\substack{k=1\\k\neq i}}^{Q} r_{k,i}\right)^{-1/2}\right] \qquad (12)$$

where E_b is the bit energy, $N_0/2$ is the noise spectral density, ϕ is the normalized (i.e., zero mean, unit variance) Gaussian cumulative distribution function, Q is the number of simultaneous users, and $r_{k,i}$ is a function of the aperiodic autocorrelation functions $C_k(l)$ and $C_i(l)$ of the codes *k* and *i* [13]:

$$r_{k,i} = 2N^2 + 4\sum_{l=1}^{N-1} C_k(l) C_i(l) + \sum_{l=1-N}^{N-1} C_k(l) C_i(l+1). \qquad (13)$$

Our perfect DFT sequences, gathered in (6) and simplified with (11), may be evaluated in a synchronous OFDM system (with an m_r input per user). However, we have decided to consider an asynchronous system (the worst case). For this reason, we have used (12) to evaluate our new bipolar codes in an asynchronous BPSK DS-CDMA communication system (with one code per user). We have selected some bipolar Gold codes, (with length *N* equal to 31, 127 and 511) to make some comparisons. In Fig. 2, it is possible to see an error probability improvement of almost 10 dB with some of our new bipolar codes derived from MOC sequences of the set defined by (6). Nevertheless, our codes are also better than the codes used in the downlink physical channels of the Universal Mobile Telecommunication System (UMTS) [14], when the Orthogonal Variable Spreading Factor (OVSF) is 512.

IV. Conclusions

A set of *N*+1 sequences of length *N* with perfect periodic autocorrelation, low cross-correlation, and low PAPR was presented. These perfect DFT sequences are derived from bipolar *m*-sequences. A method to transform these perfect DFT sequences into orthogonal sequences, with good correlation properties, was found and a code generator was also proposed. New bipolar codes (derived from mutually orthogonal complementary perfect DFT sequences) were presented as an alternative to Gold codes. All these codes derived from perfect DFT sequences families can be applied in a pre-coded version of OFDM system or in a simple DS-CDMA communication system.

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