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A procedure for computing the symmetric difference of regions defined by polygonal curves ☆

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ABSTRACT

Given any two regions A, B in the plane, defined by polygonal (simple, closed and oriented) curves, associated with their respective boundaries, we describe a procedure to compute the symmetric difference $A \oplus B$. The output is also presented in the form of polygonal curves, where in particular the curves describing the union $A \cup B$, the intersection $A \cap B$, the difference $A \setminus B$, and the complement of the difference $B \setminus A$, are also obtained. This is related with the two equivalent formulas to compute the symmetric difference, namely $A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.
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1. Introduction

In this note we provide a detailed description of a procedure to be used in calculating the symmetric difference for arbitrary regions in the plane. In fact, since the whole process is more topological than geometrical, it can be used to calculate the symmetric difference of any two regions embedded in a oriented 2-manifold, while opening the way for a future study in higher dimensions.

By a region in the plane we mean the result of taking the topological closure of the interior of an arbitrary subset of the plane, more specifically, A is a region if

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$$A = cl\bigl(int(X)\bigr)$$

for some $X \in \mathbb{R}^2$.

It is clear that every such region is completely determined by its boundary, together with a specified orientation, used to define the interior of the region. We will assume that if *walking* throughout the oriented boundary, in the positive direction, the interior is always on the left.

The boundary of a region can be determined by a family of simple closed and oriented curves in the plane. For practical reasons it is convenient to consider polygonal curves so that each component is determined by a finite sequence of vertices (Har-Peled and Smorodinsky, 2005; Kedem et al., 1986; Sheng and Meier, 1995; Franklin, 1987; Brewer and Mark, 1986; Gardan and Perrin, 1996).

One of the novelties of this work is that instead of a family of polygonal curves, each one specified by a finite sequence of vertices, we will use a graph

$$E \underset{c}{\overset{d}{\rightrightarrows}} V$$

with a symmetry

$$\varphi : E \to E.$$

A graph is simply a fourth-tuple (E, V, d, c) where E and V are sets, with the interpretation that E is the set of edges and V the set of vertices, together with two maps

$$d, c : E \to V$$

designated by domain map and codomain map, associating a starting point, $d(x) \in V$, and an endpoint, $c(x) \in V$, to each edge $x \in E$, as displayed

$$d(x) \xrightarrow{x} c(x).$$

A symmetry φ on a graph (E, V, d, c) is simply a bijection $\varphi : E \to E$ with $d\varphi = c$, that is, for every $x \in E$, we have

$$d(x) \xrightarrow{x} c(x) = d\bigl(\varphi(x)\bigr) \xrightarrow{\varphi(x)} c\bigl(\varphi(x)\bigr).$$

It is also clear that if the set of vertices is embedded in the plane then the whole graph is embedded and the result is a oriented curve. In general, if no further restrictions are imposed on the graph, it is not guaranteed that the embedded curve represents the boundary of a region (for example it could happen that the curve is not simple). Nevertheless, there is a simple procedure that can be performed on the graph in such a way that the resulting curve is the boundary of a region (it suffices to invert the orientation of some of the edges in the graph), but this will not be discussed in here because the algorithm that we are presenting does not dependent on this issue.

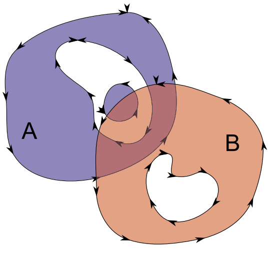
The main construction of this article is the following one. Given two graphs, G_A and G_B, each one with a symmetry and embedded in the plane (or in any other oriented 2-manifold), we provide a way to construct a new graph $G_{A,B}$ with a symmetry and an induced embedding in the plane, in such a way that: if G_A and G_B represent the boundary of two regions A and B in the plane, then the graph $G_{A,B}$ represents the region $A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$. In particular, we also provide a way (see Section 3) to separate the connected components of $G_{A,B}$ into two parts, one describing the union $A \cup B$, the other describing the intersection $A \cap B$. Moreover, applying the same procedure to the region A and the complement of B (this is done by simply inverting the edges in G_B) we obtain the components of $A \setminus B$ and the complement of $B \setminus A$.

The problem of computing boolean set operations for regions in the plane, while being of great importance with obvious applications to many fields (see for instance Requicha and Voelcker (1985), Carlbom (1987), Mantyla (1986), or more recently Martinez et al. (2009), Peng et al. (2005)) has never been considered, to our knowledge, at this level of generality where such a simple (but non-trivial) solution is possible.

This work is organized as follows. There is a section called *Problem and solution* where we make a formal statement of the problem (using the details already provided in the introduction) and provide a general overview for the proposed solution; a worked example is also presented. The section called *Graph merging* gives a detailed description for the first main step in the algorithm while the second part of it is presented in the section called *Component interpretation*. We choose to separate the two main steps of the algorithm in two different sections because they are also very different in nature. The first part is topological and only depends on some minor geometrical interpretation which is heavily used in the second part.

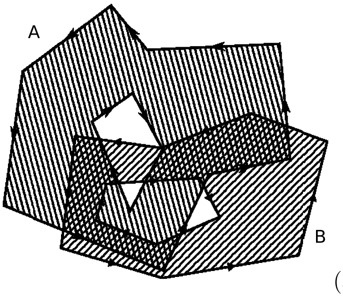
2. Problem and solution

Given any two regions A and B in the plane, as illustrated,

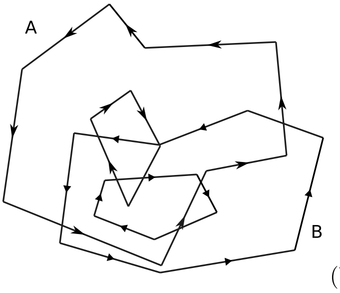


we are interested in computing the symmetric difference between A and B. This problem can be solved in many different ways. Here, we propose a method which clearly separates the steps depending only on topology from the ones that rely on geometry.

The first move is to encode the information needed to define the regions. This is done by describing an arbitrary region in the plane (as specified in the introduction) by a graph with a symmetry, in other words, each region is represented by a set of edges, with domain and codomain vertices (embedded in the plane, or in any other oriented 2-manifold) and a chaining map, called the symmetry of the graph. The next subsection provides a simple example with technical details, for the moment we may assume that the two regions A and B are described by a collection of closed and oriented polygonal curves, as illustrated below (figures (a), (b)), with the assumption that the interior of each region is always on the left hand side if walking along the directed curve. In the figure, the edges marked with long arrows belong to the boundary of region A, while the short flat arrows are used for region B.

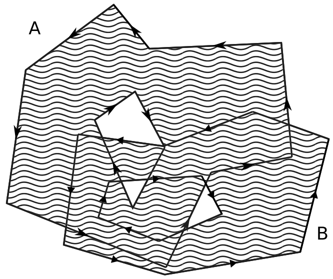


(a)

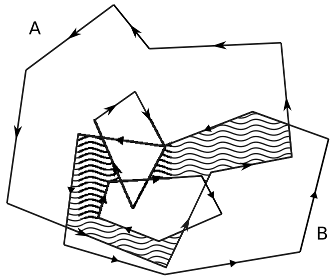


(b)

Having two graphs with a symmetry, whose vertices are embedded in the plane (or in any other oriented 2-manifold) we construct a new graph as detailed in Section 3. This new graph also has a symmetry. If each one of the input graphs is representing a region, then the connected components of the new graph can be interpreted as being part of the union (figure (c)) or part of the intersection (figure (d)) of the two regions; thus if inverting every component which is part of the intersection we obtain the description for the symmetric difference between the two input graphs (Output 3 in Section 4.1).

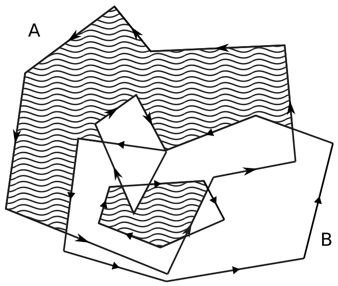


(c)

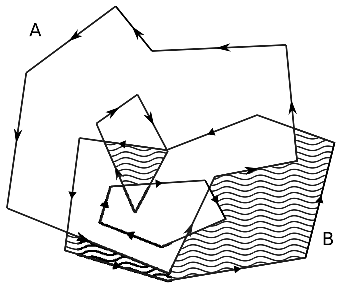


(d)

Alternatively, if the input graphs are G_A^{op} and G_B, with G_A^{op} denoting the graph G_A with every arrow inverted (specifically if $G_A = (E_A, V_A, d_A, c_A, \varphi_A)$ then $G_A^{op} = (E_A, V_A, c_A, d_A, \varphi_A^{-1})$), that is, G_A^{op} represents the set complement of the region A, then the components of the new graph are interpreted as being part either of the complement of the region $A \setminus B$ (figure (e)) or as the region $B \setminus A$ (figure (f)).



(e)

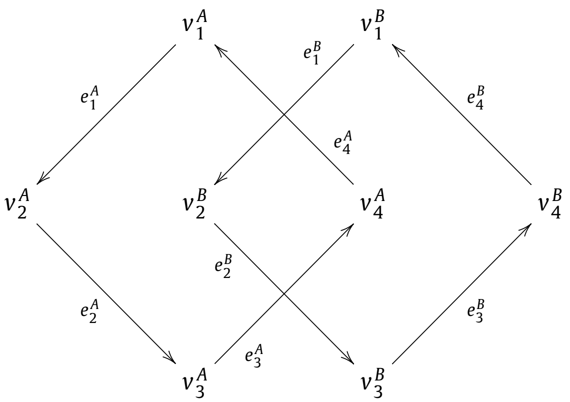


(f)

Again if inverting the components of the new graph that are part of $A \setminus B$ the result is again the symmetric difference between A and B. The interpretation process is detailed in Section 4 and illustrated below with a simple example.

2.1. Simple example

Consider the simple example of two regions A and B, both squares with non-empty intersection. The two graphs, G_A and G_B, representing each region are displayed together in the picture below.



The information used to define the graphs is specified in the following two tables, and it corresponds to (1) and (2), respectively (see next section).

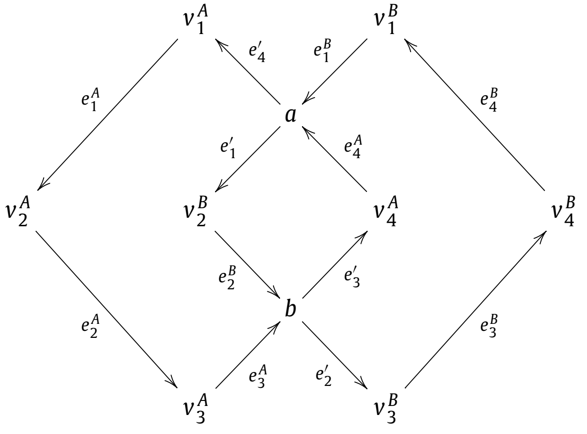
E_A	d	c	φ_A
e_1^A	v_1^A	v_2^A	e_2^A
e_2^A	v_2^A	v_3^A	e_3^A
e_3^A	v_3^A	v_4^A	e_4^A
e_4^A	v_4^A	v_1^A	e_1^A

E_B	d	c	φ_B
e_1^B	v_1^B	v_2^B	e_2^B
e_2^B	v_2^B	v_3^B	e_3^B
e_3^B	v_3^B	v_4^B	e_4^B
e_4^B	v_4^B	v_1^B	e_1^B

In this case the new graph that will be obtained is described by the table

E	d	c	φ
e_1^A	v_1^A	v_2^A	e_2^A
e_2^A	v_2^A	v_3^A	e_3^A
e_3^A	v_3^A	b	e_2'
e_4^A	v_4^A	a	e_1'
e_1^B	v_1^B	a	e_4'
e_2^B	v_2^B	b	e_2'
e_3^B	v_3^B	v_4^B	e_4^B
e_4^B	v_4^B	v_1^B	e_1^B
e_1'	a	v_2^B	e_2^B
e_2'	b	v_3^B	e_3^B
e_3'	b	v_4^A	e_4^A
e_4'	a	v_1^A	e_1^A

and is displayed in the following picture



Indeed we observe that the component

$$\left(e_1^A, e_2^A, e_3^A, e_2', e_3^B, e_4^5, e_1^B, e_4'\right)$$

represents the union, while

$$\left(e_1', e_2^B, e_3', e_4^A\right)$$

represents the intersection. The reader is now invited to consider the case where the original arrows (edges) from G_A are all inverted and observe that the resulting new graph has two components, describing precisely the complement of $A \setminus B$ and $B \setminus A$. This is so because $(A \setminus B)^c = (A \cap B^c)^c = A^c \cup B$

and $B \setminus A = B \cap A^c$, with X^c denoting the complement of X. The four possible outputs are presented in Section 4.1.

3. Graph merging

In this section we show how to merge two arbitrary graphs with symmetry (whose vertices are embedded in the plane) into a new graph with a symmetry whose cyclic components can be separated into two classes: those belonging to the union (Output 1 in Section 4.1) and those belonging to the intersection (Output 2 in Section 4.1). This later analysis is postponed to the next section.

Let

$$G_A = \left(\varphi_A \circlearrowright E_A \overset{d}{\underset{c}{\rightrightarrows}} V_A \right) \tag{1}$$

and

$$G_B = \left(\varphi_B \circlearrowright E_B \overset{d}{\underset{c}{\rightrightarrows}} V_B \right) \tag{2}$$

be two graphs with a symmetry (that is, φ_A and φ_B are invertible maps) and suppose that the vertices V_A and V_B are embedded in the plane via canonical injections

$$V_A \hookrightarrow \mathbb{R}^2,$$
$$V_B \hookrightarrow \mathbb{R}^2.$$

We will construct a new symmetric graph

$$G_{A,B} = \left(\varphi \circlearrowright E \overset{d}{\underset{c}{\rightrightarrows}} V \right)$$

in such a way that the components of φ will give the description of the regions $A \cap B$ and $A \cup B$.

3.1. Preliminary and trivial steps

The first step in defining the new graph

$$G_{A,B} = \left(\varphi \circlearrowright E \overset{d}{\underset{c}{\rightrightarrows}} V \right)$$

is to consider

$$V = V_A \cup V_B$$

and

$$E = E_A \sqcup E_B$$

the union of the vertices and the disjoint union of the edges from A and from B.

The second step is to calculate all the possible intersections between edges in A and edges in B. This is achieved by computing the pullback of the following diagram

$$\begin{array}{ccc} & & E_B \times [0,1] \\ & & \downarrow \\ E_A \times [0,1] & \longrightarrow & \mathbb{R}^2 \end{array}$$

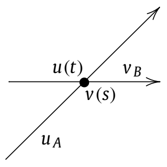
where each one of the two maps is defined by the formula

$$(u,t) \mapsto u(t) = d(u) + t\big(c(u) - d(u)\big)$$

for each $u \in E_A$ or $u \in E_B$ and $t \in [0, 1]$, assuming that the vertices $d(u)$ and $c(u)$ are canonically embedded in the plane. The resulting pullback may be presented as the set

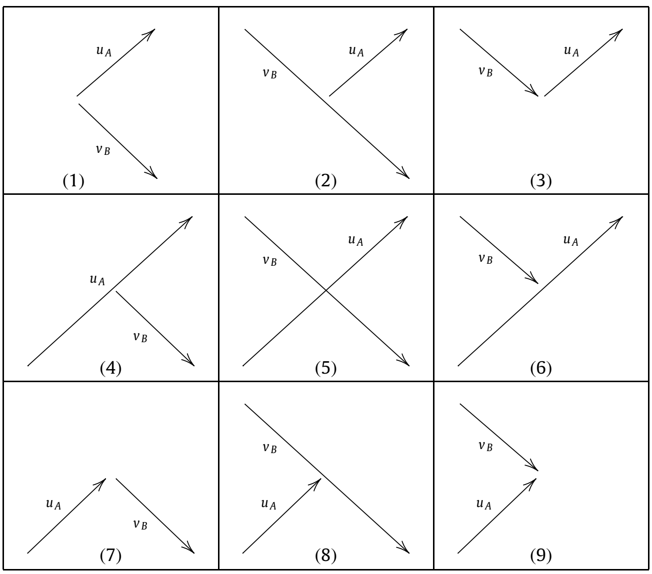
$$E_{AB} = \big\{ \big((u, t), (v, s)\big) \;\big|\; u \in E_A,\ v \in E_B,\ t, s \in [0, 1],\ u(t) = v(t) \big\}$$

as illustrated in the picture below.



3.2. Non-trivial step with nine cases to investigate

There are nine possible cases with intersections of different type that need to be analyzed, but only some of them are considered, after a systematic analysis.



The following table presents the possible configurations of intersections $((u, t), (v, s))$ with $u(t) = v(s)$, where in the second and third columns we find the values of t and s, while in the table above we illustrate each one of the nine cases, three by three.

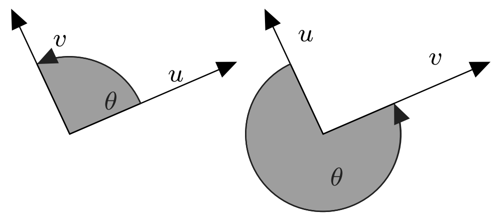
Case n	t is (in)	s is (in)
1	0	0
2	0	]0, 1[
3	0	1
4	]0, 1[	0
5	]0, 1[	]0, 1[
6	]0, 1[	1
7	1	0
8	1	]0, 1[
9	1	1

Next we present a systematic analysis of each case:

Case 1 is ignored because it will be considered as case 9;
Case 2 is ignored because it will be considered as case 8;
Case 3 is ignored because it will be considered as case 9;
Case 4 is ignored because it will be considered as case 6;
Case 5 is considered: given $((u,t),(v,s)) \in E_{AB}$, with $u \in E_A$, $v \in E_B$, $t,s \in]0,1[$, we compute

$$\sigma(u,v) = \begin{cases} 0 & \text{if } \sin(\theta) = 0 \\ 1 & \text{if } \sin(\theta) > 0 \\ -1 & \text{if } \sin(\theta) < 0 \end{cases}$$

where $\theta \in [0, 2\pi[$ is the angle measured in the trigonometric positive direction from u to v, as illustrated

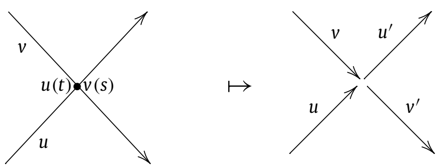


with $u \in E_A$ represented as the vector in $\mathbb{R}^2$ determined by the endpoints of u, that is,

$$d(u) \xrightarrow{u} c(u)$$

and similarly for $v_B \in E_B$ (another way to obtain $\sigma(u,v)$ is to calculate the cross product $u \times v$ of vectors in the space, using the so-called right hand rule).
Now we analyze the value of $\sigma(u,v)$:

(a) if $\sigma(u,v) = 0$, then do nothing (the two edges are parallel and the intersection is ignored);
(b) if $\sigma(u,v) \neq 0$, then create a new vertex in V, embedded in $\mathbb{R}^2$ as $u(t) = v(s)$, associate to it the number $\sigma(u,v)$ (this information will be used in the next section), and create two new edges in E, denoted by u' and v', while altering the existing ones u and v, as follows:
   (i) $d(u) \xrightarrow{u} u(t) = v(s) \xleftarrow{v} d(v)$,
   (ii) $c(u) \xleftarrow[u']{} u(t) = v(s) \xrightarrow[v']{} c(v)$ as pictured

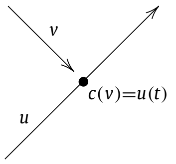


   (iii) with $\varphi(v) = u'$, $\varphi(u) = v'$, $\varphi(u') = \varphi_A(u)$ and $\varphi(v') = \varphi_B(v)$.

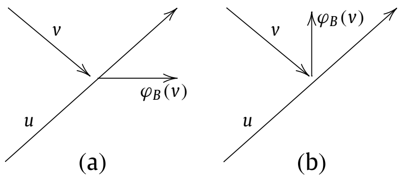
Case 6 is considered: given $((u,t),(v,s)) \in E_{AB}$, with $u \in E_A$, $t \in]0,1[$, $v \in E_B$ and $s = 1$, that is

$$u(t) = c(v) = v(1)$$

as displayed



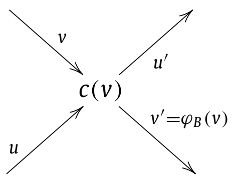
we have to investigate the relation between v and u, via $\sigma(u,v)$ as defined in case 5, as well as the relation between $\varphi_B(v)$ and u, in order to decide if $u(t)=c(v)$ is a crossing point (3(a)) or a tangent point (3(b)).



(3)

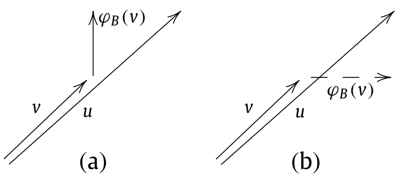
Assuming that $\sigma(u,v)\neq 0$ and $\sigma(u,\varphi_B(v))\neq 0$, $u(t)=v(1)=c(v)$ is a crossing point if and only if $\sigma(u,v)=\sigma(u,\varphi_B(v))$ and it is a tangent point if and only if $\sigma(u,v)=-\sigma(u,\varphi_B(v))$. Thus we have the following analysis:

(a) $\sigma(u,v)\neq 0$ and $\sigma(u,\varphi_B(v))\neq 0$
  (i) if $\sigma(u,v)=-\sigma(u,\varphi_B(v))$, then do nothing (tangent point);
  (ii) if $\sigma(u,v)=\sigma(u,\varphi_B(v))$, then perform the following operations: split u as



that is, u becomes $d(u)\xrightarrow{u}u(t)$ and a new $u(t)\xrightarrow{u'}c(u)$ is created; update the symmetry $\varphi: E\to E$ with $\varphi(v)=u'$, $\varphi(u)=\varphi_B(v)=v'$, $\varphi(u')=\varphi_A(u)$ and $\varphi(v')=\varphi_B(v')$.

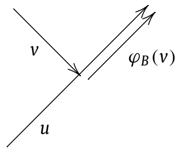
(b) If $\sigma(u,v)=0$, then do nothing; this case is ignored because, as also explained in case 9, at this point (with v parallel to u)



(4)

we may assume that if there is a crossing (4(b)), it was already considered in a situation of the form $\sigma(u,\varphi(v))=0$, see figure (5).

(c) If $\sigma(u,v)\neq 0$ and $\sigma(u,\varphi(v))=0$ then we adopt a similar procedure as the one used in the case 9(c), with u in the place of $\varphi(u)$ as displayed



(5)

in order to determine if $c(v)$ is a crossing point;

(i) if $c(v) = u(t)$ is a tangent point then do nothing;

(ii) if $c(v) = u(t)$ is a crossing point then do the same as in case (a)(ii) just above.

Case 7 is ignored because it is treated as case 9;

Case 8 is considered in a similar way as case 6, with $\varphi_A(u)$ and v in the place of u and $\varphi_B(v)$;

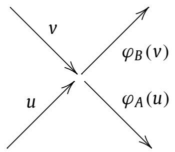
Case 9 is considered: given $((u,t),(v,s)) \in E_{AB}$ with $u \in E_A$, $v \in E_B$ and $t = 1 = s$, that is

$$c(u) = c(v),$$

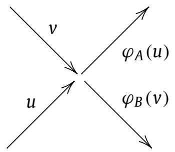
we have the following cases:

(a) $\sigma(u,v) \neq 0$ and $\sigma(\varphi_A(u), \varphi_B(v)) \neq 0$

(i) if $\sigma(u,v) = -\sigma(\varphi_A(u), \varphi_B(v))$ then do nothing ($c(u) = c(v)$ is a tangent point)



(ii) if $\sigma(u,v) = \sigma(\varphi_A(u), \varphi_B(v))$ then it is a crossing point



and we have to update φ, defining $\varphi(v) = \varphi_A(u)$ and $\varphi(u) = \varphi_B(v)$;

(b) if $\sigma(u,v) = 0$ then do nothing, by the same reason as the one explained in case 6(b);

(c) if $\sigma(u,v) \neq 0$ and $\sigma(\varphi_A(u), \varphi_B(v)) = 0$ then we have to determine if $c(u) = c(v)$ is a crossing point or a tangent point; to do so we adopt the following procedure:

(i) let $u' \leftarrow u$ (u' take the value of u) and $v' \leftarrow v$ and do repeatedly:

(ii)

$$\text{if } c(u') \in v', \quad \text{then } u' \leftarrow \varphi_A(u'),$$
$$\text{if } c(v') \in u', \quad \text{then } v' \leftarrow \varphi_B(v');$$

(iii) until $\sigma(u',v') \neq 0$;

if $\sigma(u',v') = \sigma(u,v)$ then $c(u) = c(v)$ is a crossing point, do the same as in (a)(ii); if, on the contrary, $\sigma(u',v') = -\sigma(u,v)$ then $c(u) = c(v)$ is a tangent point and do nothing.

Notice that the procedure outlined in (c) works only for the case where θ, the angle from u to v, is 0; for the case of $\theta = \pi$ we have to replace φ in the second line of (c)(ii), with φ^{-1} to obtain

$$\text{if } c(v') \in u', \quad \text{then } v' \leftarrow \varphi_B^{-1}(v').$$

This completes the analysis of the possible 9 cases and results in a new graph $G_{A,B} = (E, V, d, c, \varphi)$ with $V = (V_A \cup V_B) \sqcup V_{A,B}$, where $V_{A,B}$ is the set of the vertices added in the case 5(b), and where in the union $V_A \cup V_B$ we also identify the vertices that have the same canonical embedding in the plane. The set of edges is $E = E_A \sqcup E_B \sqcup X$, with X the set of edges u', v', that are added in the case 5(b)(ii), the edges u' that are added in case 6(a)(ii), and the edges v' that are added in the case 8(a)(ii). The maps $d, c : E \to V$ are defined for the new edges as specified while changing for the existing ones in the cases 5(b)(i), 6(a)(ii) and 8(a)(ii). The map $\varphi : E \to E$ is defined accordingly to φ_A or φ_B for the existing edges, and then updated as described in the cases 5(b)(iii), 6(a)(ii) and 9(a)(ii). The procedure also describes how to define φ for the edges in X.

4. Component interpretation

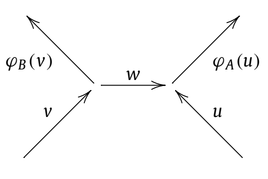
The result $G_{A,B}$ of the procedure described in the previous section will define two closed, non-intersecting polygonal curves, which represent $A \cup B$ and $A \cap B$. We will now see how to identify the components of $G_{A,B}$ that belong to the intersection and those which belong to the union.

Let C be a connected component of $G_{A,B}$.

Case A: If C contains edges from both A and B, or (equivalently) contains newly added vertices and edges, or (also equivalently) has vertices where a crossing of A and B occurs, then at those vertices $\sigma(u, v) = \sigma(\varphi_A(u), \varphi_B(v)) \neq 0$. Moreover, the component C will either contain u and $\varphi_B(v)$ or v and $\varphi_A(u)$. The sign of $\sigma(u, v)$ (or $\sigma(\varphi_B(u), \varphi_B(v))$) gives us the information we seek:

- (a) if negative:
  - (i) the component containing u and $\varphi_B(v)$ belongs to the union $A \cup B$, and
  - (ii) the component containing $\varphi_A(u)$ and v is part of the intersection $A \cap B$;
- (b) if positive:
  - (i) the component containing $\varphi_A(u)$ and v is part of the union $A \cup B$, and
  - (ii) the component containing u and $\varphi_B(v)$ belongs to the intersection $A \cap B$.

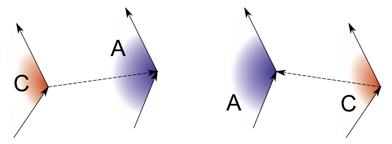
Case B: If C contains only edges from either A or B, or (equivalently) there are no vertices in C where a crossing of A with B occurs, then the above procedure cannot be applied. Let us consider, without loss of generality, that C contains edges from B only. Now consider the hypothetical edges connecting one vertex from C to one vertex in A, and find the shortest such edge, w. Let A' be the cyclic component in G_A containing the corresponding vertex of w. We define u and $\varphi_A(u)$ as before, as the edges in A' incident and emergent to or from that vertex (see figure below).



We proceed by determining $\sigma(w, u)$ and $\sigma(w, \varphi_A(u))$ with σ defined as in case 5 of the previous section.

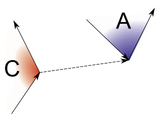
- (a) If $\sigma(w, u)$ and $\sigma(w, \varphi_A(u))$ have the same sign, then
  - (i) if they are both strictly negative, C lies in the complement of A, and belongs to the union;
  - (ii) if they are strictly positive, then C is in the interior of A, and thus C belongs to the intersection.

  Both situations are illustrated in the figures below:

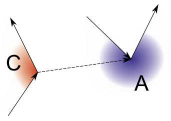


(b) If $\sigma(w,u)\cdot\sigma(w,\varphi_A(u))\leqslant 0$, we must additionally consider $\sigma(u,\varphi(u))$. Note that since both A and B have no self-intersections, u and $\varphi_A(u)$ are never anti-parallel, and so $\sigma(u,\varphi_A(u))$ will always be non-zero.

(i) If $\sigma(v,\varphi_A(v))>0$, C belongs to the union;



(ii) If $\sigma(u,\varphi_A(u))<0$, w (and C) lies in the interior of A, and so C belongs to the intersection.



If all the edges in C are from A, then we apply the same procedure as above using B in the place of A, φ_B in the place of φ_A, and v in the place of u.

4.1. Possible outputs

As already remarked in Section 2, there are many possible ways of computing the symmetric difference of two regions, and this phenomenon is visible in the algorithm. Given any two graphs with symmetry G_A and G_B (whose vertices are embedded in the plane) a new graph $G_{A,B}$ is constructed, Section 3. Each connected component of the new graph can be interpreted as belonging to the union or belonging to the intersection of the two regions defined by G_A and G_B. This means that the algorithm has four possible outputs.

Output 1: The union of A and B.
This corresponds to the subgraph of $G_{A,B}$ whose components were identified in cases A(a)(i), A(b)(i), B(a)(i) and B(b)(i).

Output 2: The intersection of A and B.
This corresponds to the subgraph of $G_{A,B}$ whose components were identified in cases A(a)(ii), A(b)(ii), B(a)(ii) and B(b)(ii).

Output 3: The symmetric difference of A and B.
This corresponds to the graph which is obtained from $G_{A,B}$ by inverting the components identified in the cases A(a)(ii), A(b)(ii), B(a)(ii) and B(b)(ii).

Output 4: The complement of the symmetric difference of A and B.
This corresponds to the graph which is obtained from $G_{A,B}$ by inverting the components identified in the cases A(a)(i), A(b)(i), B(a)(i) and B(b)(i).

In order to obtain $A\setminus B$ and $B\setminus A$ we simply observe that $A\setminus B=A\cap B^c$ and $(B\setminus A)^c=A\cup B^c$. So, we just have to apply the same procedure using B^c instead of B, and in the end take complement (i.e., invert the orientation) of the components defining the union (Output 4).

5. Applications

The algorithm presented has been developed for use in the context of generation of toolpaths for additive layered fabrication. In this application, a triangulated closed surface is decomposed in parallel horizontal slices, to be fabricated on top of each other, with each slice defined by it's polygonal boundary. It is necessary to determine whether each slice is fully supported by the underlying slices; if it isn't, we need to compute the boundary for the unsupported region, which is precisely the symmetric difference of the current and previous slices.

6. Conclusion

We conclude with a short analysis for the complexity of the outlined algorithm. The procedure described in Section 2 runs in $O(n \times m)$, with n and m the number of edges in G_A and G_B, corresponding to the comparison of the edges from A and from B. The algorithm outlined in Section 3, searches for vertices where an intersection occurs (already computed in Section 2), with complexity of at most $O(n + m)$, and has to find the closest vertices for each component falling in case B, with at most $O(n \times m)$ of complexity. This means that the overall process runs with complexity $O(n \times m)$.

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