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Editor’s Choice

Spectral theory for the fractal Laplacian in the context of h-sets

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An h-set is a nonempty compact subset of the Euclidean n-space which supports a finite Radon measure for which the measure of balls centered on the subset is essentially given by the image of their radius by a suitable function h. In most cases of interest such a subset has Lebesgue measure zero and has a fractal structure.

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ with $\Gamma \subset \Omega$. Let

$$B := (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma ,$$

where $(-\Delta)^{-1}$ is the inverse of the Dirichlet Laplacian in Ω and tr^Γ is, say, a trace type operator. The operator B, acting in convenient function spaces in Ω, is studied. Estimations for the eigenvalues of B are presented, and generally shown to be dependent on h, and the smoothness of the associated eigenfunctions is discussed. Some results on Besov spaces of generalised smoothness on $\mathbb{R}^n$ and on domains which were obtained in the course of this work are also presented, namely pointwise multipliers, the existence of a universal extension operator, interpolation with function parameter and mapping properties of the Dirichlet Laplacian.

1 Introduction

Let $h : (0, 1] \to \mathbb{R}^+$ denote a continuous monotone increasing function such that $h(0^+) = 0$. Assume that there exist a nonempty compact set $\Gamma \subset \mathbb{R}^n$ and a finite Radon measure μ with Γ as support and such that

$$\mu(B(\gamma, r)) \sim h(r), \quad 0 < r \le 1, \quad \gamma \in \Gamma, \tag{1.1}$$

where $B(\gamma, r)$ denotes any open ball with center in γ and radius r. Such a set Γ is called an h-set. Two well-known classes of h-sets are d-sets and (d, ψ)-sets. In these particular cases the function h is

$$h(r) = r^d \quad \text{and} \quad h(r) = r^d \psi(r), \tag{1.2}$$

respectively, where $d > 0$ and $\psi : (0, 1] \to \mathbb{R}^+$ is a monotone function such that

$$\psi(2^{-j}) \sim \psi(2^{-2j}), \quad j \in \mathbb{N}.$$

We refer to [5], [8], where h-sets were studied and a characterisation of the functions h for which there exist h-sets was obtained.

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Let Ω be a bounded $\mathcal{C}^\infty$ domain in $\mathbb{R}^n$ such that $\Gamma \subset \Omega$ and let $-\Delta$ be the Dirichlet Laplacian in Ω. We study the operator

$$B := (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma, \tag{1.3}$$

acting in convenient function spaces in Ω, where $(-\Delta)^{-1}$ is the inverse of the Dirichlet Laplacian in Ω and

$$\mathrm{tr}^\Gamma = \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma, \tag{1.4}$$

where tr_Γ is an extension of the operator

$$\varphi \to \varphi|_\Gamma, \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

and id^Γ, which will be formally defined later, identifies elements of $L_p(\Gamma)$ with tempered distributions. The operator B was already studied in the case where Γ is d-set by Triebel in [38, Chapter 5] and [39, Chapter 3]. The case where Γ is a (d, ψ)-set was studied by Edmunds and Triebel in [15], [16] and by Moura in [29]. As it was mentioned in these works, in the case $n = 2$ the operator B has physical relevance: it describes the vibration of a drum where the whole mass is distributed on Γ. This is the reason why the study of this subject is usually called the *fractal drum* problem. So in this paper the *fractal drum* problem in the context of h-sets is studied, extending the results for d-sets and (d, ψ)-sets. More precisely, several spectral properties of B, acting in suitable function spaces, are proved, see Proposition 7.1 and Theorem 8.7. In particular, in the latter theorem, which may be considered the main result of this paper, we study what than be called the Dirichlet Laplacian on fractal h-sets, obtaining, among other results, a representation by Green's function, smoothness properties for their eigenfunctions and the asymptotic behavior of the corresponding eigenvalues. This, which can be seen in (8.9), shows that the function h clearly influences such a behavior, except when $n = 2$.

As a by-product of studying the operator B some other interesting results for Besov spaces of generalised smoothness were obtained: a result on pointwise multipliers for these spaces (Proposition 5.4 and Corollary 5.6), the existence of a universal extension operator acting from Besov spaces on a class of domains into corresponding function spaces on $\mathbb{R}^n$ (Theorems 5.15 and 5.17) and mapping properties for the Dirichlet Laplacian (Theorem 5.21). In the proof of some of these results interpolation (with a function parameter), an important tool in the theory of function spaces, was applied (we refer to Theorem 5.2 and to Propositions 5.19 and 5.20).

The paper is organised as follows. In Section 2 we collect some general notation and in Section 3 some notions and properties related to sequences and functions that often appear in the context of function spaces of generalised smoothness. In Section 4 we present definitions and results on function spaces on $\mathbb{R}^n$ and on domains. In Section 5 we study the Laplacian acting in Besov spaces of generalised smoothness, applying interpolation with a function parameter, results on pointwise multipliers and the existence of a universal extension operator acting from function spaces on domains to corresponding spaces on $\mathbb{R}^n$. In Section 6 we collect definitions and properties of h-sets and operators defined in connection with these fractal sets. In Sections 7 and 8 we study the operator B, i.e., the Dirichlet Laplacian on fractal h-sets.

2 General notation

In this section some standard notation and useful definitions are presented. We write $\mathbb{N}$ for the set of all natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all integers, real and complex numbers, respectively. As usually, $\mathbb{R}^n$, $n \in \mathbb{N}$, stands for the n-dimensional real Euclidean space and, given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x|$ stands for the Euclidean norm of x. We write $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, to represent the collection of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$, $j = 1, \ldots, n$. For $\alpha \in {\mathbb{N}_0}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$ and the derivatives $D^\alpha := \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n})$ have the usual meaning. If there is no additional information, when we speak about *functions* we are considering complex-valued functions. References with respect to measurability and integrability should be understood in the Lebesgue sense. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for its Lebesgue measure in $\mathbb{R}^n$. An open ball with center in $x \in \mathbb{R}^n$ and radius $r > 0$ is denoted by $B(x, r)$. If $E \subset \mathbb{R}^n$, then ∂E denotes the boundary of E and $\overline{E}$ stands for the closure of E. If $E \subset \mathbb{R}^n$ and $r > 0$ we denote by E_r the r-neighborhood of E, i.e., the collection of all points x in $\mathbb{R}^n$ such that there is $y \in E$ for which $|x - y| < r$.

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable and rapidly decreasing functions in $\mathbb{R}^n$, equipped with the usual topology, and by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, the space of all tempered distributions on $\mathbb{R}^n$.

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\mathcal{F}\varphi$, or $\widehat{\varphi}$, represents the Fourier transform of φ, i.e.,

$$(\mathcal{F}\varphi)(\xi) := \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

and $\mathcal{F}^{-1}\varphi$, or $\check{\varphi}$, represents the inverse Fourier transform of φ,

$$(\mathcal{F}^{-1}\varphi)(\xi) := \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \varphi(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$ we will use the notation

$$\varphi(D)f := \mathcal{F}^{-1}(\varphi \mathcal{F} f).$$

As usual, *domain* stands for nonempty open set. If Ω is a domain in $\mathbb{R}^n$ then $\mathcal{D}(\Omega)$ is the collection of all compactly supported complex-valued C^∞ functions on Ω and $\mathcal{D}'(\Omega)$ stands for the dual space of all distributions on Ω. If Ω is a domain in $\mathbb{R}^n$ then $L_p(\Omega)$ denotes the collection of all complex-valued Lebesgue measurable functions in Ω such that

$$\|f|L_p(\Omega)\| := \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}$$

(with the usual modification ess $\sup_{x\in\Omega} |f(x)|$ if $p = \infty$) is finite.

Furthermore, if $0 < q \le \infty$ then ℓ_q is the collection of all complex sequences $a = (a_j)_{j\in\mathbb{N}_0}$ such that

$$\|a|\ell_q\| := \left(\sum_{j=0}^{\infty} |a_j|^q\right)^{1/q}$$

(with the standard adaptation $\sup_{j\in\mathbb{N}_0} |a_j|$ if $q = \infty$) is finite.

If $(f_j)_{j\in\mathbb{N}_0}$ is a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, then

$$\|(f_j)_{j\in\mathbb{N}_0}|\ell_q(L_p)\| := \left(\sum_{j=0}^{\infty} \|f_j|L_p\|^q\right)^{1/q},$$

with the appropriate modification if $q = \infty$ and

$$\|(f_j)_{j\in\mathbb{N}_0}|L_p(\ell_q)\| := \left(\int_{\mathbb{R}^n} \|f_j(x)|\ell_q\|^p dx\right)^{1/p},$$

with the usual adaptation if $p = \infty$.

We use the symbol "$\lesssim$" in

$$a_k \lesssim b_k \quad \text{or} \quad \phi(r) \lesssim \psi(r)$$

always to mean that there is a positive number c_1 such that

$$a_k \le c_1 b_k \quad \text{or} \quad \phi(r) \le c_1 \psi(r),$$

for all admitted values of the discrete variable k or the continuous variable r, where $(a_k)_k$, $(b_k)_k$ are nonnegative sequences and ϕ, ψ are nonnegative functions. We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \phi(r) \sim \psi(r),$$

respectively for

$$a_k \lesssim b_k \quad \text{and} \quad b_k \lesssim a_k \quad \text{or} \quad \phi(r) \lesssim \psi(r) \quad \text{and} \quad \psi(r) \lesssim \phi(r).$$

In what follows log is always taken with respect to base 2.

If A and B are quasi-Banach spaces and f is a linear map defined on A, then we will write

$$f : A \mapsto B$$

to mean that f is a continuous linear operator acting from A to B.

Further notation will be introduced whenever it is needed.

3 Admissible sequences and functions

In this section some definitions, notation and properties of sequences, functions and corresponding indices that will be applied along the paper are presented.

Definition 3.1 Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers. We say that σ is an *admissible sequence* if there are positive constants d_0, d_1 such that

$$d_0\sigma_j \le \sigma_{j+1} \le d_1\sigma_j, \quad j \in \mathbb{N}_0.$$

We introduce two particular kinds of admissible sequences.

Example 3.2

(i) We will denote by (s) the (admissible) sequences defined by

$$(s) := (2^{js})_{j\in\mathbb{N}_0}, \quad s \in \mathbb{R}. \tag{3.1}$$

(ii) Let $\psi : (0,1] \to \mathbb{R}$ be a positive monotone function such that $\psi(2^{-2j}) \sim \psi(2^{-j})$, for all $j \in \mathbb{N}$. We will denote by (s,ψ) the sequences

$$(s,\psi) := (2^{js}\psi(2^{-j}))_{j\in\mathbb{N}_0}, \quad s \in \mathbb{R},$$

which are also admissible sequences.

Notation 3.3 Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $\beta = (\beta_j)_{j\in\mathbb{N}_0}$ be admissible sequences and $\alpha \in \mathbb{R}$. In what follows we denote by σ^α and $\sigma\beta$ the sequences defined by

$$\sigma^\alpha = (\sigma_j^\alpha)_{j\in\mathbb{N}_0} \quad \text{and} \quad \sigma\beta = (\sigma_j\beta_j)_{j\in\mathbb{N}_0}, \tag{3.2}$$

respectively. It can be easily verified that both σ^α and $\sigma\beta$ are admissible sequences.

In this work function spaces of generalised smoothness will be considered. In this context usually some convenient indices are considered to play the role of the regularity index usually denoted by s. We will use the indices that we describe next.

Definition 3.4 Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence and

$$\underline{\sigma}_j := \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j := \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k}, \quad j \in \mathbb{N}_0.$$

The *lower* and *upper Boyd indices* of the sequence σ are defined, respectively, by

$$\underline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \underline{\sigma}_j}{j} \quad \text{and} \quad \overline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \overline{\sigma}_j}{j}.$$

Remark 3.5 For an admissible sequence σ, the sequence $(\log \overline{\sigma}_j)_{j\in\mathbb{N}_0}$ is sub-additive. This justifies the definition of $\overline{s}(\sigma)$. As $\log \underline{\sigma}_j = -\log(\overline{\sigma^{-1}})_j$, $\underline{s}(\sigma)$ also makes sense.

We remark that if σ and β are admissible sequences such that $\sigma \sim \beta$, then their Boyd indices coincide.

If σ is an admissible sequence, then for each $\delta > 0$ there are two positive constants $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$ such that for all $j, k \in \mathbb{N}_0$,

$$c_1 2^{(\underline{s}(\sigma)-\delta)j} \le \frac{\sigma_{j+k}}{\sigma_k} \le c_2 2^{(\overline{s}(\sigma)+\delta)j}. \tag{3.3}$$

This property follows immediately from Definitions 3.1 and 3.4.

In the next proposition we use the notation given in Example 3.2 and Notation 3.3.

Proposition 3.6 *Let σ and β be admissible sequences and $\alpha \in \mathbb{R}$. Then*

(i) $\underline{s}(\sigma\beta) \ge \underline{s}(\sigma) + \underline{s}(\beta)$ *and* $\overline{s}(\sigma\beta) \le \overline{s}(\sigma) + \overline{s}(\beta)$;

(ii) $\underline{s}((\alpha)\sigma) = \alpha + \underline{s}(\sigma)$ *and* $\overline{s}((\alpha)\sigma) = \alpha + \overline{s}(\sigma)$;

(iii) $\underline{s}(\sigma^\alpha) = \alpha\underline{s}(\sigma)$ *and* $\overline{s}(\sigma^\alpha) = \alpha\overline{s}(\sigma)$, *if* $\alpha > 0$;

(iv) $\underline{s}(\sigma^\alpha) = \alpha\overline{s}(\sigma)$ *and* $\overline{s}(\sigma^\alpha) = \alpha\underline{s}(\sigma)$, *if* $\alpha < 0$.

Proof. The properties given follow from Definition 3.4. We present the proof of (i). The proof of (ii)–(iv) is similar. Let σ and β be admissible sequences. Then

$$\underline{(\sigma\beta)}_j = \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}\beta_{j+k}}{\sigma_k\beta_k} \ge \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \inf_{k\in\mathbb{N}_0} \frac{\beta_{j+k}}{\beta_k} = \underline{\sigma}_j \underline{\beta}_j.$$

Hence

$$\underline{s}(\sigma\beta) = \lim_{j\to\infty} \frac{\log \underline{(\sigma\beta)}_j}{j} \ge \lim_{j\to\infty} \frac{\log(\underline{\sigma}_j \underline{\beta}_j)}{j} = \underline{s}(\sigma) + \underline{s}(\beta). \qquad \square$$

The following classes of functions will be considered.

Definition 3.7

(i) We denote by $\mathcal{B}$ the collection of all continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\overline{f}(t) := \sup_{s>0} \frac{f(st)}{f(s)} < \infty, \quad \text{for all} \quad t > 0. \tag{3.4}$$

(ii) We denote by $\mathbf{B}$ the collection of all $f \in \mathcal{B}$ such that

$$f(t^{-1}) = f(t)^{-1}, \quad \text{for all} \quad t > 0.$$

Remark 3.8 In the results in [13] and [28], the class $\mathcal{B}$ was considered. In [4] for technical reasons an additional condition was considered, resulting in the class $\mathbf{B}$.

Definition 3.9 Let $f \in \mathcal{B}$. The lower and upper Boyd indices of f are defined, respectively, by

$$\underline{S}(f) := \lim_{t\to 0} \frac{\log \overline{f}(t)}{\log t} \quad \text{and} \quad \overline{S}(f) := \lim_{t\to\infty} \frac{\log \overline{f}(t)}{\log t}.$$

Remark 3.10 The function $\overline{f}$ defined in (3.4) is submultiplicative and Lebesgue measurable. Therefore $\underline{S}(f)$ and $\overline{S}(f)$ are well-defined and satisfy

$$-\infty < \underline{S}(f) \le \overline{S}(f) < +\infty.$$

For properties of these indices we refer to [13] and [28].

For our purposes it is convenient to associate to all given admissible sequence a function in $\mathbf{B}$.

Definition 3.11 Let σ be an admissible sequence. We denote by $\mathbf{B}\sigma$ the collection of all functions $f \in \mathbf{B}$ such that

$$\sigma_j \sim f(2^j), \quad j \in \mathbb{N}_0.$$

The next example, given in [4], guarantees that for any admissible sequence σ the set $\mathbf{B}_\sigma$ is nonempty.

Example 3.12 Let $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ be an admissible sequence. The function $f : (0,\infty) \to (0,\infty)$ defined by

$$f(z) = \begin{cases} \sigma_0^{-1}\big((\sigma_{j+1} - \sigma_j)(2^{-j}z - 1) + \sigma_j\big), & z \in [2^j, 2^{j+1}), \quad j \in \mathbb{N}_0 \\ f(z^{-1})^{-1}, & z \in (0,1) \end{cases}$$

belongs to $\mathbf{B}$ and, for all $j \in \mathbb{N}_0$, $f(2^j) = \sigma_j/\sigma_0$, i.e., $f \in \mathbf{B}_\sigma$.

The result we present next (cf. [4, p. 384, Proposition 3.9]) relates the Boyd indices of an admissible sequence with the corresponding indices for functions in $\mathbf{B}_\sigma$.

Proposition 3.13 *Let σ be an admissible sequence and $f \in \mathbf{B}_\sigma$. Then*

$$\underline{s}(\sigma) = \underline{S}(f) \quad \textit{and} \quad \overline{s}(\sigma) = \overline{S}(f).$$

4 Function spaces

In this section we present the function spaces on $\mathbb{R}^n$ and on domains where the operators will act. We also present some properties, namely equivalent quasi-norms, sufficient conditions for embeddings and results on duality.

Definition 4.1 Let $1 < p < \infty$.

(i) Let $m \in \mathbb{N}$. The *Sobolev space* on $\mathbb{R}^n$, $W_p^m(\mathbb{R}^n)$, is given by

$$W_p^m(\mathbb{R}^n) := \left\{ f \in L_p(\mathbb{R}^n) : \big\|f|W_p^m(\mathbb{R}^n)\big\| = \sum_{|\alpha| \le m} \|D^\alpha f|L_p(\mathbb{R}^n)\| < \infty \right\}.$$

(ii) Let $s \in \mathbb{R}$. The *Bessel-potential space* on $\mathbb{R}^n$, $H_p^s(\mathbb{R}^n)$, is given by

$$H_p^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|\mathcal{F}^{-1}((1+|x|^2)^{s/2}\mathcal{F}f)|L_p(\mathbb{R}^n)\| < \infty \right\}.$$

If $p = 2$ we use the abbreviation

$$H_p^s(\mathbb{R}^n) = H^s(\mathbb{R}^n).$$

We fix some more notation.

Definition 4.2 By a *resolution of unity* we mean a sequence of compactly supported smooth function such that

(i) $\operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \le 2\}$;

(ii) $\operatorname{supp} \varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \le |x| \le 2^{j+1}\}, \quad j \in \mathbb{N}$;

(iii) for any $\alpha \in \mathbb{N}_0^n$ there exists a constant $c_\alpha > 0$ such that for any j $\in \mathbb{N}_0$

$$|D^\alpha \varphi_j(x)| \le c_\alpha 2^{-j|\alpha|} \quad \text{for any} \quad x \in \mathbb{R}^n;$$

(iv) for all $x \in \mathbb{R}^n$

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1.$$

Definition 4.3 Let $\Phi = (\varphi_j)_{j\in\mathbb{N}_0}$ be a resolution of unity and let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence.

(i) Let $0 < p, q \le \infty$. The *Besov space of generalised smoothness* on $\mathbb{R}^n$, $B^\sigma_{p,q}(\mathbb{R}^n)$, is given by

$$B^\sigma_{p,q}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \left\|f|B^\sigma_{p,q}(\mathbb{R}^n)\right\| = \|(\sigma_j\varphi_j(D)f)_{j\in\mathbb{N}_0}|\ell_q(L_p)\| < \infty\}.$$

(ii) Let $0 < p < \infty$ and $0 < q \le \infty$. The *Triebel-Lizorkin space of generalised smoothness* on $\mathbb{R}^n$, $F^\sigma_{p,q}(\mathbb{R}^n)$, is given by

$$F^\sigma_{p,q}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \left\|f|F^\sigma_{p,q}(\mathbb{R}^n)\right\| = \|(\sigma_j\varphi_j(D)f)_{j\in\mathbb{N}_0}|L_p(\ell_q)\| < \infty\}.$$

Remark 4.4 This Fourier analytic description of Besov spaces of generalised smoothness was given in [17]. In this work one can also find some information about the history of function spaces of generalised smoothness with several references. If $p = q$ we abbreviate $B^\sigma_p(\mathbb{R}^n) = B^\sigma_{p,p}(\mathbb{R}^n)$. If $\sigma = (s)$ for some $s \in \mathbb{R}$ the above spaces coincide with the usual Besov spaces usually denoted by $B^s_{p,q}(\mathbb{R}^n)$ treated in detail by Triebel in [34], [36] and [39]. We will follow the notation referred above and denote these spaces by $B^{(s)}_{p,q}(\mathbb{R}^n)$.

It is well-known that Besov and Triebel-Lizorkin spaces include classic function spaces, namely:

(i) $F^{(0)}_{p,2}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ if $1 < p < \infty$;

(ii) $F^{(m)}_{p,2}(\mathbb{R}^n) = W^m_p(\mathbb{R}^n)$ if $1 < p < \infty$ and $m \in \mathbb{N}$;

(iii) $F^{(s)}_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$ if $1 < p < \infty$ and $s \in \mathbb{R}$;

(iv) $B^{(s)}_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n)$ if $s > 0$,

where $\mathcal{C}^s(\mathbb{R}^n)$, for $s > 0$, denotes the Hölder-Zygmund spaces. For the definition we refer to [34, p. 36].

Let $0 < q, r \le \infty$ and $(q_r)' \in [1, \infty]$ be given by

$$\begin{cases} \dfrac{1}{(q_r)'} + \dfrac{1}{q} = \dfrac{1}{r}, & \text{if} \quad q > r, \\ \infty, & \text{if} \quad q \le r. \end{cases}$$

Proposition 4.5 *Let $0 < p, q, r \le \infty$ and let σ and τ be admissible sequences. Then*

$$B^\sigma_{p,q}(\mathbb{R}^n) \subset B^\tau_{p,r}(\mathbb{R}^n) \quad \textit{if} \quad \sigma^{-1}\tau \in l_{(q_r)'}. \tag{4.1}$$

In particular,

$$B^\sigma_{p,q}(\mathbb{R}^n) \subset B^\tau_{p,r}(\mathbb{R}^n) \quad \textit{if} \quad \overline{s}(\tau) < \underline{s}(\sigma). \tag{4.2}$$

Remark 4.6 The result in (4.1) was stated in [8, p. 40, Proposition 4.12]. The result in (4.2) follows from (4.1) and Remark 3.5.

For the following result we refer to [34, p. 88, 2.5.6].

Theorem 4.7 *Let $1 < p < \infty$.*

(i) *Let $s \in \mathbb{R}$. Then*

$$F^{(s)}_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n)$$

(*equivalent norms*).

(ii) *Let $m \in \mathbb{N}$. Then*

$$H^m_p(\mathbb{R}^n) = W^m_p(\mathbb{R}^n)$$

(*equivalent norms*) *and*

$$\|f|W^m_p(\mathbb{R}^n)\|^* := \|f|L_p(\mathbb{R}^n)\| + \sum_{j=1}^n \left\|\frac{\partial^m f}{\partial x_j^m}|L_p(\mathbb{R}^n)\right\|$$

is an equivalent norm in $H^m_p(\mathbb{R}^n)$.

Definition 4.8 Let Ω be a domain in $\mathbb{R}^n$. Let σ be an admissible sequence, let $A \in \{B, F\}$ and let $0 < p \leq \infty$ and $0 < q \leq \infty$, with $p < \infty$ if $A = F$.

(i) We denote by $A^\sigma_{p,q}(\Omega)$ the collection of all $f \in D'(\Omega)$ such that there is a $g \in A^\sigma_{p,q}(\mathbb{R}^n)$ with $g|_\Omega = f$. Furthermore,

$$\|f|A^\sigma_{p,q}(\Omega)\| := \inf \|g|A^\sigma_{p,q}(\mathbb{R}^n)\|,$$

where the infimum is taken over all $g \in A^\sigma_{p,q}(\mathbb{R}^n)$ such that its restriction $g|_\Omega$ to Ω coincides in $D'(\Omega)$ with f.

(ii) We denote by $\mathring{A}^\sigma_{p,q}(\Omega)$ the completion of $\mathcal{D}(\Omega)$ in $A^\sigma_{p,q}(\Omega)$

Remark 4.9 If Ω is a domain in $\mathbb{R}^n$, $1 < p < \infty$, $s \in \mathbb{R}$ and $m \in N$, then the meaning of

$$W^m_p(\Omega), \quad \mathring{W}^m_p(\Omega), \quad H^s_p(\Omega) \quad \text{and} \quad \mathring{H}^s_p(\Omega)$$

follows immediately from Theorem 4.7 and Definition 4.8.

Remark 4.10 One can easily verify that (4.1) and (4.2) in Proposition 4.5 remain valid if we write Ω instead of $\mathbb{R}^n$.

In the next theorem (cf. [8, p. 40, Proposition 4.11]) the dual spaces of the spaces $B^\sigma_{p,q}(\mathbb{R}^n)$ are determined.

Theorem 4.11 *Let σ be an admissible sequence, let $1 \leq p < \infty$, $1 \leq q < \infty$ and let p' and q' denote their conjugates. Then*

$$\left(B^\sigma_{p,q}(\mathbb{R}^n)\right)' = B^{\sigma^{-1}}_{p',q'}(\mathbb{R}^n) \tag{4.3}$$

according with the notation introduced in (3.2).

5 Interpolation, pointwise multipliers and an extension operator

As we have already mentioned, a collection of results on Besov spaces of generalised smoothness were obtained in order to study the spectral theory for the fractal Laplacian. This section is devoted to these results. An important tool in the proof of almost all of them was the interpolation with function parameter, so we start by collecting some notation and results on this subject.

5.1 Interpolation with function parameter

In [13] and [28] it was proved that Besov spaces of generalised smoothness can be obtained by interpolation of classic Besov spaces. In this section we apply this to obtain some properties for general Besov spaces, combining the known results for classic Besov spaces with interpolation. This technique was already used by other authors to extend results from classical to generalised smoothness. We refer to [3] and [12].

First we introduce some basic notation related to interpolation. For a systematic treatment of the interpolation theory in Banach spaces, including the notions and results we are applying, we refer to [37, Chapter 1]. For a collection of notions and results on real interpolation with function parameter of quasi-Banach spaces we refer to [2, Chapter 2].

We consider $\{A_0, A_1\}$ an interpolation couple, meaning that A_0 and A_1 are quasi-normed spaces both continuously embedded in some Hausdorff topological vector space. In this case

$$\|a|A_0 \cap A_1\| := \max\{\|a|A_0\|, \|a|A_1\|\}, \quad a \in A_0 \cap A_1$$

and

$$\|a|A_0 + A_1\| := \inf_{\substack{a = a_0 + a_1 \\ a_0 \in A_0,\ a_1 \in A_1}} (\|a_0|A_0\| + \|a_1|A_1\|), \quad a \in A_0 + A_1$$

define quasi-norms in $A_0 \cap A_1$ and $A_0 + A_1$, respectively. There are different approaches to construct interpolation spaces. In this work we use the real method based on the K-functional introduced by Peetre. This functional is defined by

$$K(t,a) = K(t,a,A_0,A_1) := \inf_{\substack{a = a_0 + a_1 \\ a_0 \in A_0,\ a_1 \in A_1}} (\|a_0|A_0\| + t\|a_1|A_1\|). \tag{5.1}$$

For all $t > 0$, $K(t,\cdot,A_0,A_1)$ is an equivalent quasi-norm in the space $A_0 + A_1$.

In the next definition we consider a function in $\mathcal{B}$. We recall that this class of functions was introduced in Definition 3.7(ii).

Definition 5.1 Let $\{A_0, A_1\}$ be an interpolation couple. Let $\gamma \in \mathcal{B}$ and let $0 < q \leq \infty$. We define the corresponding *interpolation space with function parameter* by

$$(A_0, A_1)_{\gamma,q} := \left\{ a \in A_0 + A_1 : \|a|(A_0, A_1)_{\gamma,q}\| = \left(\int_0^\infty [\gamma(t)^{-1} K(t,a,A_0,A_1)]^q \, \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

with the usual modification if q is not finite.

By [28, p. 194, Theorem 13], complemented by [13, p. 166, Theorem 5.3 and Remark 5.4] and [3, p. 205, Proposition 7], Besov spaces of generalised smoothness can be obtained by interpolation of classic Besov spaces with a convenient function parameter. This result is formally stated in the next theorem.

Theorem 5.2 *Let σ be an admissible sequence and let $\phi \in \mathbf{B}_\sigma$. Let $0 < p \leq \infty$ and let $0 < q_0, q_1, q \leq \infty$. Let $s_0, s_1 \in \mathbb{R}$ satisfy $s_1 < \underline{s}(\sigma) \leq \overline{s}(\sigma) < s_0$ and let γ be given by*

$$\gamma(t) = \frac{t^{\frac{s_0}{s_0 - s_1}}}{\phi\big(t^{\frac{1}{s_0 - s_1}}\big)}, \quad t \in (0, \infty). \tag{5.2}$$

Then

$$\big(B^{(s_0)}_{p,q_0}(\mathbb{R}^n), B^{(s_1)}_{p,q_1}(\mathbb{R}^n)\big)_{\gamma,q} = B^{\sigma}_{p,q}(\mathbb{R}^n). \tag{5.3}$$

Remark 5.3 Usually the results related to interpolation with a function parameter are stated in terms of functions in class $\mathcal{B}$, i.e., given $\phi \in \mathcal{B}$,

$$\big(B^{(s_0)}_{p,q_0}(\mathbb{R}^n), B^{(s_1)}_{p,q_1}(\mathbb{R}^n)\big)_{\gamma,q} = B^{\beta}_{p,q}(\mathbb{R}^n), \quad s_1 < \underline{S}(\phi) \leq \overline{S}(\phi) < s_0, \tag{5.4}$$

where $\beta = (\phi(2^j))_j$, and for p, p_i, q, q_i, $i = 0, 1$, as in Theorem 5.2 and γ as in (5.2). We state the results in terms of admissible sequences, σ. This formulation follows immediately from the classic one in (5.4), if one considers a function in $\mathbf{B}_\sigma$ (we refer to Definition 3.11) and applies Proposition 3.13.

5.2 Pointwise multipliers for Besov spaces of generalised smoothness

This subsection is devoted to multiplication properties of Besov spaces of generalised smoothness on $\mathbb{R}^n$, first, and then on domains. If g is a function on $\mathbb{R}^n$ (respectively Ω) we study whether $f \mapsto gf$ (pointwise multiplication) yields a bounded linear mapping from Besov spaces of generalised smoothness on $\mathbb{R}^n$ (respectively Ω) into themselves. For the study of these properties in the case of Besov spaces with classical smoothness we refer to [34], in particular to Section 2.8 for spaces on $\mathbb{R}^n$ and to Section 3.3.2 in the case of spaces on domains. In Section 2.8 Triebel describes in particular what is meant by gf (pointwise multiplication) when both g and f denote tempered distributions. It is also remarked that for $g \in \mathcal{S}(\mathbb{R}^n)$ the meaning of gf according to this interpretation coincides with the usual one. In the present work we shall only consider $g \in \mathcal{S}(\mathbb{R}^n)$ and so gf has the usual meaning. We study pointwise multipliers for Besov spaces of generalised smoothness applying an important tool in the theory of function spaces: interpolation.

The next theorem is based on [34, p. 140, Theorem 2.8.2(i)]. In [34] the pointwise multipliers were considered in spaces $B^{(\rho)}_{\infty,\infty}$, for ρ satisfying convenient conditions. In this work we content ourselves with the smaller class, $\mathcal{S}(\mathbb{R}^n)$, and we recall that this space is even continuously embedded in the spaces $B^{(s)}_{p,q}(\mathbb{R}^n)$, for all $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.

Theorem 5.4 *Let σ be an admissible sequence, let $0 < p, q \leq \infty$ and let ρ satisfy*

$$\rho > \max\left\{\overline{s}(\sigma), \frac{n}{p} - \underline{s}(\sigma)\right\}. \tag{5.5}$$

Then any $g \in \mathcal{S}(\mathbb{R}^n)$ is a multiplier for $B^\sigma_{p,q}(\mathbb{R}^n)$, i.e., $f \mapsto gf$ yields a bounded linear mapping from $B^\sigma_{p,q}(\mathbb{R}^n)$ into itself and there is a positive constant c such that

$$\left\|gf|B^\sigma_{p,q}(\mathbb{R}^n)\right\| \leq c\left\|g|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)\right\| \cdot \left\|f|B^\sigma_{p,q}(\mathbb{R}^n)\right\|, \tag{5.6}$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$ and all $f \in B^\sigma_{p,q}(\mathbb{R}^n)$.

Remark 5.5 For a proof of Theorem 5.4 we refer to [25]. The idea is to apply both the corresponding result for classic Besov spaces (cf. [34, Section 2.8]) and Theorem 5.2.

A corresponding result for Besov spaces of generalised smoothness on domains follows immediately from Definition 4.8(i) and Theorem 5.4.

Corollary 5.6 *Let Ω be a domain in $\mathbb{R}^n$. Let σ be an admissible sequence, let $0 < p, q \leq \infty$ and let ρ satisfy* (5.5)*. Then any $g \in \mathcal{D}(\Omega)$ is a multiplier for $B^\sigma_{p,q}(\Omega)$, i.e., $f \mapsto gf$ yields a bounded linear mapping from $B^\sigma_{p,q}(\Omega)$ into itself and there is a positive constant c such that*

$$\left\|gf|B^\sigma_{p,q}(\Omega)\right\| \leq c\left\|g_\Omega\|B^{(\rho)}_{\infty,\infty}(\mathbb{R}^n)\right| \cdot \left\|f|B^\sigma_{p,q}(\Omega)\right\|, \tag{5.7}$$

for all $g \in \mathcal{D}(\Omega)$ and all $f \in B^\sigma_{p,q}(\mathbb{R}^n)$, where we are writing g_Ω to denote the extension by zero of the function g.

5.3 A universal extension operator

In this subsection we state the existence of a universal extension operator acting from Besov spaces of generalised smoothness on bounded Lipschitz domains to corresponding spaces on $\mathbb{R}^n$. We also present the main ideas of the proof. First we introduce some definitions and fix some notation.

Definition 5.7

(i) We say that a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a *special Lipschitz domain* if there is a Lipschitz function $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega = \{x = (x', x_n) \in \mathbb{R}^n \ : \ x_n > \omega(x')\}. \tag{5.8}$$

(ii) We say that a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a *bounded Lipschitz domain* if it is a bounded domain whose boundary $\partial\Omega$ can be covered by a finite number of open balls B_j, $j = 1, \ldots, J$, centered at points of $\partial\Omega$, such that

$$B_j \cap \Omega = B_j \cap \Omega_j, \quad j = 1, \ldots, J, \tag{5.9}$$

where Ω_j are rotations of suitable special Lipschitz domains.

(iii) A bounded Lipschitz domain in the real line $\mathbb{R}$ is the interior of a finite union of disjoint bounded closed intervals.

Notation 5.8 In what follows if we assume that Ω is a special Lipschitz domain in $\mathbb{R}^n$ then we will denote by ω the Lipschitz function referred in Definition 5.7(i), particularly (5.8). We will denote by A a positive constant such that

$$|\omega(x') - \omega(y')| \leq A|x' - y'|, \quad x', y' \in \mathbb{R}^{n-1}.$$

Moreover we shall write K_A to denote the cone

$$K_A := \{(x', x_n) \in \mathbb{R}^n \ : \ |x'| < A^{-1}x_n\},$$

which satisfies the property

$$x + K_A \subset \Omega, \quad x \in \Omega. \tag{5.10}$$

Notation 5.9

(i) In this section, for $f \in \mathcal{S}(\mathbb{R}^n)$ and $0 < t \leq 1$, if nothing additionally is said, f_t denotes the function defined by

$$f_t(x) = t^{-n} f(t^{-1}x), \quad x \in \mathbb{R}^n.$$

(ii) For any $g : \Omega \to \mathbb{R}$ we denote by g_Ω its extension by zero from Ω to all $\mathbb{R}^n$.

Definition 5.10 Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and let $f \in \mathcal{S}(\mathbb{R}^n)$. We define $f * T$ by

$$\langle f * T, \varphi\rangle := \langle T, f(-\cdot) * \varphi\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Remark 5.11 For this definition we refer, for example, to [22, Tome II, p. 18], where it is mentioned that $f * T$ belongs to $\mathcal{S}'(\mathbb{R}^n)$. Applying a collection of results which can be found in [22], namely the fact that $D(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$, a Fubini-type property for tensor products of distributions and a characterisation for the convolution of distributions with functions in $D(\mathbb{R}^n)$, one can prove that $f * T$ is a function, in the sense that it corresponds to a regular distribution given by the function defined by

$$\psi(x) := \langle T, f(x - \cdot)\rangle\,, \quad x \in \mathbb{R}^n.$$

Definition 5.12 Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ and let $r > 0$. Let $g \in \mathcal{S}'(\mathbb{R}^n)$. We define

$$(\varphi_0^* g)_r(x) := \sup_{y \in \mathbb{R}^n} \frac{|\varphi_0 * g(y)|}{(1 + |x - y|)^r}, \quad x \in \mathbb{R}^n,$$

and, for $j \in \mathbb{N}$,

$$(\varphi_{2^{-j}}^* g)_r(x) := \sup_{y \in \mathbb{R}^n} \frac{|\varphi_{2^{-j}} * g(y)|}{(1 + 2^j|x - y|)^r}, \quad x \in \mathbb{R}^n,$$

where in $\varphi_{2^{-j}}$ we are following Notation 5.9. Additionally, consider an admissible sequence σ and $0 < p, q \leq \infty$. We define

$$\big\|g|B^\sigma_{p,q}(\mathbb{R}^n)\big\|^{(1)}_{\varphi_0,\varphi,r} := \sigma_0 \big\|(\varphi_0^* g)_r | L_p(\mathbb{R}^n)\big\| + \left(\sum_{j=1}^\infty \sigma_j^q \big\|(\varphi_{2^{-j}}^* g)_r | L_p(\mathbb{R}^n)\big\|^q\right)^{1/q}$$

and

$$\big\|g|B^\sigma_{p,q}(\mathbb{R}^n)\big\|^{(2)}_{\varphi_0,\varphi} := \sigma_0 \|\varphi_0 * g | L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^\infty \sigma_j^q \|\varphi_{2^{-j}} * g | L_p(\mathbb{R}^n)\|^q\right)^{1/q}, \tag{5.11}$$

with the usual modification if $q = \infty$.

Definition 5.13 Let Ω be a special Lipschitz domain in $\mathbb{R}^n$. For $\gamma \in \mathcal{D}(-K_A)$ and $f \in \mathcal{D}'(\Omega)$ we define

$$\gamma * f(x) := \langle f, \gamma(x - \cdot)\rangle, \quad x \in \Omega.$$

Remark 5.14 By (5.10), Definition 5.13 makes sense. In what follows we need to guarantee that the convolution presented in Definition 5.13 is associative, namely

$$(\gamma * \eta) * f(x) = \gamma * (\eta * f)(x), \quad \gamma, \eta \in \mathcal{D}(-K_A), \quad x \in \Omega.$$

This is a consequence of the equality

$$\gamma * f(x) = \gamma * g(x), \quad \gamma \in \mathcal{D}(-K_A), \quad x \in \Omega,$$

which holds for all $f \in \mathcal{D}'(\Omega)$ such that there is $g \in \mathcal{S}'(\mathbb{R}^n)$ with $g|_\Omega = f$. Actually, this will be the case in what follows.

Theorem 5.15 *Let Ω be a special Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$.*

(i) *There exist four functions η_0, η, θ_0 and θ in $\mathcal{S}(\mathbb{R}^n)$ supported in $-K_A$ such that*

$$\int x^\alpha \eta(x)\, dx = \int x^\alpha \theta(x)\, dx = 0, \quad \textit{for all} \quad \alpha \in {\mathbb{N}_0}^n,$$

and

$$f = \theta_0 * \eta_0 * f + \sum_{j=1}^{\infty} \theta_{2^{-j}} * \eta_{2^{-j}} * f \quad \textit{in} \quad \mathcal{D}'(\Omega) \tag{5.12}$$

for all $f \in \mathcal{D}'(\Omega)$.

(ii) *Let f belong to the restriction of $\mathcal{S}'(\mathbb{R}^n)$ to Ω and*

$$\mathscr{E} f := \theta_0 * (\eta_0 * f)_\Omega + \sum_{j=1}^{\infty} \theta_{2^{-j}} * (\eta_{2^{-j}} * f)_\Omega. \tag{5.13}$$

Then, for any admissible sequence σ and $0 < p, q \leq \infty$, $\mathscr{E}$ is a linear bounded extension operator from $B^\sigma_{p,q}(\Omega)$ into the corresponding space on $\mathbb{R}^n$.

Remark 5.16 The result stated in (i) was proved by Rychkov in [32, pp. 253–255]. In this work Rychkov also proved (ii) for classic Besov spaces, i.e., using admissible sequences $\sigma = (s)$, $s \in \mathbb{R}$. In the proof for general Besov spaces one uses the construction obtained by Rychkov and applies characterisations of these function spaces proved by Farkas and Leopold in [17]. For a detailed description of the proof we refer to [25]. Here we just present the main ideas:

(1) By [17, Theorem 4.3.4 and Remark 4.3.5, p. 36, and Lemma 4.4.5, pp. 50–51] for all $r > n/p$, there exist two convenient functions $k_0, k \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\big\|g|B^\sigma_{p,q}(\mathbb{R}^n)\big\|^{(1)}_{k_0,k,r} \sim \big\|g|B^\sigma_{p,q}(\mathbb{R}^n)\big\| \sim \big\|g|B^\sigma_{p,q}(\mathbb{R}^n)\big\|^{(2)}_{k_0,k}, \quad g \in \mathcal{S}'(\mathbb{R}^n).$$

(2) One can prove that for the functions η, η_0 in Theorem 5.15, for conveniently chosen $k_0, k \in \mathcal{S}(\mathbb{R}^n)$ and for sufficiently large r,

$$\big\|g|B^\sigma_{p,q}(\mathbb{R}^n)\big\|^{(1)}_{\eta_0,\eta,r} \leq c\big\|g|B^\sigma_{p,q}(\mathbb{R}^n)\big\|^{(1)}_{k_0,k,r}, \quad g \in \mathcal{S}'(\mathbb{R}^n).$$

(3) For $f \in \mathcal{D}'(\Omega)$, let

$$(\eta_0^* f)^\Omega_r(x) := \sup_{y\in\Omega} \frac{|\eta_0 * f(y)|}{(1+|x-y|)^r}, \quad x \in \mathbb{R}^n,$$

$$(\eta_{2^{-j}}^* f)^\Omega_r(x) := \sup_{y\in\Omega} \frac{|\eta_{2^{-j}} * f(y)|}{(1+2^j|x-y|)^r}, \quad x \in \mathbb{R}^n,$$

for $j \in \mathbb{N}$, and

$$\big\|f|B^\sigma_{p,q}(\Omega)\big\|_{\eta_0,\eta,r} := \sigma_0\big\|(\eta_0^* f)^\Omega_r|L_p(\Omega)\big\| + \left(\sum_{j=1}^{\infty} \sigma_j^q \big\|(\eta_{2^{-j}}^* f)^\Omega_r|L_p(\Omega)\big\|^q\right)^{1/q}.$$

It can be proved that, if $f \in B^\sigma_{p,q}(\Omega)$ and $g \in B^\sigma_{p,q}(\mathbb{R}^n)$, with $g|_\Omega = f$, then

$$\big\|f|B^\sigma_{p,q}(\Omega)\big\|_{\eta_0,\eta,r} \lesssim \big\|g|B^\sigma_{p,q}(\mathbb{R}^n)\big\|^{(1)}_{\eta_0,\eta,r}.$$

(4) Applying [9, p. 224. Lemma 2.1(ii)], [17, p. 36, Theorem 4.3.4 and Remark 4.3.5] and the properties of special Lipschitz domains, one proves that the expression in (5.13) converges in $\mathcal{S}'(\mathbb{R}^n)$ and, moreover,

$$\big\|\mathscr{E} f|B^\sigma_{p,q}(\mathbb{R}^n)\big\|^{(2)}_{k_0,k,r} \lesssim \big\|f|B^\sigma_{p,q}(\Omega)\big\|_{\eta_0,\eta,r}.$$

(5) By Steps 1–4, one concludes that

$$\big\|\mathscr{E}f|B^{\sigma}_{p,q}(\mathbb{R}^n)\big\| \lesssim \big\|f|B^{\sigma}_{p,q}(\Omega)\big\|.$$

By the conditions on the supports of the functions θ_0 and θ one concludes that $(\mathscr{E}f)|_{\Omega} = f$.

Theorem 5.17 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. There is a universal extension operator, i.e., there is an extension operator $\mathscr{E}$ such that, for all admissible sequences σ and $0 < p, q \leq \infty$,*

$$\mathscr{E} : B^{\sigma}_{p,q}(\Omega) \to B^{\sigma}_{p,q}(\mathbb{R}^n).$$

Remark 5.18 A complete proof can be found in [25]. The extension operator is obtained as follows. Let J, B_j and Ω_j, $j = 1, \ldots, J$ be as in Definition 5.7 (ii). For simplicity assume that the Ω_j's are special Lipschitz domains (without the need to consider rotations). By Theorem 5.15, for every $j = 1, \ldots, J$, there is a universal extension operator, which will be denoted by $\mathscr{E}_j$, acting from Besov spaces on Ω_j into the corresponding Besov spaces on $\mathbb{R}^n$.

One considers a partition of the unity on Ω constituted by $\phi \in \mathcal{D}(\Omega)$ and $\phi_j \in \mathcal{D}(B_j)$, $j = 1, \ldots, J$. Let $\delta > 0$ be such that the closure of the neighborhood $(\operatorname{supp}\phi)_\delta$, of radius δ, of $\operatorname{supp}\phi$ is a subset of Ω. Analogously, for every $j = 1, \ldots, J$ let $\delta_j > 0$ be such that the closure of the neighborhood $(\operatorname{supp}\phi_j)_{\delta_j}$ of $\operatorname{supp}\phi_j$ is a subset of B_j. Fix auxiliary smooth functions ψ and ψ_j, $j, = 1, \ldots, J$, which satisfy

$$\psi(x) = 1 \quad \text{if} \quad x \in \operatorname{supp}\phi \quad \text{and} \quad \psi_j(x) = 0 \quad \text{if} \quad x \notin (\operatorname{supp}\phi)_\delta \tag{5.14}$$

and

$$\psi_j(x) = 1 \quad \text{if} \quad x \in \operatorname{supp}\phi_j \quad \text{and} \quad \psi_j(x) = 0 \quad \text{if} \quad x \notin (\operatorname{supp}\phi_j)_{\delta_j}. \tag{5.15}$$

Let $f \in B^{\sigma}_{p,q}(\Omega)$. For $j = 1, \ldots, J$, one considers the auxiliary distributions:

$$\langle f^j, \varphi\rangle := \langle f, \phi_j\varphi\rangle, \quad \varphi \in \mathcal{D}(\Omega_j). \tag{5.16}$$

Applying Theorem 5.4 one concludes that $f^j \in B^{\sigma}_{p,q}(\Omega_j)$ and so one defines

$$\mathscr{E}f := (f\phi)_\Omega + \sum_{j=1}^{J} \psi_j(\mathscr{E}_j f^j), \tag{5.17}$$

where $(f\phi)_\Omega$ is given by

$$\langle (f\phi)_\Omega, \varphi\rangle = \langle f\phi, (\varphi\psi)|_\Omega\rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

By Definition 4.8(i) and applying Theorems 5.15 and 5.4, one finally shows that

$$\mathscr{E}f \in B^{\sigma}_{p,q}(\mathbb{R}^n), \quad \big\|\mathscr{E}f|B^{\sigma}_{p,q}(\mathbb{R}^n)\big\| \lesssim \big\|f|B^{\sigma}_{p,q}(\Omega)\big\| \quad \text{and} \quad (\mathscr{E}f)|_\Omega = f.$$

5.4 The Laplacian

In this subsection Ω denotes a bounded C^∞ domain in $\mathbb{R}^n$. For the definition of bounded C^∞ domain we refer to [34, 3.2.1, p. 191], for example.

As usual

$$-\Delta := -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is the Laplacian in Ω.

We use the notation *Dirichlet Laplacian* always with the understanding that the vanishing boundary data at $\partial\Omega$ are incorporated in the definition of the domain of $-\Delta$ in the function spaces considered, as we shall see in what follows.

In this subsection the Dirichlet Laplacian acting in Besov spaces of generalised smoothness on bounded smooth domains is studied. It is well-known that if Ω is a bounded C^∞ domain in $\mathbb{R}^n$ and

$$1 \le p \le \infty, \quad 1 \le q \le \infty \quad \text{and} \quad s > \frac{1}{p}, \tag{5.18}$$

then $-\Delta$ maps

$$\{g \in B^{(s)}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega}\, g = 0\} \quad \text{isomorphically onto} \quad B^{(s-2)}_{p,q}(\Omega). \tag{5.19}$$

This result can be found in [39, p. 255] where the following references are given: [37, Remark 1, 5.7.1], [34, 4.3.3, 4.3.4] and [31, 3.5.2, p. 130]. In this subsection this result is extended to spaces of generalised smoothness applying interpolation with function parameter. For this purpose it is convenient to prove that (5.3) holds if one considers Besov spaces on some class of domains, instead of $\mathbb{R}^n$. For that the extension operator referred in Theorem 5.17 will be applied. In [41, p. 69, Theorem 1.110] this extension operator was used to get interpolation results for function spaces on Lipschitz domains from corresponding results on $\mathbb{R}^n$, for the real interpolation method for quasi-Banach spaces and the classical complex interpolation method for Banach spaces, always in the context of classic Besov spaces. For real interpolation with a function parameter the result can be obtained by the same method.

Proposition 5.19 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let σ be an admissible sequence, let $0 < p \le \infty$ and let $0 < q_0, q_1, q \le \infty$. Let $s_0, s_1 \in \mathbb{R}$ satisfy $s_1 < \underline{s}(\sigma) \le \overline{s}(\sigma) < s_0$ and let γ be given by (5.2). Then*

$$\left(B^{(s_0)}_{p,q_0}(\Omega), B^{(s_1)}_{p,q_1}(\Omega)\right)_{\gamma,q} = B^\sigma_{p,q}(\Omega).$$

Proof. Let us temporarily use the following notation:

$$A_i = B^{(s_i)}_{p,q_i}(\mathbb{R}^n) \quad \text{and} \quad B_i = B^{(s_i)}_{p,q_i}(\Omega), \quad i = 0, 1.$$

Then

$$A_0 \subset A_1 \subset \mathcal{S}'(\mathbb{R}^n) \quad \text{and} \quad B_0 \subset B_1 \subset \mathcal{D}'(\Omega).$$

Let us denote by $\mathscr{R}$ the restriction to $\mathcal{D}'(\Omega)$ and by $\mathscr{E}$ the extension operator referred in Theorem 5.17. As $\mathscr{R}\mathscr{E}$ is the identity operator in B_i, id_{B_i}, then $\mathscr{R}$ is a retraction in $L(A_i, B_i)$ and $\mathscr{E}$ is a coretraction belonging to $\mathscr{R}$. We say that B_i is a retract of A_i.

Applying Definition 3.9 and Proposition 3.13, one can verify that, under the conditions considered, $0 < \underline{S}(\gamma) \le \overline{S}(\gamma) < 1$. So one can apply Theorem 2.2.6 and Remark 2.2.7 in [2, pp. 20–21] to conclude that $(B_0, B_1)_{\gamma,q}$ is a retract of $(A_0, A_1)_{\gamma,q}$ (with “the same mappings” $\mathscr{R}$ and $\mathscr{E}$) and

$$\|f|(B_0, B_1)_{\gamma,q}\| \sim \|\mathscr{E}f|(A_0, A_1)_{\gamma,q}\|, \quad f \in (B_0, B_1)_{\gamma,q}.$$

Let $f \in (B_0, B_1)_{\gamma,q}$. Then, by the above mentioned results,

$$\mathscr{E}f \in (A_0, A_1)_{\gamma,q} = B^\sigma_{p,q}(\mathbb{R}^n) \quad \text{and} \quad \left\|\mathscr{E}f|B^\sigma_{p,q}(\mathbb{R}^n)\right\| \sim \|f|(B_0, B_1)_{\gamma,q}\|.$$

Using the fact that $(B_0, B_1)_{\gamma,q}$ is a retract of $(A_0, A_1)_{\gamma,q}$ and Definition 4.8, we conclude that

$$f = \mathscr{R}\mathscr{E}f \in B^\sigma_{p,q}(\Omega) \quad \text{and} \quad \left\|f|B^\sigma_{p,q}(\Omega)\right\| \lesssim \|f|(B_0, B_1)_{\gamma,q}\|.$$

We prove now the reverse inclusion. Let $f \in B^\sigma_{p,q}(\Omega)$. Then, by Theorem 5.17,

$$\mathscr{E}f \in B^\sigma_{p,q}(\mathbb{R}^n) \quad \text{and} \quad \left\|\mathscr{E}f|B^\sigma_{p,q}(\mathbb{R}^n)\right\| \lesssim \left\|f|B^\sigma_{p,q}(\Omega)\right\|. \tag{5.20}$$

Again by [2, p. 20, Theorem 2.2.6] one concludes that $f = \mathscr{R}\mathscr{E}f \in (B_0, B_1)_{\gamma,q}$ and

$$\|f|(B_0, B_1)_{\gamma,q}\| \lesssim \|\mathscr{E}f|(A_0, A_1)_{\gamma,q}\|$$

and so, by (5.20), we conclude. □

Proposition 5.20 *Let Ω be a bounded C^∞ domain in let $\mathbb{R}^n$, let $0 < p \le \infty$ and let $0 < q \le \infty$. Let σ be an admissible sequence and let $s_0, s_1 \in \mathbb{R}$ be such that*

$$\frac{1}{p} + \max\left(0, (n-1)\left(\frac{1}{p} - 1\right)\right) < s_1 < \underline{s}(\sigma) \le \overline{s}(\sigma) < s_0$$

and let γ be given by (5.2). *Then*

$$\begin{aligned}&\left(\left\{f \in B^{(s_0)}_{p,q_0}(\Omega) : \mathrm{tr}_{\partial\Omega}\, f = 0\right\}, \left\{f \in B^{(s_1)}_{p,q_1}(\Omega) : \mathrm{tr}_{\partial\Omega}\, f = 0\right\}\right)_{\gamma,q}\\ &\quad = \left\{f \in B^{\sigma}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega}\, f = 0\right\}.\end{aligned} \tag{5.21}$$

Proof. Let us temporarily use the following notation

$$\begin{aligned} A_i &= B^{(s_i)}_{p,q_i}(\Omega), \quad B_i = \{f \in A_i : \mathrm{tr}_{\partial\Omega}\, f = 0\}, \quad i = 0,1, \quad \text{and}\\ B &= \{f \in A_0 + A_1 : \mathrm{tr}_{\partial\Omega}\, f = 0\}.\end{aligned}$$

To prove this result we will use the fact that B is the range of a projection in $A_0 + A_1$. We recall that we say that Q is a projection in a subspace, A, of a quasi-Banach space if it is a linear operator acting in A such that $Q^2 = Q$ in A. As $s_1 < s_0$ one can prove, applying Proposition 4.5, that $A_0 + A_1 = A_1$ (equivalent quasi-norms). Therefore

$$B = \{f \in A_1 : \mathrm{tr}_{\partial\Omega} = 0\}.$$

By [34, p. 200, Theorem 3.3.3] the operator $\mathcal{R}f := \mathrm{tr}_{\partial\Omega}\, f$ is a retraction from $B^{(s_1)}_{p,q}(\Omega)$ onto $B^{s_1-\frac{1}{p}}_{p,q}(\partial\Omega)$. For the definition of these spaces on $\partial\Omega$, which will not be applied directly in this proof, we refer to [34, p. 192, 3.2.2]. Then $\mathcal{R}$ is a continuous linear map from $B^{(s_1)}_{p,q}(\Omega)$ onto $B^{s_1-\frac{1}{p}}_{p,q}(\partial\Omega)$ and there exists a continuous linear map, say $\mathcal{G}$, from $B^{s_1-\frac{1}{p}}_{p,q}(\partial\Omega)$ into $B^{(s_1)}_{p,q}(\Omega)$ such that $\mathcal{R}\mathcal{G} = E$ is the identity in $B^{s_1-\frac{1}{p}}_{p,q}(\partial\Omega)$. Let

$$P = \mathcal{G}\mathcal{R} \quad \text{and} \quad Q = I - P,$$

where I stands for the identity operator in $B^{(s_1)}_{p,q}(\Omega)$. Both P and Q are projections in $B^{(s_1)}_{p,q}(\Omega)$, i.e., $P^2 = P$ and $Q^2 = Q$. Moreover

$$QB^{(s_1)}_{p,q}(\Omega) = \{f \in B^{(s_1)}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega}\, f = 0\} = B. \tag{5.22}$$

So let us prove (5.21). The inclusion "$\subseteq$" is immediate since $B_i \subset A_i$, $i = 0,1$. Let us prove the reverse inclusion. Let $f \in B^{\sigma}_{p,q}(\Omega)$ be such that $\mathrm{tr}_{\partial\Omega}\, f = 0$. By Proposition 5.19, $f \in (A_0, A_1)_{\gamma,q}$ and $\|f|(A_0, A_1)_{\gamma,q}\| \lesssim \left\|f|B^{\sigma}_{p,q}(\Omega)\right\|$. Using the equality $A_0 + A_1 = A_1$ one concludes that $f \in B_0 + B_1$. Let us prove that

$$\|f|(B_0, B_1)_{\gamma,q}\| \lesssim \|f|(A_0, A_1)_{\gamma,q}\|,$$

proving that there is $c > 0$ such that, for all $t > 0$,

$$K(t, f, A_0, A_1) \le cK(t, f, B_0, B_1), \tag{5.23}$$

where K is as in (5.1). Let $a_i \in A_i$ be such that $f = a_0 + a_1$. As $f \in B$ then it follows from (5.22) that $Qf = f$. Let us consider $b_i = Qa_i$, $i = 0,1$. Hence

$$b_0 + b_1 = Q(a_0 + a_1) = Qf = f \quad \text{and} \quad \|b_i|B_i\| \lesssim \|a_i|A_i\|, \quad i = 0,1,$$

which implies that (5.23) is satisfied, concluding the proof. □

Theorem 5.21 *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, let*

$$1 \le p \le \infty, \quad 1 \le q \le \infty$$

and let σ be an admissible sequence such that

$$\underline{s}(\sigma) > \frac{1}{p}.$$

Then $-\Delta$ maps

$$\left\{g \in B^{\sigma}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega}\, g = 0\right\} \quad \textit{isomorphically onto} \quad B^{\sigma(-2)}_{p,q}(\Omega),$$

where, according to the notation introduced in 3.1–3.3, $\sigma(-2) = (\sigma_j 2^{-2j})_{j\in\mathbb{N}_0}$.

Proof. Let $\phi \in \mathbf{B}_\sigma$. Let $s_0, s_1 \in \mathbb{R}$ be such that

$$\frac{1}{p} < s_1 < \underline{s}(\sigma) \le \overline{s}(\sigma) < s_0$$

and let γ be as in (5.2). Then, by Proposition 5.20, (5.21) holds. The function given by $\psi(t) = t^{-2}\phi(t)$ belongs to $\mathbf{B}_{\sigma(-2)}$. Moreover,

$$s_1 - 2 < \underline{s}(\sigma) - 2 = \underline{s}(\sigma(-2)) \le \overline{s}(\sigma(-2)) = \overline{s}(\sigma) - 2 < s_0 - 2$$

and for all $t > 0$

$$\gamma(t) = \frac{t^{\frac{s_0-2}{(s_0-2)-(s_1-2)}}}{\psi t^{\left(\frac{1}{(s_0-2)-(s_1-2)}\right)}}.$$

So, by Proposition 5.19,

$$\left(B^{(s_0-2)}_{p,q_0}(\Omega), B^{(s_1-2)}_{p,q_1}(\Omega)\right)_{\gamma,q} = B^{\sigma(-2)}_{p,q}(\Omega). \tag{5.24}$$

By (5.18)–(5.19), for $i = 0, 1$, $-\Delta$ maps

$$\left\{g \in B^{(s_i)}_{p,q}(\Omega) : \mathrm{tr}_{\partial\Omega}\, g = 0\right\} \quad \text{isomorphically onto} \quad B^{(s_i-2)}_{p,q}(\Omega), \tag{5.25}$$

therefore, using the interpolation property, the result follows from (5.21), (5.24) and (5.25). □

6 Fractal *h*-sets and traces

In this section h-sets are defined and some of their properties are presented. Besov spaces of generalised smoothness on h-sets and some operators connected with them are given.

6.1 *h*-sets

In the designation h-*set*, the h denotes a function, which shall be in a convenient class of functions.

Definition 6.1 Let $\mathbb{H}$ denote the class of all continuous monotone increasing functions $h : (0,\infty) \to (0,\infty)$ such that $h(0^+) = 0$. We refer to $\mathbb{H}$ as the set of all *gauge functions*.

Notation 6.2 In what follows, for $h \in \mathbb{H}$, we denote by $\boldsymbol{h}$ the sequence

$$\boldsymbol{h} := (h(2^{-j}))_{j\in\mathbb{N}_0}. \tag{6.1}$$

Definition 6.3 Let $h \in \mathbb{H}$ and Γ be a nonempty compact set of $\mathbb{R}^n$. We say that Γ is an *h-set* if there exists a finite Radon measure μ such that

$$\operatorname{supp} \mu = \Gamma$$

and

$$\mu(B(\gamma, r)) \sim h(r), \quad 0 < r \leq 1, \quad \gamma \in \Gamma.$$

Then we say that h is a *measure function* (in $\mathbb{R}^n$) and that μ is an *h-measure* (related to Γ).

Remark 6.4 The h-measures are also designated by *isotropic measures* (cf. [41, p. 95]).

If the function h is given by

$$h(r) = r^d \psi(r), \quad 0 < r \leq 1,$$

where $0 < d \leq n$ and $\psi : (0,1] \to \mathbb{R}^+$ is a monotone function such that

$$\psi(2^{-j}) \sim \psi(2^{-2j}), \quad \text{for all} \;\; j \in \mathbb{N},$$

then we say that Γ is a *(d, ψ)-set*. If, additionally, $\psi \sim 1$ then we say that Γ is a *d-set*. Therefore the class of h-sets is a generalisation of the class of (d, ψ)-sets, which is itself a generalisation of d-sets. There are many authors studying these classes of fractal sets, both in fractal geometry and in the theory of function spaces. In the case of d-sets we refer to [21], [26] and [38] for example. For (d, ψ)-sets we refer to [15], [29] and [30]. As far as h-sets are concerned we refer to [7], [8], [20] and [23].

Some well-known self-similar fractals are examples of d-sets, namely the Cantor set in $\mathbb{R}^1$ is a d-set for $d = \log 2/ \log 3$ and the von Koch curve in $\mathbb{R}^2$ is a d-set for $d = \log 4/ \log 3$.

Shifting from d-sets and (d, ψ)-sets to h-sets it is often convenient to find appropriate numbers to play the role of the number d. In our results we usually consider the lower and upper Boyd indices of the sequence $\boldsymbol{h}$ given in (6.1). Next we present some other indices, considered namely in [7].

Definition 6.5 Let $h \in \mathbb{H}$. We define the *upper* and the *lower orders of* h, respectively, by

$$\overline{\omega}(h) = \limsup_{r \to 0} \frac{\log h(r)}{\log r} \quad \text{and} \quad \underline{\omega}(h) = \liminf_{r \to 0} \frac{\log h(r)}{\log r}.$$

The results in the next proposition were proved in [5].

Proposition 6.6 *Let $h \in \mathbb{H}$ and $n \in \mathbb{N}$. Consider an h-set Γ and μ an h-measure* (*related to* Γ).
Then

(i) *The measure μ is equivalent to $\mathcal{H}^h_\Gamma$, where $\mathcal{H}^h_\Gamma$ is the restriction of the Hausdorff measure $\mathcal{H}^h$ in $\mathbb{R}^n$ to Γ.*

(ii) *The Hausdorff dimension of μ coincides with the lower order of h, $\underline{\omega}(h)$.*

Remark 6.7 The property in (i) was proved in [5, p. 22, Theorem 1.7.6] and the property (ii) in [5, p. 29, Theorem 1.8.2].

We now present a definition about a geometric property of sets. It is useful when working with traces on Besov spaces on $\mathbb{R}^n$.

Definition 6.8 A nonempty Borel set Γ satisfies the *ball condition* (or *porosity condition*) if there exists a number $0 < \eta < 1$ with the following property:

for any ball $B(\gamma, r)$ with $\gamma \in \Gamma$ and $0 < r \leq 1$ there is a ball $B(x, \eta r)$ centred at $x \in \mathbb{R}^n$ such that

$$B(x, \eta\, r) \subset B(\gamma, r) \quad \text{and} \quad B(x, \eta r) \cap \overline{\Gamma} = \emptyset.$$

The next proposition can be found in [10] and follows from [39, pp. 139–140, Proposition 9.18] and the properties of Boyd indices, namely (3.3).

Proposition 6.9 *Let $\Gamma \subset \mathbb{R}^n$ be an h-set and let $\boldsymbol{h}$ be the sequence given in* (6.1). *Then Γ satisfies the ball condition if, and only if,*

$$\underline{s}(\boldsymbol{h}) > -n. \tag{6.2}$$

Remark 6.10 If $h(r) = r^d$, $r > 0$, then (6.2) is equivalent to $d < n$.

6.2 Traces and Besov spaces on *h*-sets

In this section we define the operator trace and we apply it to define Besov spaces of generalised smoothness on h-sets.

Let μ be an h-measure in $\mathbb{R}^n$ according to Definition 6.3. For $0 < p \leq \infty$, $L_p(\Gamma, \mu)$ (or simply $L_p(\Gamma)$) denotes the usual complex quasi-Banach space (Banach if $p \geq 1$) with respect to the related measure μ, quasi-normed by

$$\|f|L_p(\Gamma,\mu)\| := \left(\int_\Gamma |f(\gamma)|^p \, d\mu(\gamma) \right)^{1/p},$$

with the usual modification if $p = \infty$.

Definition 6.11 Let Γ be an h-set and let us fix an admissible sequence σ. Let $0 < p, q < \infty$. Suppose that there exists a positive constant c such that

$$\|\varphi|_\Gamma \mid L_p(\Gamma)\| \leq c \left\| \varphi \mid B^\sigma_{p,q}(\mathbb{R}^n) \right\|, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{6.3}$$

Let us consider $f \in B^\sigma_{p,q}(\mathbb{R}^n)$. As $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^\sigma_{p,q}(\mathbb{R}^n)$, there is a sequence $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\varphi_j \to f, \quad \text{as} \quad j \to \infty, \quad \text{in} \quad B^\sigma_{p,q}(\mathbb{R}^n).$$

By (6.3) the sequence $\{\varphi_j|_\Gamma\}_{j\in\mathbb{N}_0}$ converges in $L_p(\Gamma)$ to an element which we call *trace of* f and we denote by $\operatorname{tr}_\Gamma f$.

The results stated in the next proposition were proved in [8].

Proposition 6.12 *Let $h \in \mathbb{H}$ and let Γ be an h-set in $\mathbb{R}^n$. Let $0 < p < \infty$. Recall that*

$$\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}} = \left(h(2^{-j})^{1/p} 2^{nj/p} \right)_j.$$

(i) *Let $0 < q \leq \min(1,p)$. Then there exists*

$$\operatorname{tr}_\Gamma : B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,q}(\mathbb{R}^n) \to L_p(\Gamma).$$

If, additionally, Γ satisfies the ball condition, then

$$\operatorname{tr}_\Gamma B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,q}(\mathbb{R}^n) = L_p(\Gamma).$$

(ii) *Let $0 < q \leq \infty$ and let σ be an admissible sequence with $\underline{s}(\sigma) > 0$. Then there exists*

$$\operatorname{tr}_\Gamma : B^{\sigma\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,q}(\mathbb{R}^n) \to L_p(\Gamma),$$

where $\sigma\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}} = (\sigma_j h(2^{-j})^{1/p} 2^{nj/p})_j$.

Remark 6.13 Assertion (i) was stated and proved in [8, pp. 47–49, Theorem 5.9] where it was proved that (6.3) holds for the spaces in (i). Assertion (ii) follows from (i) and from the following inclusion

$$B^{\sigma\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,\min(1,p)}(\mathbb{R}^n),$$

which follows from Proposition 4.5. Proposition 6.12 can be extended to $p = \infty$, because if $p = \infty$ and $\underline{s}(\sigma) > 0$ then, by Proposition 4.5 and by [31, p. 32, Theorem 1],

$$B^\sigma_{\infty,q}(\mathbb{R}^n) \hookrightarrow B^{(0)}_{\infty,1}(\mathbb{R}^n) \hookrightarrow \mathcal{C}(\mathbb{R}^n),$$

where $\mathcal{C}(\mathbb{R}^n)$ is the space of all bounded and uniformly continuous functions in $\mathbb{R}^n$, normed in the usual way, the trace of $f \in B^\sigma_{\infty,q}(\mathbb{R}^n)$ being then defined as the pointwise restriction.

6.3 An identification operator

Next we present an operator which identifies an element of $L_p(\Gamma)$, where Γ is an h-set in $\mathbb{R}^n$ and $1 \leq p \leq \infty$, with a tempered distribution. We collect some results on this operator, proved by Bricchi, and we relate it with the trace operator treated previously.

Definition 6.14 Let $1 \le p \le \infty$ and let Γ be an h-set in $\mathbb{R}^n$. We define

$$\mathrm{id}^\Gamma : L_p(\Gamma) \to \mathcal{S}'(\mathbb{R}^n) \tag{6.4}$$

by

$$\langle \mathrm{id}^\Gamma f, \varphi\rangle := \int_\Gamma f(\gamma)(\varphi|_\Gamma)(\gamma)d\mu(\gamma), \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{6.5}$$

Definition 6.15 Let Γ be a compact set in $\mathbb{R}^n$, let σ be an admissible sequence and let $0 < p, q \le \infty$. Then we define

$$B^{\sigma,\Gamma}_{p,q}(\mathbb{R}^n) := \big\{ f \in B^\sigma_{p,q}(\mathbb{R}^n) : \langle f, \varphi\rangle = 0 \text{ if } \varphi \in \mathcal{S}(\mathbb{R}^n),\ \varphi|_\Gamma = 0 \big\}.$$

The following result was proved in [8, p. 44, Theorem 5.4].

Proposition 6.16 *Let Γ be an h-set in $\mathbb{R}^n$, let $1 \le p \le \infty$ and let p' denote the conjugate exponent of p. Recall that $\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}} = \big(h(2^{-j})^{-\frac{1}{p'}} 2^{-\frac{nj}{p'}}\big)_j$. Then*

$$\mathrm{id}^\Gamma : L_p(\Gamma) \to B^{\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}},\Gamma}_{p,\infty}(\mathbb{R}^n). \tag{6.6}$$

If, additionally, $1 < p \le \infty$ and Γ satisfies the ball condition, then

$$\mathrm{id}^\Gamma L_p(\Gamma) = B^{\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}},\Gamma}_{p,\infty}(\mathbb{R}^n).$$

The next result follows immediately from Propositions 6.12 and 6.16.

Proposition 6.17 *Let Γ be an h-set in $\mathbb{R}^n$, let $1 \le p \le \infty$ and let p' denote the conjugate exponent of p. Let*

$$\mathrm{tr}^\Gamma := \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma\,.$$

Then

$$\mathrm{tr}^\Gamma : B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,1}(\mathbb{R}^n) \to B^{\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}_{p,\infty}(\mathbb{R}^n),$$

where $\boldsymbol{h}$ is as in (6.1).

Let $g \in (L_p(\Gamma))'$ and let $1 \le p < \infty$. There is a uniquely determined $g^* \in L_{p'}(\Gamma)$ such that

$$\langle g, f\rangle = \int_\Gamma f(\gamma) g^*(\gamma)\, d\mu(\gamma), \quad \text{for all} \quad f \in L_p(\Gamma), \tag{6.7}$$

where

$$\|g|(L_p(\Gamma))'\| = \|g^*|L_{p'}(\Gamma)\|. \tag{6.8}$$

This will be applied in the next proposition.

Proposition 6.18 *Let Γ be an h-set in $\mathbb{R}^n$, let $1 \le p < \infty$, $0 < q < \infty$ and let σ be an admissible sequence such that*

$$\underline{s}(\sigma) > \frac{1}{p}(n + \overline{s}(\boldsymbol{h})). \tag{6.9}$$

Then

$$\mathrm{tr}'_\Gamma = \mathrm{id}^\Gamma, \tag{6.10}$$

where id^Γ is considered acting in $L_{p'}(\Gamma)$, tr'_Γ denotes the dual operator of

$$\mathrm{tr}_\Gamma : B^\sigma_{p,q}(\mathbb{R}^n) \to L_p(\Gamma) \tag{6.11}$$

and where by (6.10) *we mean*

$$\langle \mathrm{tr}'_\Gamma g, v\rangle = \langle \mathrm{id}^\Gamma g^*, v\rangle, \quad v \in B^\sigma_{p,q}(\mathbb{R}^n) \tag{6.12}$$

for $g \in (L_p(\Gamma))'$ and $g^* \in L_{p'}(\Gamma)$ according to (6.7)–(6.8), Proof. It follows from (6.9), (4.2) and Proposition 6.12 that (6.11) makes sense. Therefore for the dual operator, tr'_Γ, we have

$$\mathrm{tr}'_\Gamma : (L_p(\Gamma))' \to (B^\sigma_{p,q}(\mathbb{R}^n))'.$$

By the definition of dual operator and (6.7), for all $v \in B^\sigma_{p,q}(\mathbb{R}^n)$,

$$\langle \mathrm{tr}'_\Gamma\, g, v\rangle = \langle g, \mathrm{tr}_\Gamma\, v\rangle = \int_\Gamma (\mathrm{tr}_\Gamma\, v)(\gamma) g^*(\gamma)\, d\mu(\gamma).$$

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\langle \mathrm{tr}'_\Gamma\, g, \varphi\rangle = \int_\Gamma (\varphi_{|\Gamma})(\gamma) g^*(\gamma)\, d\mu(\gamma) = \langle \mathrm{id}^\Gamma\, g^*, \varphi\rangle.$$

Let $v \in B^\sigma_{p,q}(\mathbb{R}^n)$ and $(\varphi_j)_{j\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ be such that

$$\|v - \varphi_j | B^\sigma_{p,q}(\mathbb{R}^n)\| \to 0 \quad \text{when} \quad j \to \infty.$$

Then

$$|\langle \mathrm{tr}'_\Gamma\, g, v - \varphi_j\rangle| = |\langle g, \mathrm{tr}_\Gamma\, v - \varphi_j|_\Gamma\rangle| \le \|g|(L_p(\Gamma))'\| \cdot \|\,\mathrm{tr}_\Gamma\, v - \varphi_j|_\Gamma | L_p(\Gamma)\|. \tag{6.13}$$

As $\|g|(L_p(\Gamma))'\|$ is finite, it follows from the definition of trace that the expression in (6.13) converges to 0 when $j \to \infty$. Hence

$$\langle \mathrm{tr}'_\Gamma\, g, v\rangle = \lim_{j\to\infty} \langle \mathrm{tr}'_\Gamma\, g, \varphi_j\rangle = \lim_{j\to\infty} \langle \mathrm{id}^\Gamma\, g^*, \varphi_j\rangle = \langle \mathrm{id}^\Gamma\, g^*, v\rangle. \tag{6.14}$$

We justify the last equality in (6.14): as $g^* \in L_{p'}(\Gamma)$, then, by Proposition 6.16 and Theorem 4.11,

$$\mathrm{id}^\Gamma\, g^* \in B^{\boldsymbol{h}^{-\frac{1}{p}}(n)^{-\frac{1}{p}}}_{p',\infty}(\mathbb{R}^n) = \left(B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,1}(\mathbb{R}^n)\right)' \subset \big(B^\sigma_{p,q}(\mathbb{R}^n)\big)'.$$

Hence

$$|\langle \mathrm{id}^\Gamma\, g^*, v - \varphi_j\rangle| \le \big\|\,\mathrm{id}^\Gamma\, g^* \big| \big(B^\sigma_{p,q}(\mathbb{R}^n)\big)'\big\| \cdot \big\|v - \varphi_j \big| B^\sigma_{p,q}(\mathbb{R}^n)\big\| \to 0 \quad \text{when} \quad j \to \infty,$$

concluding the proof. ☐

Remark 6.19 Let $\Gamma \subset \mathbb{R}^n$ be an h-set satisfying the ball condition. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ such that $\Gamma \subset \Omega$. Consider $0 < p, q \le \infty$ and an admissible sequence σ such that $\underline{s}(\sigma) > 0$. Let $f \in B^\sigma_{p,q}(\Omega)$ and $g, u \in B^\sigma_{p,q}(\mathbb{R}^n)$ be such that $g|_\Omega = u|_\Omega = f$. If for all such g and u one has $\mathrm{tr}_\Gamma\, g = \mathrm{tr}_\Gamma\, u$, then one can consider $\mathrm{tr}_\Gamma\, f := \mathrm{tr}_\Gamma\, g$, i.e., we can consider the operator trace acting in Besov spaces on Ω. Let $(\varphi_j)_j, (\psi_j)_j \subset \mathcal{S}(\mathbb{R}^n)$ converge in $B^\sigma_{p,q}(\mathbb{R}^n)$ to g and u, respectively. Let χ be a smooth function such that $\chi(x) = 1$ in a neighborhood of Γ and $\operatorname{supp}\chi \subset \Omega$. By Theorem 5.4, χg and χu are also elements of $B^\sigma_{p,q}(\mathbb{R}^n)$ and the sequences $(\chi\varphi_j)_j$ and $(\chi\psi_j)_j$ converge in $B^\sigma_{p,q}(\mathbb{R}^n)$ to χg and χu, respectively. By Definition 6.11,

$$tr_\Gamma(\chi g) = \mathrm{tr}_\Gamma\, g \quad \text{and} \quad tr_\Gamma(\chi u) = \mathrm{tr}_\Gamma\, u.$$

So one just needs to prove that $\chi g = \chi u$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\langle \chi g, \varphi\rangle = \langle g, \chi\varphi\rangle = \langle f, \chi\varphi\rangle = \langle u, \chi\varphi\rangle = \langle \chi u, \varphi\rangle.$$

So

$$\mathrm{tr}_\Gamma\, B^\sigma_{p,q}(\Omega) := \mathrm{tr}_\Gamma\, B^\sigma_{p,q}(\mathbb{R}^n)$$

and all the results stated for the trace operator acting in Besov spaces on $\mathbb{R}^n$ remain valid if one replaces $\mathbb{R}^n$ by Ω. Analogously for the results on the operators id^Γ and tr^Γ.

7 The operator *B*

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let Γ be an h-set such that $\Gamma \subset \Omega$. We define

$$B := (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma, \tag{7.1}$$

where $(-\Delta)^{-1}$ denotes the inverse of the Dirichlet Laplacian presented in Section 5. In this section we study the operator B acting on Besov spaces of generalised smoothness on Ω.

By Proposition 6.17, Remark 6.19 and Theorem 5.21, if

$$1 \le p \le \infty \quad \text{and} \quad n-2 < -\overline{s}(\boldsymbol{h}),$$

then

$$B : B^{\boldsymbol{h}^{\frac{1}{p}}(n)^{\frac{1}{p}}}_{p,1}(\Omega) \to B^{(2)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}_{p,\infty}(\Omega),$$

where $\boldsymbol{h}$ is as in (6.1).

The following proposition extends part of the results in [30, p. 123, Lemma 4.1.4].

Proposition 7.1 *Let $h \in \mathbb{H}$ be a strictly increasing function. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let Γ be an h-set such that $\Gamma \subset \Omega$ and*

$$(n-2)_+ < -\overline{s}(\boldsymbol{h}) \le -\underline{s}(\boldsymbol{h}) \le n. \tag{7.2}$$

Let

$$1 \le p \le \infty, \quad 0 < q \le \infty$$

and let σ be an admissible sequence such that

$$\frac{n+\overline{s}(\boldsymbol{h})}{p} < \underline{s}(\sigma) \le \overline{s}(\sigma) < 2 - \frac{n+\overline{s}(\boldsymbol{h})}{p'}. \tag{7.3}$$

Then, the operator

$$B := (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma$$

is compact in $B^\sigma_{p,q}(\Omega)$. Moreover, if u is an eigenfunction associated to an eigenvalue ρ of B, $\rho \neq 0$, then

$$u \in B^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega). \tag{7.4}$$

Recall that $(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'} = (2^{2j}h(2^{-j})^{-1/p'}2^{-nj/p'})_j$.

Remark 7.2 As we are assuming that $-\overline{s}(\boldsymbol{h}) > n-2$, then

$$2 - \frac{n+\overline{s}(\boldsymbol{h})}{p'} = 2 - (n+\overline{s}(\boldsymbol{h})) + \frac{n+\overline{s}(\boldsymbol{h})}{p} > \frac{n+\overline{s}(\boldsymbol{h})}{p}$$

and so condition (7.3) makes sense.

Proof. We prove that B is a compact operator acting in $B^\sigma_{p,q}(\Omega)$. We factorize B as follows

$$B = \mathrm{id}_4 \circ (-\Delta)^{-1} \circ \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma \circ \mathrm{id}_3 \circ \mathrm{id}_2 \circ \mathrm{id}_1,$$

where, for $s_0, s_1 \in ((n+\overline{s}(\boldsymbol{h}))/p, \underline{s}(\sigma))$, with $s_0 > s_1$,

$$\mathrm{id}_1 : B^\sigma_{p,q}(\Omega) \hookrightarrow B^{(s_0)}_{p,q}(\Omega), \tag{7.5}$$

$$\mathrm{id}_2 : B^{(s_0)}_{p,q}(\Omega) \hookrightarrow B^{(s_1)}_{p,q}(\Omega), \tag{7.6}$$

$$\mathrm{id}_3 : B^{(s_1)}_{p,q}(\Omega) \hookrightarrow B^{\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,p}(\Omega), \tag{7.7}$$

$$\mathrm{tr}_\Gamma : B^{\boldsymbol{h}^{1/p}(n)^{1/p}}_{p,p}(\Omega) \to L_p(\Gamma), \tag{7.8}$$

$$\mathrm{id}^\Gamma : L_p(\Gamma) \to B^{\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega), \tag{7.9}$$

$$(-\Delta)^{-1} : B^{\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega) \to B^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega), \tag{7.10}$$

$$\mathrm{id}_4 : B^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega) \hookrightarrow B^\sigma_{p,q}(\Omega). \tag{7.11}$$

The embeddings (7.5) and (7.7) can be justified by Proposition 4.5 and the property in (3.3). Applying [41, p. 67, Theorem 1.107], as $s_0 > s_1$, one concludes that the embedding in (7.6) is compact. The continuity of the mapping in (7.8) follows from Proposition 6.12. The expressions in (7.9) and (7.10) are justified by Proposition 6.16 and Theorem 5.21, respectively.

Finally, (7.11) follows from Proposition 4.5 and Remark 4.10 making use of (7.3).

From the previous factorization of B, (7.4) follows immediately. □

Remark 7.3 Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let Γ be an h-set such that $\Gamma \subset \Omega$ and

$$(n-2)_+ < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) \leq n.$$

Then taking $p = q = 2$ and $\sigma = (1)$, the conditions of Proposition 7.1 are satisfied. Hence, B is a compact operator in $H^1(\Omega)$. Now let $1 < p \leq \infty$. Although the sequence $\sigma = (2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'} = (2^{2j} h(2^{-j})^{-1/p'} 2^{-nj/p'})_j$ does not necessarily satisfy (7.3), one can conclude, from the proof of Proposition 7.1, that the operator B is compact in $B^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega)$, considering the factorization

$$B = (-\Delta)^{-1} \circ \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma \circ \mathrm{id}_3 \circ \mathrm{id}_2 \circ \mathrm{id}_1 \circ \mathrm{id}_4,$$

where the operators involved are as in (7.5)–(7.11).

8 The operator *B* in *$\mathring{H}^1$ (Ω)*

In this section we consider the operator B acting in $\mathring{H}^1(\Omega)$. We refer to Definition 4.8(ii) and Remarks 4.4 and 4.9. We prove that B acting in this space is self-adjoint, nonnegative, compact and generated by a sesquilinear form. We deal with the asymptotic distribution of the eigenvalues of B and present some results on the associated eigenfunctions.

Definition 8.1 Let Ω be a domain in $\mathbb{R}^n$. We define, for $f, g \in \mathring{H}^1(\Omega)$,

$$(f,g)_{\mathring{H}^1(\Omega)} := \sum_{j=1}^{n} \int_\Omega \frac{\partial f}{\partial x_j}(x) \frac{\partial \overline{g}}{\partial x_j}(x)\, dx, \tag{8.1}$$

where $\frac{\partial f}{\partial x_j}$ and $\frac{\partial f}{\partial x_j}$ denote, for $j \in \{1, \dots, n\}$, the weak derivatives of first order of f and $\overline{g}$, respectively.

The following result follows from Friedrich's inequality (cf. [35, p. 357], for example) and it can be found in [38, p. 195].

Proposition 8.2 *Let Ω be a bounded domain in $\mathbb{R}^n$. For all $f \in \mathring{H}^1(\Omega)$*

$$\|f|H^1(\Omega)\|^2 \sim (f,f)_{\mathring{H}^1(\Omega)}.$$

Remark 8.3 In what follows, for technical reasons, in the space $\mathring{H}^1(\Omega)$ we will not consider the norm inherited from $H^1(\Omega)$, but the equivalent norm given by

$$\|f|\mathring{H}^1(\Omega)\| := \sqrt{(f,f)_{\mathring{H}^1(\Omega)}} = \sqrt{\sum_{j=1}^{n} \int_\Omega \left| \frac{\partial f}{\partial x_j}(x) \right|^2 dx}, \quad f \in \mathring{H}^1(\Omega), \tag{8.2}$$

which is justified by Proposition 8.2.

In the proof of the following theorem we will apply the notion of approximation numbers of an operator. We recall the definition.

Definition 8.4 Let X and Y be normed vector spaces. Given $L \in L(X, Y)$, rank L denotes the dimension of the range of L. Let $T \in L(X, Y)$ and $k \in \mathbb{N}$. The *kth approximation number*, $a_k(T)$, of T is defined by

$$a_k(T) := \inf\{\|T - L\| : L \in L(X, Y) \text{ and } \operatorname{rank} L < k\}.$$

In the following proof we will also apply the next result.

Proposition 8.5 *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$. There is a function G,*

$$G : \overline{\Omega} \times \Omega \to \mathbb{R},$$

usually called Green's function, such that

(i) *for all $x^0 \in \Omega$ and $\varepsilon > 0$,*

$$G(x^0, \cdot) \in C^\infty(\Omega \setminus \overline{B(x^0, \varepsilon)});$$

(ii) *for all $x, y \in \Omega$, with $x \neq y$, $G(x, y) = G(y, x)$;*

(iii) *if $n \geq 3$*

$$0 < G(x, y) \lesssim |x - y|^{2-n}, \quad x, y \in \Omega, \quad x \neq y,$$

and, if $n = 2$,

$$0 < G(x, y) \lesssim \max_{z \in \partial\Omega} \ln |z - x| - \ln |x - y|, \quad x, y \in \Omega, \quad x \neq y;$$

(iv) *$G(x, y) = 0$, for all $x \in \partial\Omega$ and $y \in \Omega$;*

(v) *for all $\varphi \in \mathcal{D}(\Omega)$,*

$$(-\Delta)^{-1}\varphi(x) = \int_\Omega G(x, y)\varphi(y)\, dy, \quad x \in \Omega. \tag{8.3}$$

Therefore, for all $f \in H^{-1}(\Omega)$,

$$(-\Delta)^{-1} f = \lim_{j\to\infty} \int_\Omega G(\cdot, y)\varphi_j(y)\, dy \quad \textit{in} \quad \overset{\circ}{H}{}^1(\Omega),$$

where

$$(\varphi_j)_{j\in\mathbb{N}_0} \subset \mathcal{D}(\Omega) \quad \textit{with} \quad f = \lim_{j\to\infty} \varphi_j \quad \textit{in} \quad H^{-1}(\Omega). \tag{8.4}$$

Remark 8.6 In (8.4) we used the fact that $\mathcal{D}(\Omega)$ is dense in $H^{-1}(\Omega)$, which can be justified as follows: the restriction of $\mathcal{S}(\mathbb{R}^n)$ to Ω is dense in $H^{-1}(\Omega)$. Any function in the restriction of $\mathcal{S}(\mathbb{R}^n)$ to Ω can be approximated in $L_2(\Omega)$ by functions belonging to $\mathcal{D}(\Omega)$. But this is also an approximation in $H^{-1}(\Omega)$.

For the results in the previous proposition we refer to [19, pp. 160–163, 273], [39, p. 299].

In connection with the Green's function we also refer to [1, pp. 10–13], [35, pp. 145, 194–196], [39, pp. 243–244] and [41, p. 301].

The next theorem extends the results in [39, Theorem 19.7] and [16, Theorem 2.28]. We also refer to [30, Theorem 4.1.7].

Theorem 8.7 *Let $h \in \mathbb{H}$ be a strictly increasing function. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$, and let Γ be an h-set such that $\Gamma \subset \Omega$ and*

$$n - 2 < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) < n. \tag{8.5}$$

Then B is a nonnegative compact self-adjoint operator in $\mathring{H}^1(\Omega)$ with null-space

$$N(B) = \{f \in \mathring{H}^1(\Omega) : \mathrm{tr}_\Gamma\, f = 0\}. \tag{8.6}$$

Moreover, B is generated by the sesquilinear form

$$(Bf, g)_{\mathring{H}^1(\Omega)} = \int_\Gamma (\mathrm{tr}_\Gamma\, f)(\gamma)\overline{(\mathrm{tr}_\Gamma\, g)(\gamma)}\, d\mu(\gamma), \quad f, g \in \mathring{H}^1(\Omega), \tag{8.7}$$

with (8.1) *as the scalar product in $\mathring{H}^1(\Omega)$. Furthermore, B is given by*

$$Bf = \int_\Gamma G(\cdot, \gamma)(\mathrm{tr}_\Gamma\, f)(\gamma)\, d\mu(\gamma), \quad f \in \mathring{H}^1(\Omega), \tag{8.8}$$

where G is the Green's function referred in Proposition 8.5.

Let ρ_k denote the positive eigenvalues of B repeated according to multiplicity and ordered by decreasing order of their magnitude, and let u_k denote related eigenfunctions,

$$Bu_k = \rho_k u_k, \quad k \in \mathbb{N}.$$

(i) *The largest eigenvalue is simple, i.e.,*

$$\rho_1 > \rho_2 \geq \rho_3 \ldots$$

and, for all $k \in \mathbb{N}$,

$$\rho_k \sim k^{-1} H(k^{-1})^{2-n}, \tag{8.9}$$

where H denotes the inverse function of h.

(ii) *The eigenfunctions u_k are* (*classical*) *harmonic functions in $\Omega \setminus \Gamma$,*

$$\Delta u_k(x) = 0 \quad \textit{if} \quad x \in \Omega \setminus \Gamma. \tag{8.10}$$

(iii) *Let $1 < p \leq \infty$, $\varepsilon \in \mathbb{R}$ and let p' be the conjugate exponent of p. Then*

$$u_k \in B^{(\varepsilon)(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}_{p,\infty}(\Omega) \quad \textit{if, and only if,} \quad \varepsilon \leq 0.$$

Recall that $(\varepsilon)(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'} = (2^{\varepsilon j} 2^{2j} h(2^{-j})^{-1/p'} 2^{-nj/p'})_j$.

(iv) *The eigenfunctions $u_1(x)$ have no zeros in Ω*

$$u_1(x) = cu(x) \quad \textit{with} \quad c \in \mathbb{C} \quad \textit{and} \quad u(x) > 0 \quad \textit{if} \quad x \in \Omega.$$

Proof. *Step 1:*

Let us prove that B is a compact operator acting in $\mathring{H}^1(\Omega)$. We factorize B as

$$B = (-\Delta)^{-1} \circ \mathrm{id}_3 \circ \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma,$$

where

$$\mathrm{tr}_\Gamma \ : \ \mathring{H}^1(\Omega) \to L_2(\Gamma), \tag{8.11}$$

$$\mathrm{id}^\Gamma \ : \ L_2(\Gamma) \to B^{\boldsymbol{h}^{-1/2}(n)^{-1/2}}_{2,\infty}(\Omega), \tag{8.12}$$

$$\mathrm{id}_3 \ : \ B^{\boldsymbol{h}^{-1/2}(n)^{-1/2}}_{2,\infty}(\Omega) \hookrightarrow H^{-1}(\Omega), \tag{8.13}$$

$$(-\Delta)^{-1} \ : \ H^{-1}(\Omega) \to \mathring{H}^1(\Omega). \tag{8.14}$$

By [40, p. 13, Theorem 1], the operator tr_Γ in (8.11) is compact. Proposition 6.16 justifies (8.12) and the embedding (8.13) follows from (4.2), because

$$\underline{s}\left(\boldsymbol{h}^{-1/2}(n)^{-1/2}\right) = -\frac{\overline{s}(\boldsymbol{h})}{2} - \frac{n}{2} > \frac{n-2}{2} - \frac{n}{2} = -1 = \overline{s}((-1)).$$

By [39, p. 255]

$$\overset{\circ}{H}{}^1(\Omega) = \{f \in H^1(\Omega) : \mathrm{tr}_{\partial\Omega}\, f = 0\}.$$

Therefore, applying Theorem 5.21, we obtain (8.14).

Step 2: Let us prove (8.8). Let $f \in \overset{\circ}{H}{}^1(\Omega)$. Then, by (8.11)–(8.13), $\mathrm{id}^\Gamma(\mathrm{tr}_\Gamma\, f)$ belongs to $H^{-1}(\Omega)$. Let $(\psi_j)_{j\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ converge to $\mathrm{id}^\Gamma(\mathrm{tr}_\Gamma\, f)$ in $H^{-1}(\Omega)$. Let us consider $\chi_k \in \mathcal{D}(\Omega)$, $k \in \mathbb{N}$, such that

$$0 \le \chi_k \le 1, \quad \chi_k(x) = 1 \ \text{ if } \ x \in \Gamma, \quad \text{and} \quad \operatorname{supp}\chi_k \subset \Gamma_{1/k}, \tag{8.15}$$

where $\Gamma_{1/k} = \{x \in \mathbb{R}^n : \mathrm{dist}(x,\Gamma) < 1/k\}$. Applying Corollary 5.6 one concludes that, for all $k \in \mathbb{N}$,

$$\lim_{j\to\infty}(\chi_k\psi_j) = \chi_k\, \mathrm{id}^\Gamma(\mathrm{tr}_\Gamma\, f) \quad \text{in} \quad H^{-1}(\Omega).$$

By Definition 6.14 and (8.15)

$$\chi_k\, \mathrm{id}^\Gamma(\mathrm{tr}_\Gamma\, f) = \mathrm{id}^\Gamma(\mathrm{tr}_\Gamma\, f), \quad k \in \mathbb{N}.$$

By Proposition 8.5, for each $k \in \mathbb{N}$,

$$Bf = (-\Delta)^{-1}\, \mathrm{id}^\Gamma(\mathrm{tr}_\Gamma\, f) = \lim_{j\to\infty}\int_\Omega G(\cdot,y)\chi_k(y)\psi_j(y)\,dy \quad \text{in} \quad \overset{\circ}{H}{}^1(\Omega).$$

So, for each $k \in \mathbb{N}$, there is $A_k \subset \mathbb{R}^n$ with $|A_k| = 0$ and there is a subsequence of $(\chi_k\psi_j)_{j\in\mathbb{N}}$, say $(\chi_k\psi_{\sigma_k(j)})_{j\in\mathbb{N}}$, such that

$$Bf(x) = \lim_{j\to\infty}\int_\Omega G(x,y)\chi_k(y)\psi_{\sigma_k(j)}(y)\,dy, \quad \text{for all} \quad x \in \Omega\setminus A_k. \tag{8.16}$$

Let $A := \bigcup_{k=1}^\infty A_k$. Then $|A| = 0$ and (8.16) is satisfied for all $k \in \mathbb{N}$ and all $x \in \Omega\setminus A$. But we remark that for each k we may have a different subsequence $(\chi_k\psi_{\sigma_k(j)})_j$. As we want to obtain a representation for Bf, which is a regular distribution, we may exclude from consideration sets with Lebesgue measure equal to 0. So it is enough to consider $x \in \Omega\setminus(A\cup\Gamma)$. Let us consider such x. We fix $k \in \mathbb{N}$ such that $2/k < \min\{\mathrm{dist}(x,\Gamma), \mathrm{dist}(\Gamma,\partial\Omega)\}$ and we consider $\theta_k \in \mathcal{D}(\Omega)$ satisfying

$$0 \le \theta_k \le 1, \quad \theta_k(y) = 1 \quad \text{if} \quad y \in \Gamma_{1/k}, \quad \text{and} \quad \operatorname{supp}\theta_k \subset \Gamma_{2/k}.$$

Hence

$$\begin{aligned} Bf(x) &= \lim_{j\to\infty}\int_\Omega G(x,y)\chi_k(y)\psi_{\sigma_k(j)}(y)\,dy \\ &= \lim_{j\to\infty}\int_\Omega G(x,y)\theta_k(y)\chi_k(y)\psi_{\sigma_k(j)}(y)\,dy. \end{aligned}$$

We remark that $G(x,\cdot)\theta_k \in \mathcal{D}(\Omega)$ and that $(\chi_k\psi_{\sigma_k(j)})_j$ converges to $\mathrm{id}^\Gamma(\mathrm{tr}_\Gamma\, f)$ in $H^{-1}(\Omega)$ and, so, also in $\mathcal{D}'(\Omega)$. Thus

$$\begin{aligned} Bf(x) &= \lim_{j\to\infty}\left\langle \chi_k\psi_{\sigma_k(j)}, G(x,\cdot)\theta_k\right\rangle \\ &= \langle \mathrm{id}^\Gamma(\mathrm{tr}_\Gamma\, f), G(x,\cdot)\theta_k\rangle \\ &= \int_\Gamma(\mathrm{tr}_\Gamma\, f)(\gamma)G(x,\gamma)\,d\mu(\gamma) \end{aligned}$$

proving (8.8).

Step 3: Let us prove (8.7). Let $f \in \mathring{H}^1(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Then

$$\begin{aligned}(Bf,\varphi)_{\mathring{H}^1(\Omega)} &= \sum_{j=1}^{n} \left\langle \frac{\partial Bf}{\partial x_j}, \frac{\partial \overline{\varphi}}{\partial x_j} \right\rangle \\ &= \left\langle -\Delta Bf, \overline{\varphi} \right\rangle \\ &= \left\langle \operatorname{tr}^{\Gamma} f, \overline{\varphi} \right\rangle \\ &= \int_{\Gamma} (\operatorname{tr}_{\Gamma} f)(\gamma) \overline{(\varphi|_{\Gamma}(\gamma))}\, d\mu(\gamma).\end{aligned}$$

Let $f, g \in \mathring{H}^1(\Omega)$ and let $(\varphi_t)_{t\in\mathbb{N}_0} \subset \mathcal{D}(\Omega)$ converge to g in $\mathring{H}^1(\Omega)$. Then

$$\begin{aligned}&\left| (Bf,g)_{\mathring{H}^1(\Omega)} - \int_{\Gamma} (\operatorname{tr}_{\Gamma} f)(\gamma)\overline{(\operatorname{tr}_{\Gamma} g)(\gamma)}\, d\mu(\gamma) \right| \\ &\quad \le \lim_{t\to\infty} \int_{\Gamma} \left| (\operatorname{tr}_{\Gamma} f)(\gamma) \overline{(\operatorname{tr}_{\Gamma} \varphi_t - \operatorname{tr}_{\Gamma} g)(\gamma)} \right| d\mu(\gamma) \\ &\quad \le \lim_{t\to\infty} \| \operatorname{tr}_{\Gamma} f | L_2(\Gamma)\| \cdot \| \operatorname{tr}_{\Gamma}(\varphi_t - g) | L_2(\Gamma)\| \\ &\quad \le c\, \| \operatorname{tr}_{\Gamma} f | L_2(\Gamma)\| \lim_{t\to\infty} \|\varphi_t - g | H^1(\Omega)\| \;=\; 0.\end{aligned}$$

Hence, we proved (8.7) and we also conclude (8.6) and that B is a nonnegative self-adjoint operator in $\mathring{H}^1(\Omega)$.

Step 4: We prove (8.9). We can easily check that conditions $n \ge 2$ and (8.5) guarantee that [40, p. 14, Theorem 2] can be applied to conclude that the approximation numbers of the compact operator

$$\operatorname{tr}_{\Gamma} : \mathring{H}^1(\Omega) \to L_2(\Gamma)$$

satisfy

$$a_k(\operatorname{tr}_{\Gamma}) \sim k^{-\frac{1}{2}} H(k^{-1})^{1-\frac{n}{2}}, \quad k \in \mathbb{N}.$$

By (6.10) and applying the known assertion about the relation between the approximation numbers of dual operators (cf. [14, p. 55, Proposition II, 2.5], for example),

$$a_k(\operatorname{id}^{\Gamma}) = a_k(\operatorname{tr}_{\Gamma}) \sim k^{-\frac{1}{2}} H(k^{-1})^{1-\frac{n}{2}}, \quad k \in \mathbb{N}.$$

Hence, again by the properties of the approximation numbers (cf. [14, p. 53, Proposition II, 2.2]),

$$a_{2k}(B) \le c\, a_k(\operatorname{tr}_{\Gamma}) a_k(\operatorname{id}^{\Gamma}) \sim k^{-1} H(k^{-1})^{2-n}, \quad k \in \mathbb{N}.$$

By the properties of h and as $a_k(B) = \rho_k$ (cf. [14, p 91, Theorem II, 5.10]) we get to the "$\lesssim$" part of (8.9).

Let us prove the "$\gtrsim$" part of (8.9). Let ρ_k be a positive eigenvalue of B. Then $\sqrt{\rho_k}$ is an eigenvalue for the operator $\sqrt{B}$. So again by [14, p. 91, Theorem II, 5.10] it is sufficient to prove that

$$a_k(\sqrt{B}) \gtrsim k^{-\frac{1}{2}} H(k^{-1})^{1-\frac{n}{2}}, \quad k \in \mathbb{N}. \tag{8.17}$$

As $\Gamma \subset \Omega$, $\operatorname{dist}(\Gamma, \partial\Omega) > 0$. Let $j_0 \in \mathbb{N}$ be such that $2^{-j_0} < \operatorname{dist}(\Gamma, \partial\Omega)$. For all $j \in \mathbb{N}$, $j \ge j_0$, there is $\{\gamma^{j,m}\}_{m=1}^{M_j} \subset \Gamma$ such that the balls $B(\gamma^{j,m}, 2^{-j})$, $m = 1, \ldots, M_j$, are pairwise disjoint subsets of Ω and $M_j \sim (h(2^{-j}))^{-1}$. The latter equivalence follows from [6, p. 26, Lemma 8.8].

Let $\delta \in (0, 1/2)$ and let ω be a C^{∞} nonnegative function such that

$$\operatorname{supp} \omega \subset \{x \in \mathbb{R}^n : |x| \le 1/2\} \quad \text{and} \quad \omega(x) > 0, \quad \text{if} \quad |x| \le \delta.$$

Let

$$\omega^{j,m}(x) := \omega(2^j(x - \gamma^{j,m})), \quad j \ge j_0, \quad m = 1, \ldots, M_j,$$

and

$$f_j(x) := \sum_{m=1}^{M_j} c_{j,m}\,\omega^{j,m}(x), \quad c_{j,m} \in \mathbb{C}, \quad j \geq j_0. \tag{8.18}$$

Then, as $\operatorname{supp}\omega^{j,m} \subset B(\gamma^{j,m}, 2^{-j})$, by (8.2) and by the properties of $\{\gamma^{j,m}\}_{m=1}^{M_j}$,

$$\|f_j|\mathring{H}^1(\Omega)\|^2 = \sum_{m=1}^{M_j} |c_{j,m}|^2 \cdot \|\omega^{j,m}|\mathring{H}^1(\Omega)\|^2.$$

By the conditions on the support of ω and applying the homogeneity property (cf. [11]),

$$\|\omega^{j,m}|\mathring{H}^1(\Omega)\| \sim \|\omega^{j,m}|H^1(\mathbb{R}^n)\| \sim 2^{j(1-\frac{n}{2})}\|\omega|H^1(\mathbb{R}^n)\|.$$

Therefore

$$\|f_j|\mathring{H}^1(\Omega)\| \sim 2^{j(1-\frac{n}{2})}\left(\sum_{m=1}^{M_j}|c_{j,m}|^2\right)^{1/2}. \tag{8.19}$$

For all $j \geq j_0$, as $\{B(\gamma^{j,m}, 2^{-j})\}_m$ are pairwise disjoint,

$$\begin{aligned}
\|\operatorname{tr}_\Gamma f_j|L_2(\Gamma)\|^2 &= \sum_{m=1}^{M_j}|c_{j,m}|^2\int_\Gamma(\omega^{j,m}(\gamma))^2\,d\mu(\gamma)\\
&\geq \sum_{m=1}^{M_j}|c_{j,m}|^2\int_{\Gamma\cap B(\gamma^{j,m},\delta 2^{-j})}(\omega^{j,m}(\gamma))^2\,d\mu(\gamma)\\
&\geq \inf_{|x|\leq\delta}\omega(x)^2\cdot\sum_{m=1}^{M_j}|c_{j,m}|^2\cdot\mu(B(\gamma^{j,m},\delta 2^{-j}))\\
&\sim h(2^{-j})\sum_{m=1}^{M_j}|c_{j,m}|^2.
\end{aligned}$$

Analogously one can obtain the reverse inequality. So

$$\|\operatorname{tr}_\Gamma f_j|L_2(\Gamma)\| \sim h(2^{-j})^{1/2}\left(\sum_{m=1}^{M_j}|c_{j,m}|^2\right)^{1/2}. \tag{8.20}$$

Let $T \in L(\mathring{H}^1(\Omega), \mathring{H}^1(\Omega))$ be such that $\operatorname{rank} T < M_j$. Then there is $(c_{j,m})_{m=1}^{M_j} \subset \mathbb{C}$ such that

$$\sum_{m=1}^{M_j} c_{j,m}T\omega^{j,m} = 0, \tag{8.21}$$

where there is at least one $m = 1, \ldots, M_j$ for which $c_{j,m} \neq 0$. Let f_j be given by (8.18). By (8.19)–(8.21),

$$\|\sqrt{B} - T\| \geq \frac{\|\sqrt{B}f_j|\mathring{H}^1(\Omega)\|}{\|f_j|\mathring{H}^1(\Omega)\|} \sim h(2^{-j})^{1/2}2^{-j(1-\frac{n}{2})}. \tag{8.22}$$

Now (8.17) follows from the equivalence $M_j \sim (h(2^{-j}))^{-1}$, $j \in \mathbb{N}$, and the properties of h.

Step 5: We prove (ii). Let ρ_k be a positive eingenvalue for the operator B and let u_k be an associated eigenfunction. Hence

$$\rho_k^{-1}\,\mathrm{tr}^\Gamma\, u_k = (-\Delta)u_k \in B_{2,\infty}^{\boldsymbol{h}^{-1/2}(n)^{-1/2}}(\Omega).$$

As $\mathrm{supp}(\mathrm{tr}^\Gamma\, u_k) \subset \Gamma$ then $\Delta u_k = 0$ in $\mathcal{D}'(\Omega \setminus \Gamma)$. It is a well-known result of the theory of distributions that a distributional solution of the Laplace equation is also a classic solution (cf. [37, Section 6.4.1, Lemma, p. 414]), so (8.10) is proved.

Step 6: We prove (iii). Let us prove the "if" part. Let

$$1 < p_0 < p_1 \le \infty \quad \text{and} \quad s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$$

and assume that (7.3) is satisfied considering $\sigma = (s_i)$ and $p = p_i$, for $i = 0, 1$. Then

$$B_{p_0,q}^{(s_0)}(\Omega) \subset B_{p_1,q}^{(s_1)}(\Omega) \subset B_{p_0,q}^{(s_1)}(\Omega). \tag{8.23}$$

For the first inclusion in (8.23) we refer to [34, 2.7.1, p. 129] and for the second one we refer to [30, p. 120, Proposition 4.1.2]. Let us temporarily denote by $B(p_i, s_j)$, $i, j = 0, 1$, the operator B considered acting in $B_{p_i,q}^{(s_j)}(\Omega)$. By (8.23), the eigenvalues and the related eigenfunctions of $B(p_0, s_0)$ are also eigenvalues and related eigenfunctions for $B(p_1, s_1)$ and $B(p_0, s_1)$. Now let $\rho \neq 0$ be an eigenvalue for $B(p_0, s_1)$ and let u be an associated eigenfunction. Then, by Proposition 7.1, $u \in B_{p_0,\infty}^{(2)\boldsymbol{h}^{-1/p_0'}(n)^{-1/p_0'}}(\Omega)$. The embedding (7.11) with $\sigma = (s_0)$ and $p = p_0$ holds and this can be justified as in the proof of Proposition 7.1, because (7.3) is satisfied. So $u \in B_{p_0,q}^{(s_0)}(\Omega)$ and thus ρ and u are an eigenvalue and an associated eigenfunction, respectively, for $B(p_0, s_0)$ and $B(p_1, s_1)$. Therefore the eigenvalues $\rho \neq 0$ and the associated eigenfunctions of $B(p_0, s_0)$, $B(p_1, s_1)$ and $B(p_0, s_1)$ coincide.

Let $1 < p \le \infty$. If $p = 2$ it is immediate by Proposition 7.1 that if u is an eigenfucntion for B then $u \in B_{2,\infty}^{(2)\boldsymbol{h}^{-1/2}(n)^{-1/2}}(\Omega)$. Otherwise, we consider a number s such that (7.3) is satisfied for $\sigma = (s)$. Using pairs as (p_0, s_0) and (p_1, s_1) and repeating this construction a finite number of times one can "reach" the pair (p, s) starting from $(2, 1)$, which corresponds to the space $B_{2,2}^{(1)}(\Omega) = H^1(\Omega)$. One can replace $H^1(\Omega)$ by $\overset{\circ}{H}{}^1(\Omega)$, which does not influence the above arguments. Therefore one concludes that the eigenfunctions of B belong to $B_{p,\infty}^{(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}(\Omega)$ and so, by Proposition 4.5, to all spaces $B_{p,\infty}^{(\varepsilon)(2)\boldsymbol{h}^{-1/p'}(n)^{-1/p'}}(\Omega)$, with $\varepsilon \le 0$.

Now let us prove the "only if" part. Assume that there are $\varepsilon > 0$, $1 < p \le \infty$ and an eigenfunction u_k associated to a positive eigenvalue ρ_k of the operator B such that

$$u_k \in B_{p,\infty}^{(2+\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}(\Omega).$$

Let $\varphi \in \mathcal{D}(\Omega)$ with $\varphi|_\Gamma = 0$. Then

$$-\rho_k\langle \Delta u_k, \varphi\rangle \;=\; \langle \mathrm{tr}^\Gamma\, u_k, \varphi\rangle \;=\; \langle \mathrm{id}^\Gamma \circ \mathrm{tr}_\Gamma\, u_k, \varphi\rangle \;=\; \int_\Gamma (\mathrm{tr}_\Gamma\, u_k)(\gamma)\varphi(\gamma)\, d\mu(\gamma) = 0$$

and so

$$\Delta u_k \in B_{p,\infty}^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}(\Omega)$$

and

$$\langle \Delta u_k, \psi\rangle = 0 \quad \text{for all} \quad \psi \in \mathcal{D}(\Omega) \quad \text{such that} \quad \varphi|_\Gamma = 0. \tag{8.24}$$

Let us consider two smooth positive functions η_1 and η_2 such that $\eta_1(x) + \eta_2(x) = 1$, for all $x \in \mathbb{R}^n$,

$$\mathrm{supp}\,\eta_1 \subset \Omega \quad \text{and} \quad \mathrm{supp}\,\eta_2 \subset (\mathbb{R}^n \setminus \Omega)_r, \tag{8.25}$$

where $(\mathbb{R}^n \setminus \Omega)_r$ denotes a neighborhood of $\mathbb{R}^n \setminus \Omega$ such that $\Gamma \cap (\mathbb{R}^n \setminus \Omega)_r = \emptyset$. We define

$$\langle \widetilde{\Delta u_k}, \varphi \rangle := \langle \Delta u_k, (\eta_1 \varphi)|_\Omega \rangle, \quad \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{8.26}$$

Let us prove that

$$\widetilde{\Delta u_k} \in B_{p,\infty}^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}},\Gamma}(\mathbb{R}^n) \quad \text{and} \quad (\widetilde{\Delta u_k})|_\Omega = \Delta u_k, \tag{8.27}$$

where we are using the notation introduced in Definition 6.15. Let $\psi \in \mathcal{D}(\Omega)$ and $\widetilde{\psi}$ denote the extension of ψ to $\mathbb{R}^n$ by zero. Then

$$\langle (\widetilde{\Delta u_k})|_\Omega, \psi \rangle = \langle \widetilde{\Delta u_k}, \widetilde{\psi} \rangle = \langle \Delta u_k, (\eta_1 \widetilde{\psi})|_\Omega \rangle.$$

By (8.24) and (8.25), one concludes that $\langle \Delta u_k, (\eta_2 \widetilde{\psi})|_\Omega \rangle = 0$ and so

$$\langle (\widetilde{\Delta u_k})|_\Omega, \psi \rangle = \langle \Delta u_k, (\eta_1 \widetilde{\psi} + \eta_2 \widetilde{\psi})|_\Omega \rangle = \langle \Delta u_k, \psi \rangle,$$

proving the second part of (8.27). Let us prove the first part. Let $g \in B_{p,\infty}^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}(\mathbb{R}^n)$ be such that $g|_\Omega = \Delta u_k$. By Theorem 5.4, $\eta_1 g \in B_{p,\infty}^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}(\mathbb{R}^n)$. By the definition of $\widetilde{\Delta u_k}$ one can easily conclude that $\widetilde{\Delta u_k} = \eta_1 g$ and so $\widetilde{\Delta u_k} \in B_{p,\infty}^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}}}(\mathbb{R}^n)$. Now let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\varphi|_\Gamma = 0$. By (8.24)–(8.26), one can conclude that $\langle \widetilde{\Delta u_k}, \varphi \rangle = 0$ and so (8.27) is proved.

Let $\delta \in (0, \varepsilon p'/2)$. Let us apply (4.1) to conclude that

$$B_{p,\infty}^{(\varepsilon)\boldsymbol{h}^{-\frac{1}{p'}}(n)^{-\frac{1}{p'}},\Gamma}(\mathbb{R}^n) \subset B_{p,\infty}^{\left(-\frac{n-(\underline{\omega}(h)+\delta)}{p'}\right),\Gamma}(\mathbb{R}^n). \tag{8.28}$$

By Definition 6.5 there is $j_0 \in \mathbb{N}$ such that

$$\underline{\omega}(h) - \delta \leq \frac{\log h(2^{-j})}{\log 2^{-j}}, \quad j \geq j_0.$$

Then, for all $j \geq j_0$,

$$2^{-\varepsilon j} h(2^{-j})^{\frac{1}{p'}} 2^{\frac{nj}{p'}} 2^{-\frac{n-(\underline{\omega}(h)+\delta)}{p'} j} \leq 2^{-\varepsilon j} 2^{\frac{-j(\underline{\omega}(h)-\delta)}{p'}} 2^{\frac{nj}{p'}} 2^{-\frac{n-(\underline{\omega}(h)+\delta)}{p'} j} \leq 2^{-j(\varepsilon - \frac{2\delta}{p'})}. \tag{8.29}$$

Now (8.28) follows from (4.1) and (8.29). As was mentioned in Proposition 6.6, the Hausdorff dimension of an h-set Γ, $\dim_{\mathcal{H}}\Gamma$, is equal to $\underline{\omega}(h)$. By [38, p. 130, Theorem 17.8],

$$\dim_{\mathcal{H}}\Gamma = \sup\left\{ t \;:\; B_{p,\infty}^{(-\frac{n-t}{p'}),\Lambda}(\mathbb{R}^n) \text{ is nontrivial for some compact } \Lambda \subset \Gamma \right\},$$

where we are using the notation introduced in Definition 6.15. Therefore, as

$$\underline{\omega}(h) + \delta > \sup\left\{ t \;:\; B_{p,\infty}^{(-\frac{n-t}{p'}),\Lambda}(\mathbb{R}^n) \text{ is nontrivial for some compact } \Lambda \subset \Gamma \right\},$$

both spaces in (8.28) are trivial. So $\widetilde{\Delta u_k} = 0$ and $\Delta u_k = 0$. By Theorem 5.21, $-\Delta$ is an isomorphism. Then $u_k = 0$, which is a contradiction.

Step 7: Let $\rho = \rho_1$ be the largest positive eigenvalue of B. Let us prove that a nontrivial function $v \in \overset{\circ}{H}{}^1(\Omega)$ is an eigenfunction of B associated to ρ if, and only if,

$$\int_\Gamma |(\mathrm{tr}_\Gamma\, v)(\gamma)|^2 d\mu(\gamma) = \rho \sum_{j=1}^n \int_\Omega \left| \frac{\partial v}{\partial x_j}(x) \right|^2 dx. \tag{8.30}$$

The “only if” part is immediate (cf. (8.7)). We prove the “if” one. First, we introduce some more notation: we denote by $(f_k)_{k\in\mathbb{N}} \subset \mathring{H}^1(\Omega)$ an orthonormal system of eigenfunctions associated to the corresponding eigenvalues ρ_k, i.e., $Bf_k = \rho_k f_k$, $k \in \mathbb{N}$, and by M the closed subspace generated by $(f_k)_k$, so that

$$\mathring{H}^1(\Omega) = M \oplus N(B)$$

(cf. Theorem 4.4 in [33, p. 357], for example). Let $v \in \mathring{H}^1(\Omega)$, $v \neq 0$, satisfy (8.30). So there exist $z \in M$ and $g \in N(B)$ such that

$$v = z + g. \tag{8.31}$$

As $z \in M$, there exist $(\alpha_k)_k \subset \mathbb{C}$ such that

$$z = \sum_{k=1}^{\infty} \alpha_k f_k \quad \text{and} \quad \|z|\mathring{H}^1(\Omega)\|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2.$$

Using the fact that $Bg = 0$ one can conclude that $(z, g)_{\mathring{H}^1(\Omega)} = 0$. Therefore

$$\sum_{j=1}^{n} \int_\Omega \left|\frac{\partial v}{\partial x_j}(x)\right|^2 dx = \|z|\mathring{H}^1(\Omega)\|^2 + \|g|\mathring{H}^1(\Omega)\|^2 \geq \|z|\mathring{H}^1(\Omega)\|^2. \tag{8.32}$$

One can easily obtain

$$\sqrt{B}\, v = \sum_{k=1}^{\infty} \sqrt{\rho_k}\, \alpha_k f_k. \tag{8.33}$$

Therefore, by (8.7) and (8.33),

$$\| \operatorname{tr}_\Gamma v | L_2(\Gamma)\|^2 = \sum_{k=1}^{\infty} \rho_k |\alpha_k|^2. \tag{8.34}$$

Now applying (8.30)–(8.34) we obtain

$$\sum_{k=1}^{\infty} (\rho_k - \rho)|\alpha_k|^2 \geq 0.$$

As ρ is the largest eigenvalue we conclude that, for all $k \in \mathbb{N}_0$ such that $\rho_k \neq \rho$, $\alpha_k = 0$. So there is $K \in \mathbb{N}_0$ such that

$$z = \sum_{k=1}^{K} \alpha_k f_k,$$

where f_k, $k = 1, \ldots, K$, are eigenfunctions associated to ρ. Hence z is also an eigenfunction associated to ρ and, consequently,

$$Bv = \rho z + Bg = \rho z. \tag{8.35}$$

Hence

$$(Bv, v)_{\mathring{H}^1(\Omega)} = (\rho z, z + g)_{\mathring{H}^1(\Omega)} = \rho \|z|\mathring{H}^1(\Omega)\|^2$$

and, applying (8.7),

$$\| \operatorname{tr}_\Gamma v | L_2(\Gamma)\|^2 = \rho \|z|\mathring{H}^1(\Omega)\|^2. \tag{8.36}$$

It follows from (8.30), (8.32) and (8.36) that $g = 0$ and so, by (8.31) and (8.35), we conclude that v is an eigenfunction associated to ρ.

Step 8: We prove part of (iv), i.e., we prove that there is a function u such that

$$Bu = \rho u \quad \text{and} \quad u(x) > 0, \quad x \in \Omega. \tag{8.37}$$

Let v be an eigenfunction associated to ρ. Then (8.30) is satisfied for v and, consequently, also for $\overline{v}$ (instead of v). So, by what was proved in the previous step, $\overline{v}$ is also an eigenfunction associated to ρ. Thus $\operatorname{Re} v$ and $\operatorname{Im} v$ are also eigenfunctions, if they are different from zero. So we may assume that v is real. Let us prove that $|v|$ is also an eigenfunction associated to ρ.

By Nikodym's Theorem (cf. [27, p. 8, Theorem 1]), $|v| \in \overset{\circ}{H}{}^1(\Omega)$. Let us prove that (8.30) is satisfied when one replaces v by $|v|$. By the property referred in [14, p. 220, Proposition 2.6] or [18, p. 152, Lemma 7.6], $|v| \in H^1(\Omega)$ and, for all $\alpha \in \mathbb{N}_0{}^n$, $|\alpha| = 1$,

$$D^\alpha |v(x)| = \begin{cases} D^\alpha v(x), & v(x) > 0, \\ 0, & v(x) = 0, \\ -D^\alpha v(x), & v(x) < 0. \end{cases} \tag{8.38}$$

It follows immediately from (8.38) that

$$\sum_{j=1}^n \int_\Omega \left| \frac{\partial |v|}{\partial x_j}(x) \right|^2 dx = \sum_{j=1}^n \int_\Omega \left| \frac{\partial v}{\partial x_j}(x) \right|^2 dx. \tag{8.39}$$

Next we prove that

$$\| \operatorname{tr}_\Gamma v | L_2(\Gamma) \| = \| \operatorname{tr}_\Gamma |v| | L_2(\Gamma) \|. \tag{8.40}$$

As v is an eigenfunction, it follows from Step 6, (3.3), (4.1) and (iv) in Remark 4.4 that, for $\delta \in (0, 2 - n - \overline{s}(\boldsymbol{h}))$, $v \in \mathcal{C}^{2-n-\overline{s}(\boldsymbol{h})-\delta}(\Omega)$. Hence v is a continuous function in $\overline{\Omega}$ and, consequently, $\operatorname{tr}_\Gamma v = v_{|\Gamma}$. This follows from Proposition 3.4.2 and Theorem 3.4.15 in [5, p. 107 and p. 114]. As v is continuous, $|v|$ is continuous and so $\operatorname{tr}_\Gamma |v| = |v|_{|\Gamma}$ and now (8.40) follows immediately.

Therefore, by (8.7), (8.39) and (8.40),

$$\rho \sum_{j=1}^n \int_\Omega \left| \frac{\partial |v|}{\partial x_j}(x) \right|^2 dx = \int_\Gamma |(\operatorname{tr}_\Gamma |v|)(\gamma)|^2 d\mu(\gamma)$$

and so, by the previous step, $|v|$ is also an eigenfunction associated to ρ and the function $\omega = |v| - v$ satisfies $B\omega = \rho\omega$. As ω is continuous in $\overline{\Omega}$, $\operatorname{tr}_\Gamma \omega = \omega|_\Gamma \geq 0$. By what was done in Step 2, there is a set A with Lebesgue measure 0 such that

$$\rho\, \omega(x) = (-\Delta)^{-1} \circ \operatorname{tr}^\Gamma \omega(x) = \int_\Gamma G(x, \gamma)(\operatorname{tr}_\Gamma \omega)(\gamma)\, d\mu(\gamma), \tag{8.41}$$

for all $x \in \Omega \setminus (A \cup \Gamma)$. Let us prove that the equality (8.41) holds for all $x \in \Omega$. We present the proof for $n \geq 3$. The case $n = 2$ can be studied similarly. First we prove that (8.41) holds for all $x \in \Gamma$. We recall that, as $\underline{s}(\boldsymbol{h}) > -n$, then, by Proposition 6.9, Γ satisfies the ball condition. Thus, by Definition 6.8, there is a number $\eta \in (0,1)$ such that for all $\gamma^0 \in \Gamma$, for all $j \in \mathbb{N}$, there is a ball $B(z_j, \eta 2^{-j})$ contained in $B(\gamma^0, 2^{-j})$ which does not intersect Γ. As $|A| = 0$, there is, for all $j \in \mathbb{N}$, $x_j \in B(z_j, \eta 2^{-j-1}) \setminus A$. Let us fix $\gamma^0 \in \Gamma$ and a sequence $(x_j)_j$ as we have just described. As $(x_j)_j$ converges to γ^0, ω is a continuous function and $(x_j)_j \in \Omega \setminus (A \cup \Gamma)$,

$$\rho\omega(\gamma^0) = \lim_{j\to\infty} \rho\omega(x_j) = \lim_{j\to\infty} \int_\Gamma G(x_j, \gamma)(\operatorname{tr}_\Gamma \omega)(\gamma)\, d\mu(\gamma). \tag{8.42}$$

One can also prove that

$$\lim_{j\to\infty} G(x_j, \gamma)(\operatorname{tr}_\Gamma \omega)(\gamma) = G(\gamma^0, \gamma)(\operatorname{tr}_\Gamma \omega)(\gamma), \quad \text{for all} \quad \gamma \in \Gamma \setminus \{\gamma^0\}. \tag{8.43}$$

Let us prove that the functions $G(x_j,\cdot)\,\mathrm{tr}_\Gamma\,\omega$ can be estimated from above by a μ-integrable function. Suppose that j and γ are such that $2^{-j+1}\le|\gamma-\gamma^0|$. Thus

$$|x_j-\gamma|\ge|\gamma-\gamma^0|-|x_j-\gamma^0|\ge|\gamma-\gamma^0|-2^{-j}\ge2^{-1}|\gamma-\gamma^0|. \tag{8.44}$$

Now we suppose that $2^{-j+1}>|\gamma-\gamma^0|$. Then

$$|x_j-\gamma|\ge\eta2^{-j-1}>\eta2^{-2}|\gamma-\gamma^0|. \tag{8.45}$$

So, by (8.44)–(8.45) and by Proposition 8.5, as $\mathrm{tr}_\Gamma\,\omega\ge0$,

$$G(x_j,\gamma)(\mathrm{tr}_\Gamma\,\omega)(\gamma)\lesssim|\gamma^0-\gamma|^{2-n}(\mathrm{tr}_\Gamma\,\omega)(\gamma),\quad j\in\mathbb{N},\quad\gamma\in\Gamma. \tag{8.46}$$

We prove that the function on the right of (8.46) is μ-integrable: let $\delta\in(0,2-n-\overline{s}(\boldsymbol{h}))$. Then

$$\begin{aligned}\int_\Gamma|\gamma^0-\gamma|^{2-n}(\mathrm{tr}_\Gamma\,\omega)(\gamma)\,d\mu(\gamma)&\lesssim\max_{\lambda\in\Gamma}\omega(\lambda)\sum_{j=0}^{\infty}\int_{c_0 2^{-(j+1)}<|\gamma-\gamma^0|\le c_0 2^{-j}}|\gamma-\gamma^0|^{2-n}\,d\mu(\gamma)\\&\lesssim\max_{\lambda\in\Gamma}\omega(\lambda)\sum_{j=0}^{\infty}2^{-j(2-n-\overline{s}(\boldsymbol{h})-\delta)}\\&\lesssim\max_{\lambda\in\Gamma}\omega(\lambda).\end{aligned} \tag{8.47}$$

By (8.42)–(8.43), (8.46)–(8.47) and the Dominated Convergence Theorem (cf. [24, p. 141, Theorem 5.8], for example), we conclude that (8.41) holds for all $x\in\Gamma$.

Now let $x\in A\setminus\Gamma$. As Γ is closed, $a:=\mathrm{dist}(x,\Gamma)>0$. We consider the open ball $B(x,a/2)$ and an arbitrary sequence $(x_j)_j\subset B(x,a/2)$ convergent to x. The function ω is continuous and, for each $\gamma\in\Gamma$ fixed, the function $G(\cdot,\gamma)$ is continuous. So, as x and x_j, $j\in\mathbb{N}$, are not elements of Γ,

$$\lim_{j\to\infty}G(x_j,\gamma)(\mathrm{tr}_\Gamma\,\omega)(\gamma)=G(x,\gamma)(\mathrm{tr}_\Gamma\,\omega)(\gamma),\quad\text{for all }\gamma\in\Gamma.$$

As $\omega\ge0$,

$$G(x_j,\gamma)(\mathrm{tr}_\Gamma\,\omega)(\gamma)\lesssim|x_j-\gamma|^{2-n}(\mathrm{tr}_\Gamma\,\omega)(\gamma)\le2^{n-2}a^{2-n}(\mathrm{tr}_\Gamma\,\omega)(\gamma),$$

which is independent of j and μ-integrable:

$$\int_\Gamma2^{n-2}a^{2-n}(\mathrm{tr}_\Gamma\,\omega)(\gamma)\,d\mu(\gamma)\lesssim2^{n-2}a^{2-n}\max_{\lambda\in\Gamma}\omega(\lambda)\mu(\Gamma)<\infty.$$

Again by the Dominated Convergence Theorem and by the continuity of ω one concludes that (8.41) holds.

Now we prove that either $\omega>0$ or $\omega=0$. We suppose that there is $x\in\Omega$ such that $\omega(x)>0$. So by (8.41), there is $\lambda\in\Gamma$ such that $(\mathrm{tr}_\Gamma\,\omega)(\lambda)>0$. As ω is continuous and Γ is an h-set, there is a set of positive μ-measure around λ where $\mathrm{tr}_\Gamma\,\omega>0$. But in this case, as $G(x,\gamma)>0$ for all $\gamma\in\Gamma\setminus\{x\}$ and $\mathrm{tr}_\Gamma\,\omega\ge0$, by (8.41), $\omega(y)>0$ for all $y\in\Omega$.

If $\omega>0$ then v is strictly negative and $u=-v$ is the eigenfunction we are looking for. If $\omega=0$ then $v\ge0$. Applying to v the arguments we used for ω we conclude that either $v=0$ or $v>0$. But as v is an eigenfunction, we must have $v>0$ and so $u=v$ is a positive eigenfunction.

Step 9: Let us prove that the largest eigenvalue is simple. That, together with Step 8, will also prove (iv). Assume that ρ is not simple. Let u and v be $\mathring{H}^1(\Omega)$-orthogonal eigenfunctions associated to ρ. By the previous step, the real functions $\mathrm{Re}\,u$, $\mathrm{Im}\,u$, $\mathrm{Re}\,v$ and $\mathrm{Im}\,v$ are also eigenfunctions associated to ρ, if they are different from 0. Assume that in this subset of functions there are two linearly independent functions. Then one can obtain two $\mathring{H}^1(\Omega)$-orthogonal real eigenfunctions associated to ρ.

Now assume that there are no two linearly independent functions in that set of 4 functions. So, in particular, $\mathrm{Re}\,u$ and $\mathrm{Im}\,u$ are linearly dependent. Then, either $\mathrm{Im}\,u=0$, or there is a real number α such that $\mathrm{Re}\,u=\alpha\mathrm{Im}\,u$. Therefore, either $u=\mathrm{Re}\,u$, or $u=(\alpha+i)\mathrm{Im}\,u$. In both cases there is a complex number $z_1\ne0$ such that z_1u

is a real function. Analogously one concludes that there is $z_2 \in \mathbb{C} \setminus \{0\}$ such that $z_2 v$ is a real function. So there are two $\mathring{H}^1(\Omega)$-orthogonal real eigenfunctions associated to ρ. In the previous step it was also proved that a real eigenfunction associated to the largest eigenvalue is strictly positive or strictly negative. Thus there exist a pair of $\mathring{H}^1(\Omega)$-orthogonal strictly positive eigenfunctions associated to ρ. To simplify the notation let us assume that u and v are such functions. Then

$$0 = (\rho v, u)_{\mathring{H}^1(\Omega)} = (Bv, u)_{\mathring{H}^1(\Omega)} = \int_\Gamma (\mathrm{tr}_\Gamma\, v)(\gamma)(\mathrm{tr}_\Gamma\, u)(\gamma)\, d\mu(\gamma),$$

which is a contradiction. □

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