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**BOOK OF EXTENDED  
ABSTRACTS**

## Some representations for the eigenfunctions of the time-fractional wave operator

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**Abstract:** In this work we present some new representations for the eigenfunctions of the time-fractional wave operator with the time-fractional derivative in the Caputo sense.

**keywords:** Time-fractional wave operator; Eigenfunctions; Caputo fractional derivative.

**MSC2010:** 35R11; 26A33; 35C15.

### 1 Introduction

The time-fractional wave equations are obtained from the standard wave equation by replacing the time derivative by a fractional derivative of order  $\beta \in ]1, 2]$ . This equation models anomalous wave propagation phenomena and has been studied over the last years by several authors. In the one-dimensional it has been studied comprehensively in several papers (see e.g. [5] and references therein). For the multidimensional case there are some works in this direction (see e.g. [2, 3] and references therein). In [2] it was obtained a representation for the eigenfunctions of the time-fractional diffusion-wave equation in terms of Kampé de Fériet and Lauricella series. In this paper, we focused in the one-dimensional case and we show alternative representations for the eigenfunctions of the time-fractional wave operator in terms of radial integrals, H-functions of two variables, and double series. Moreover, we present some plots of the eigenfunctions for some particular choices of the order of derivation and the eigenvalue.

### 2 Preliminaries

In this section we recall some basic definitions about fractional derivatives and special functions that are necessary for this work.

**Definition 1** (cf. [6, Def. 4]) *Let  $(a, b)$  be a finite or infinite interval on the real line  $\mathbb{R}$  and  $\alpha > 0$ . Also let  $\psi$  be a monotone increasing and positive function on  $(a, b)$ , having a continuous derivative  $\psi'$  in  $(a, b)$ . The left Riemann-Liouville fractional integral of a function  $f$  with respect to another function  $\psi$  on  $[a, b]$  is given by*

$$\left(I_{a^+}^{\alpha, \psi} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(w) (\psi(t) - \psi(w))^{\alpha-1} f(w) dw, \quad t > a. \quad (1)$$

Next, we give the definition of the so-called  $\psi$ -Hilfer fractional derivative of a function  $f$  with respect to another function.

**Definition 2** (cf. [6, Def. 7]) Let  $\alpha > 0$  and  $m = [\alpha] + 1$ , where  $[\alpha]$  denotes the integer part of  $\alpha$ . Let also  $I = [a, b]$  be a finite or infinite interval on the real line and  $f, \psi \in C^m[a, b]$  two functions such that  $\psi$  is a positive monotone increasing function and  $\psi'(t) \neq 0$ , for all  $t \in I$ . The left  $\psi$ -Hilfer left fractional derivative  ${}^H\mathbb{D}_{t,a^+}^{\alpha,\mu;\psi}$  of order  $\alpha$  and type  $\mu \in [0, 1]$  is defined by

$$\left({}^H\mathbb{D}_{a^+}^{\alpha,\mu;\psi} f\right)(t) = I_{a^+}^{\mu(m-\alpha),\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^m I_{a^+}^{(1-\mu)(m-\alpha),\psi} f(t).$$

We observe that when  $\mu = 1$  and  $\psi(t) = t$  with  $t \in \mathbb{R}^+$ , we recover the left Caputo fractional derivative of order  $\alpha > 0$  on  $[a, b] \subset \mathbb{R}$  (see [6, Sec. 5]), which is defined by

$$\left({}^CD_{a^+}^\alpha f\right)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(w)}{(t-w)^{\alpha-m+1}} dw. \quad (2)$$

One of the integral representations presented in this work involves the bivariate Mittag-Leffler function which has the following double series representation:

$$E_{(a_1,a_2),b}(z_1, z_2) = \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} \frac{(l_1+l_2)!}{l_1!l_2!} \frac{z_1^{l_1} z_2^{l_2}}{\Gamma(b+a_1l_1+a_2l_2)}.$$

We recall the H-function of two variables, which is defined via a double Mellin-Barnes type integral by (see [1])

$$\begin{aligned} H_{p_1,q_1;p_2,q_2;p_3,q_3}^{0,n_1;m_2,n_2;m_3,n_3} \left[ \begin{array}{c} z_1 \\ z_2 \end{array} \left| \begin{array}{c} (a_j; \alpha_j, A_j)_{1,p_1}; (c_j, \gamma_j)_{1,p_2}; (e_j, E_j)_{1,p_3} \\ (b_j; \beta_j, B_j)_{1,q_1}; (d_j, \delta_j)_{1,q_2}; (f_j, F_j)_{1,q_3} \end{array} \right. \right] \\ = \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_2} \int_{\mathcal{L}_1} \phi(s, w) \phi_1(s) \phi_2(w) z_1^s z_2^w ds dw, \end{aligned}$$

where

$$\begin{aligned} \phi(s, w) &= \frac{\prod_{i=1}^{n_1} \Gamma(1 - a_i + \alpha_i s + A_i w)}{\prod_{i=n_1+1}^{p_1} \Gamma(a_i - \alpha_i s - A_i w) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j s + B_j w)}, \\ \phi_1(s) &= \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \delta_j s) \prod_{i=1}^{n_2} \Gamma(1 - c_i + \gamma_i s)}{\prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + \delta_j s) \prod_{i=n_2+1}^{p_2} \Gamma(c_i - \gamma_i s)}, \quad \phi_2(w) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j w) \prod_{i=1}^{n_3} \Gamma(1 - e_i + E_i w)}{\prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + F_j w) \prod_{i=n_3+1}^{p_3} \Gamma(e_i - E_i w)}, \end{aligned}$$

and where an empty product is interpreted as 1,  $x, y \in \mathbb{C}$ ,  $m_i, n_i, p_i, q_i \in \mathbb{Z}$  such that  $0 \leq m_i \leq q_i$ ,  $0 \leq n_i \leq p_i$  ( $i = 1, 2, 3$ );  $a_i, b_j, c_i, d_j, e_i, f_j \in \mathbb{C}$ ,  $\alpha_i, A_i, \beta_j, B_j, \gamma_i, \delta_j, E_i, F_j \in \mathbb{R}^+$  and the sequence of parameters  $(a_j)$ ,  $(b_j)$ ,  $(c_j)$ ,  $(d_j)$ ,  $(e_j)$  and  $(f_j)$  are restricted that none of the poles of the integrand coincide. For more details about the convergence conditions of this function the interested reader can consult [1].

### 3 Integral and series representations for the eigenfunctions

In this section, we present some representations for the eigenfunctions of the time-fractional wave operator in terms of double Mellin-Barnes integrals, H-functions of two variables, and double series. Let us start considering the eigenfunction equation for the time-fractional wave operator

$$\left({}^CD_{t,0^+}^\beta - c_0^2 \partial_{xx}^2\right) u_\lambda^\beta(x, t) = \lambda u_\lambda^\beta(x, t), \quad (3)$$

subject to the following initial conditions

$$u_\lambda^\beta(x, 0) = \delta(x) \quad \text{and} \quad \partial_t u_\lambda^\beta(x, 0) = 0, \quad (4)$$

where  $x \in \mathbb{R}$ ,  $t \in \mathbb{R}^+$ ,  $\lambda \in \mathbb{C}$ ,  $\delta(x)$  is the distributional Dirac delta functions, and  ${}^C\partial_{t,0^+}^\beta$  is the partial time-fractional derivative in the Caputo sense of order  $1 < \beta \leq 2$  (see (2)). Problem (3)-(4) can be seen as a particular case of the problem studied in [4] where the authors solved the Cauchy problem associated to the time-fractional telegraph equation with  $\psi$ -Hilfer derivatives in  $\mathbb{R}^n \times \mathbb{R}^+$ :

$$c_2 {}^H\partial_{t,a^+}^{\alpha_2, \mu_2; \psi} u(x, t) + c_1 {}^H\partial_{t,a^+}^{\alpha_1, \mu_1; \psi} u(x, t) - c_0^2 \Delta_x u(x, t) + d u(x, t) = q(x, t), \quad (5)$$

with appropriate initial conditions. If we consider in (5)

$$n = 1, \quad c_2 = 1, \quad c_1 = -\lambda, \quad d = 0, \quad q(x, t) = 0, \quad \psi(t) = t, \quad \mu_1 = \mu_2 = 1, \quad \alpha_2 = \beta, \quad \alpha_1 = 0 \quad (6)$$

we recover immediately the one-dimensional case (3). In this sense, assuming (6) and the initial conditions (4) in some of the results presented in [4] we obtain the following representations of  $u_\lambda^\beta$ :

- In the form of radial integrals:

$$u_\lambda^\beta(x, t) = \frac{1}{\pi} \sum_{p=0}^{+\infty} (\lambda t^\beta)^p \int_0^{+\infty} E_{\beta, \beta p+1}^{p+1}(-c_0^2 w^2 t^\beta) \cos(|x|w) dw$$

$$- \frac{\lambda t^\beta}{\pi} \sum_{p=0}^{+\infty} (\lambda t^\beta)^p \int_0^{+\infty} E_{\beta, \beta(1+p)+1}^{p+1}(-c_0^2 w^2 t^\beta) \cos(|x|w) dw;$$

- In the form of H-functions of two variables:

$$u_\lambda^\beta(x, t) = \frac{1}{\sqrt{\pi}|x|} H_{1,1;1,0;0,1}^{0,1;0,1;1,0} \left[ \begin{matrix} \frac{4c_0^2 t^\beta}{|x|^2} \\ -\lambda t^\beta \end{matrix} \middle| \begin{matrix} (0; 1, 1); \left(\frac{1}{2}, 1\right); -- \\ (0; \beta, \beta); --; (0, 1) \end{matrix} \right]$$

$$- \frac{\lambda t^\beta}{\sqrt{\pi}|x|} H_{1,1;1,0;0,1}^{0,1;0,1;1,0} \left[ \begin{matrix} \frac{4c_0^2 t^\beta}{|x|^2} \\ -\lambda t^\beta \end{matrix} \middle| \begin{matrix} (0; 1, 1); \left(\frac{1}{2}, 1\right); -- \\ (-\beta; \beta, \beta); --; (0, 1) \end{matrix} \right]; \quad (7)$$

- In the form of double series:

$$u_\lambda^\beta(x, t) = \frac{1}{2c_0 t^{\frac{\beta}{2}}} \sum_{p=0}^{+\infty} \sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p}{\Gamma\left(1 - \frac{\beta}{2} + \beta p - \frac{\beta r}{2}\right) p! r!} (\lambda t^\beta)^p \left(-\frac{|x|}{c_0 t^{\frac{\beta}{2}}}\right)^r$$

$$- \frac{\lambda t^\beta}{2c_0} \sum_{p=0}^{+\infty} \sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p}{\Gamma\left(1 + \frac{\beta}{2} + \beta p - \frac{\beta r}{2}\right) p! r!} (\lambda t^\beta)^p \left(-\frac{|x|}{c_0 t^{\frac{\beta}{2}}}\right)^r. \quad (8)$$

## 4 Graphical representation of the eigenfunctions

In this section, we present and discuss some plots of the eigenfunctions  $u_\lambda^\beta$  for some particular choices of the fractional parameter  $\beta$  and the eigenvalue  $\lambda$ . For the generation of the plots we consider the series representation (8) with  $c_0 = 1$ ,  $x \in [-3, 3]$ , and  $t \in [0, 2]$ .

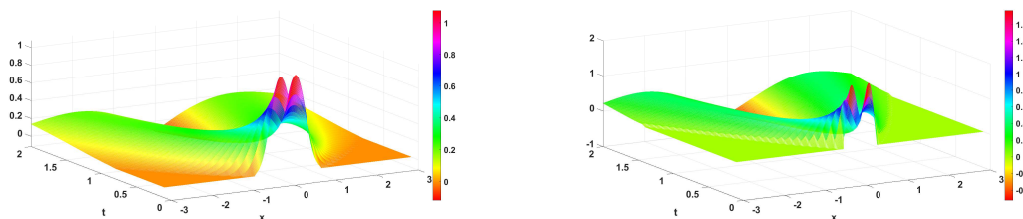


Figure 1: Plots of  $u_\lambda^\beta$  for  $\beta = 1.25$  and  $\lambda = 0.5$  (right), and  $\beta = 1.50$  and  $\lambda = 1$  (left).

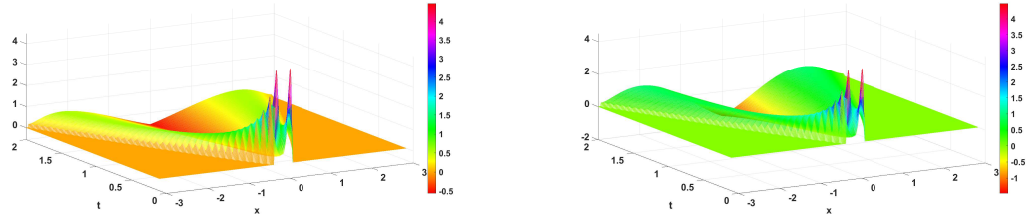


Figure 2: Plots of  $u_\lambda^\beta$  for  $\beta = 1.75$  and  $\lambda = 1$  (right), and  $\beta = 1.75$  and  $\lambda = 2$  (left).

From the analysis of Figures 1 and 2 we conclude that the eigenfunctions attain two symmetric maxima which decrease with time, and attains a minimum at  $x = 0$ . For all values of  $\lambda$  the eigenfunctions are continuous and differentiable with respect to the space variable. The obtained plots are in agreement with those obtained in [2].

## 5 Conclusions

In this work, we presented several representations for the eigenfunctions of the time-fractional wave equation in  $\mathbb{R} \times \mathbb{R}^+$ : in the form of radial series, in terms of H-functions of two variables, and as double series. The representation (7) coincides with the one presented in [2] when  $n = 1$  (see expression (3.6)). For particular values of  $\beta$  and  $\lambda$ , some plots of the eigenfunctions were presented and discussed.

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