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The fractal Dirichlet Laplacian

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Abstract An h-set is a non-empty compact subset of the Euclidean n-space which supports a finite Radon measure for which the measure of balls centered on the subset is essentially given by the image of their radius by a suitable function h. In most cases of interest such a subset has Lebesgue measure zero and has a fractal structure.

We prove the existence of solutions for the so-called fractal Dirichlet problem for such h-sets.

The "construction" of the solutions is based on the consideration of complete o.n. systems for convenient function spaces on the fractals. These systems are obtained combining properties of traces with integral representations (obtained with the help of Green's functions) for the so-called fractal Laplacian.

Keywords Fractals · Function spaces · Dirichlet problem · Laplacian · h-sets · Traces

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1 Introduction

Let $h : (0, 1] \to \mathbb{R}^+$ denote a continuous monotone increasing function such that $h(0^+) = 0$. Assume that there exist a non-empty compact set $\Gamma \subset \mathbb{R}^n$ and a finite

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Radon measure μ such that

$$\operatorname{supp}\mu = \Gamma \quad \text{and} \quad \mu(B(\gamma, r)) \sim h(r), \quad 0 < r \leq 1, \ \gamma \in \Gamma. \tag{1.1}$$

Such a set Γ is called an h-set. Two well-known classes of h-sets are d-sets and (d, ψ)-sets. In these particular cases the function h is

$$h(r) = r^d \quad \text{and} \quad h(r) = r^d \psi(r), \tag{1.2}$$

respectively, where $d > 0$ and $\psi : (0, 1] \to \mathbb{R}^+$ is a monotone function such that

$$\psi(2^{-j}) \sim \psi(2^{-2j}), \quad j \in \mathbb{N}.$$

We refer to [3–5], where h-sets were studied and a characterisation of the functions h for which there exist h-sets was obtained.

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ such that $\Gamma \subset \Omega$.

In this paper we prove the existence of a solution for the fractal Dirichlet problem for h-sets, i.e., we prove that, under some conditions, given g in a convenient function space on an h-set $\Gamma \subset \Omega$, there exists u such that

$$u \in H^1(\Omega), \quad \Delta u(x) = 0 \quad \text{in } \Omega \setminus \Gamma,$$

and

$$\operatorname{tr}_{\partial\Omega} u = 0, \qquad \operatorname{tr}_\Gamma u = g.$$

We prove the existence of a solution making use of the results for the operator B defined by

$$B := (-\Delta)^{-1} \circ \operatorname{tr}^\Gamma, \tag{1.3}$$

acting in convenient function spaces in Ω, where $(-\Delta)^{-1}$ is the inverse of the Dirichlet Laplacian in Ω and

$$\operatorname{tr}^\Gamma = \operatorname{id}^\Gamma \circ \operatorname{tr}_\Gamma, \tag{1.4}$$

where tr_Γ is an extension of the operator

$$\varphi \to \varphi_{|\Gamma}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

and id^Γ, which will be formally defined later, identifies elements of $L_p(\Gamma)$ with tempered distributions.

Applying some results on the fractal Laplacian (i.e., on the operator B), complete o.n. systems for convenient function spaces are obtained and used to construct a solution for the fractal Dirichlet problem.

The operator B was already studied in the case where Γ is a d-set by Triebel in [27, Chap. 5] and [28, Chap. 3]. The case where Γ is a (d, ψ)-set was studied by Edmunds and Triebel in [9] and by Moura in [19]. The case where Γ is an h-set was studied in [6]. As it was mentioned in these works, in the case $n = 2$ the operator B has physical relevance: it describes the vibration of a drum where the whole mass is distributed on Γ.

The fractal Dririchlet problem was studied by Triebel for the case of d-sets (cf. [28, Chap. 3, Sect. 20]).

The paper is organised as follows. In Sect. 2 we collect some general notation and we collect some definitions and results on function spaces on domains. In Sect. 3 fractal h-sets and their properties are presented. In Sect. 4 we collect some results on the fractal Laplacian, represented previously by B. Finally, in Sect. 5 we prove the existence of at least one solution for the fractal Dirichlet Laplacian.

2 Preliminaries

In this chapter we fix some notation. In the first section we present some general notation. In the second one we present some of the function spaces considered in this paper.

2.1 General notation

We introduce some standard notation and useful definitions. We write $\mathbb{N}$ for the set of all natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of all real and complex numbers, respectively. As usually, $\mathbb{R}^n$, $n \in \mathbb{N}$, stands for the n-dimensional real Euclidean space and, given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $|x|$ stands for the Euclidean norm of x. We write $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, to represent the collection of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_j \in \mathbb{N}_0$, $j = 1, \ldots, n$. For $\alpha \in {\mathbb{N}_0}^n$, $D^\alpha := \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n})$ denotes the classic or weak partial derivative of order α.

If there is no additional information, when we speak about "functions" we are considering complex-valued functions and references with respect to measurability and integrability should be understood in the Lebesgue sense. We write supp to denote the support of a function, a distribution or a measure.

If $E \subset \mathbb{R}^n$, then ∂E denotes the boundary of E and $\overline{E}$ stands for the closure of E.

We use the symbol "$\lesssim$" in

$$a_k \lesssim b_k \quad \text{or} \quad \phi(r) \lesssim \psi(r)$$

always to mean that there is a positive number c_1 such that

$$a_k \leq c_1 b_k \quad \text{or} \quad \phi(r) \leq c_1 \psi(r)$$

for all admitted values of the discrete variable k or the continuous variable r, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and ϕ, ψ are non-negative functions. We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \phi(r) \sim \psi(r)$$

for

$$a_k \lesssim b_k \quad \text{and} \quad b_k \lesssim a_k \quad \text{or} \quad \phi(r) \lesssim \psi(r) \quad \text{and} \quad \psi(r) \lesssim \phi(r).$$

In what follows log is always taken with respect to the base 2.

If T is an operator we denote by $N(T)$ the null-space of T.

If S denotes a subspace of a space with a scalar product then we write $S^{\perp}$ to represent the orthogonal complement of S.

All unimportant constants are denoted by c and may sometimes represent different constants in a single chain of inequalities. Sometimes we distinguish them using different representations (c_1 and c', for instance).

As usual, "domain" stands for non-empty "open set". In this paper we consider bounded C^∞ domains in $\mathbb{R}^n$. For the definition of bounded C^∞ domain we refer to [24, Sect. 3.2.1, p. 191], for example.

2.2 Function spaces

We write $C^\infty(\mathbb{R}^n)$ to denote the class of all infinitely differentiable functions. We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all rapidly decreasing functions in $C^\infty(\mathbb{R}^n)$, equipped with the usual topology, and by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, the space of all tempered distributions on $\mathbb{R}^n$.

If Ω is a domain in $\mathbb{R}^n$ then $L_p(\Omega)$, $0 < p < \infty$, denotes the collection of all complex-valued Lebesgue measurable functions in Ω such that

$$\|f \mid L_p(\Omega)\| := \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}$$

is finite. Let $1 < p < \infty$, $m \in \mathbb{N}$ and $s \in \mathbb{R}$. We write $W_p^m(\mathbb{R}^n)$ and $H_p^s(\mathbb{R}^n)$ to denote, respectively, the Sobolev and the Bessel-potential spaces on $\mathbb{R}^n$. If Ω is a domain, then the spaces $W_p^m(\Omega)$ and $H_p^s(\Omega)$ are defined by restriction of the corresponding spaces on $\mathbb{R}^n$. For more details we refer to [6], for example. In this paper we will work with the spaces $W_2^1(\Omega) = H_2^1(\Omega)$ and $H_2^{-1}(\Omega)$. We will abbreviate the notation denoting these spaces by $H^1(\Omega)$ and $H^{-1}(\Omega)$, respectively.

If Ω is a domain, $\mathcal{D}(\Omega)$ is the collection of all compactly supported complex-valued C^∞ functions in Ω and $\mathcal{D}'(\Omega)$ stands for the dual space of all distributions on Ω. We denote by $\mathring{H}^1(\Omega)$ the completion of $\mathcal{D}(\Omega)$ in $\mathring{H}^1(\Omega)$.

Let us collect some more notation and results with respect to the space $\mathring{H}^1(\Omega)$.

Definition 2.1 Let Ω be a domain in $\mathbb{R}^n$. We define, for $f, g \in \mathring{H}^1(\Omega)$,

$$(f, g)_{\mathring{H}^1(\Omega)} := \sum_{j=1}^n \int_\Omega \frac{\partial f}{\partial x_j}(x) \frac{\partial \overline{g}}{\partial x_j}(x) dx. \tag{2.1}$$

Proposition 2.2 *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$.*

(i) *Then*

$$\mathring{H}^1(\Omega) = \{f \in H^1(\Omega) : \mathrm{tr}_{\partial\Omega} f = 0\}.$$

(ii) *For all $f \in \mathring{H}^1(\Omega)$*

$$\|f \mid H^1(\Omega)\|^2 \sim (f, f)_{\mathring{H}^1(\Omega)}.$$

Remark 2.3 The characterisation given in (i) can be found in [28, p. 255]. The result (ii) follows from Friedrich's inequality (cf. [25, p. 357], for example) and it can be found in [27, p. 195].

Remark 2.4 In what follows, for technical reasons, in the space $\mathring{H}^1(\Omega)$ we will not consider the norm inherited from $H^1(\Omega)$, but the equivalent norm given by

$$\|f \mid \mathring{H}^1(\Omega)\| := \sqrt{(f, f)_{\mathring{H}^1(\Omega)}} = \sqrt{\sum_{j=1}^{n} \int_{\Omega} \left| \frac{\partial f}{\partial x_j}(x) \right|^2 dx}, \quad f \in \mathring{H}^1(\Omega). \tag{2.2}$$

That this can be done follows from Proposition 2.2, (ii).

3 Fractal *h*-sets

3.1 Definition and some properties

In this paper a particular class of fractals are considered: the so-called h-sets. In this section we introduce h-sets and present some of their properties.

In the designation *h-set*, the h denotes a function, which shall be a gauge function.

Definition 3.1 Let $\mathbb{H}$ denote the class of all continuous monotone increasing functions $h : (0, \infty) \to (0, \infty)$ such that $h(0^+) = 0$. We refer to $\mathbb{H}$ as the set of all *gauge functions*.

Definition 3.2 Let $h \in \mathbb{H}$ and Γ be a non-empty compact set of $\mathbb{R}^n$. We say that Γ is an *h-set* if there exists a finite Radon measure μ such that

$$\operatorname{supp} \mu = \Gamma$$

and

$$\mu(B(\gamma, r)) \sim h(r), \quad 0 < r \leq 1, \ \gamma \in \Gamma.$$

Then we say that h is a *measure function* (in $\mathbb{R}^n$) and that μ is an *h-measure* (related to Γ).

Remark 3.3 The h-measures are also designated by *isotropic measures* (cf. [29, p. 95]).

If the function h is given by

$$h(r) = r^d \psi(r), \quad 0 < r \leq 1,$$

where $0 < d \leq n$ and $\psi : (0, 1] \to \mathbb{R}^+$ is a monotone function such that

$$\psi(2^{-j}) \sim \psi(2^{-2j}), \quad \text{for all } j \in \mathbb{N}_0,$$

then we say that Γ is a (d, ψ)*-set*. If, additionally, $\psi \sim 1$ then we say that Γ is a d*-set*. Therefore the class of h-sets is a generalisation of the class of (d, ψ)-sets, which is itself a generalisation of d-sets. There are many authors studying these classes of fractal sets, both in fractal geometry and in the theory of function spaces. In the case of d-sets we refer to [14, 17] and [27] for example. For (d,ψ)-sets we refer to [9, 20] and [19]. As far as h-sets are concerned we refer to [4, 5, 13] and [16].

Some well-known self-similar fractals are examples of d-sets, namely the Cantor set in $\mathbb{R}^1$ is a d-set for $d = \log 2/\log 3$ and the von Koch curve in $\mathbb{R}^2$ is a d-set for $d = \log 4/\log 3$.

Bricchi characterised in [4] which functions h are measure functions with the following outcome.

Theorem 3.4 *Let $h \in \mathbb{H}$. Then h is a measure function in $\mathbb{R}^n$ if, and only if, there exists a gauge function $\widetilde{h} \sim h$ such that*

$$\frac{\tilde{h}(2^{-(j+k)})}{\tilde{h}(2^{-j})} \geq 2^{-kn}, \quad j,k \in \mathbb{N}_0.$$

Remark 3.5 In the case of d-sets the result above reads as follows: let $n \in \mathbb{N}$ and $d > 0$. Then there exists a d-set if and only if $d \leq n$.

Shifting from d-sets and (d,ψ)-sets to h-sets it is often convenient to find appropriate numbers to play the role of the number d. In this paper we consider indices of a sequence obtained from the function h.

Notation 3.6 In what follows, for $h \in \mathbb{H}$, we denote by $\boldsymbol{h}$ the sequence

$$\boldsymbol{h} := (h(2^{-j}))_{j\in\mathbb{N}_0}. \tag{3.1}$$

Definition 3.7 Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be a sequence of positive numbers and assume that there exist positive $d_0, d_1 > 0$ such that

$$d_0\sigma_j \leq \sigma_{j+1} \leq d_1\sigma_j, \quad j \in \mathbb{N}_0. \tag{3.2}$$

Let

$$\underline{\sigma}_j := \inf_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\sigma}_j := \sup_{k\in\mathbb{N}_0} \frac{\sigma_{j+k}}{\sigma_k}, \quad j \in \mathbb{N}_0.$$

The *lower* and *upper Boyd indices* of the sequence σ are defined, respectively, by

$$\underline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \underline{\sigma}_j}{j} \quad \text{and} \quad \overline{s}(\sigma) := \lim_{j\to\infty} \frac{\log \overline{\sigma}_j}{j}.$$

Remark 3.8 The sequences satisfying the conditions referred in Definition 3.7 are usually called *admissible sequences*.

If $h \in \mathbb{H}$ and if there is an h-set, then $\boldsymbol{h}$ is an admissible sequence.

For an admissible sequence σ, the sequence $(\log \overline{\sigma}_j)_{j\in\mathbb{N}_0}$ is sub-additive. This justifies the definition of $\overline{s}(\sigma)$. As $\log \underline{\sigma}_j = -\log \overline{(\sigma^{-1})}_j$, $\underline{s}(\sigma)$ also makes sense.

We remark that if σ and β are admissible sequences such that $\sigma \sim \beta$, then their Boyd indices coincide.

We will apply frequently the following property: for each $\delta > 0$ there are two positive constants $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$ such that for all $j, k \in \mathbb{N}_0$,

$$c_1 2^{(\underline{s}(\sigma)-\delta)j} \leq \frac{\sigma_{j+k}}{\sigma_k} \leq c_2 2^{(\overline{s}(\sigma)+\delta)j}. \tag{3.3}$$

Admissible sequences and Boyd indices usually appear in the context of spaces of generalised smoothness. We refer to [7], for example, where some useful properties of these indices are given.

We now present a definition about a geometric property of sets.

Definition 3.9 A non-empty Borel set Γ satisfies the *ball condition* (or *porosity condition*) if there exists a number $0 < \eta < 1$ with the following property:

for any ball $B(\gamma, r)$ with $\gamma \in \Gamma$ and $0 < r \leq 1$ there is a ball $B(x, \eta r)$ centred at $x \in \mathbb{R}^n$ such that

$$B(x, \eta r) \subset B(\gamma, r) \quad \text{and} \quad B(x, \eta r) \cap \overline{\Gamma} = \emptyset.$$

In the next proposition a necessary and sufficient condition for an h-set to satisfy the ball condition is stated. It follows from [28, pp. 139–140, Proposition 9.18].

Proposition 3.10 *Let $\Gamma \subset \mathbb{R}^n$ be an h-set. Then Γ satisfies the ball condition if, and only if,*

$$\underline{s}(\boldsymbol{h}) > -n, \tag{3.4}$$

with $\underline{s}(\boldsymbol{h})$ according to Definitions 3.7 and (3.1).

Remark 3.11 If $h(r) = r^d$, $r > 0$, then (3.4) is equivalent to $d < n$.

3.2 Trace and identification operators

In this section we introduce function spaces on h-sets, as trace of function spaces on domains.

Let μ be an h-measure in $\mathbb{R}^n$ according to Definition 3.2. As usually, $L_2(\Gamma, \mu)$ (or simply $L_2(\Gamma)$) denotes the usual complex Banach space with respect to the related measure μ, quasi-normed by

$$\|f \mid L_2(\Gamma, \mu)\| := \left(\int_\Gamma |f(\gamma)|^2 d\mu(\gamma) \right)^{1/2}.$$

The next proposition is a consequence of [3, pp. 99–102, Theorem 3.3.1].

Proposition 3.12 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain in $\mathbb{R}^n$ and Γ be an h-set such that $\Gamma \subset \Omega$. Assume that*

$$n - 2 < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) < n. \tag{3.5}$$

Then there exists a positive number c such that

$$\|\varphi|_\Gamma \,|\, L_2(\Gamma)\| \le c\, \|\varphi \,|\, \mathring{H}^1(\Omega)\|, \quad \varphi \in \mathcal{D}(\Omega). \tag{3.6}$$

Definition 3.13 Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain and $\Gamma \subset \Omega$ be an h-set. Assume that (3.5) is satisfied. Let $f \in \mathring{H}^1(\Omega)$. Then there is $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{D}(\Omega)$ such that $\varphi_j \to f$ as $j \to \infty$ in $\mathring{H}^1(\Omega)$.

We define

$$\mathrm{tr}_\Gamma\, f := \lim_{j\to\infty} (\varphi_{j\,|\Gamma}) \quad \text{in } L_2(\Gamma). \tag{3.7}$$

Remark 3.14 The existence of the limit in (3.7) is justified by Proposition 3.12.

Next we define a function space on h-sets.

Definition 3.15 Let Ω be a bounded C^∞ domain and $\Gamma \subset \Omega$ be an h-set. Assume that (3.5) is satisfied. Let σ be the sequence

$$\sigma := (2^j h(2^{-j})^{-\frac{1}{2}} 2^{-\frac{nj}{2}})_{j\in\mathbb{N}_0}. \tag{3.8}$$

We define

$$\mathbb{H}^\sigma(\Gamma) = \mathrm{tr}_\Gamma\ \mathring{H}^1(\Omega) \tag{3.9}$$

endowed with the quasi-norm

$$\|f \,|\, \mathbb{H}^\sigma(\Gamma)\| = \inf \|g \,|\, \mathring{H}^1(\Omega)\|,$$

where the infimum is taken over all $g \in \mathring{H}^1(\Omega)$ such that $\mathrm{tr}_\Gamma\, g = f$.

Remark 3.16 The spaces presented in Definition 3.15 are included in the Besov spaces on h-sets considered in [13], which are particular class of Besov spaces on fractal h-sets considered in [3, 16] and [7]. Generally, given an admissible sequence σ with $\underline{s}(\sigma > 0)$ (we refer to Definition 3.7 and Remark 3.8), $0 < p, q \le \infty$ and an h-set $\Gamma \subset \mathbb{R}^n$ such that $\underline{s}(\boldsymbol{h}) > -n$, then

$$\mathbb{B}^\sigma_{p,q}(\Gamma) = \mathrm{tr}_\Gamma\ B^{\sigma \boldsymbol{h}^{1/p}(n)^{1/p}}_{p,q}(\mathbb{R}^n), \tag{3.10}$$

where

$$\sigma \boldsymbol{h}^{1/p}(n)^{1/p} = (\sigma_j h(2^{-j})^{\frac{1}{p}} 2^{\frac{jn}{p}})_{j\in\mathbb{N}_0},$$

the spaces on the right side of (3.10) are Besov spaces of generalised smoothness on $\mathbb{R}^n$ and the definition of tr_Γ is analogous to Definition 3.13. For the definition and several characterisations and properties of Besov spaces of generalised smoothness on $\mathbb{R}^n$ we refer to [3, 8, 10, 11, 15, 18] by Bricchi [2, 19, 21] and [7].

Next we present an operator which identifies an element of $L_p(\Gamma)$, where Γ is an h-set, with a distribution on a bounded C^∞ domain, Ω, which contains Γ.

Definition 3.17 Let $1 \le p \le \infty$ and Γ be an h-set in $\mathbb{R}^n$. We define

$$\mathrm{id}^\Gamma : L_p(\Gamma) \to \mathcal{D}'(\Omega) \tag{3.11}$$

by

$$\langle \mathrm{id}^\Gamma f, \varphi \rangle := \int_\Gamma f(\gamma)(\varphi_{|\Gamma})(\gamma) d\mu(\gamma), \quad \varphi \in \mathcal{D}(\Omega). \tag{3.12}$$

The result in the next proposition is a consequence of [3, p. 92, Theorem 3.2.8].

Proposition 3.18 *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and $\Gamma \subset \Omega$ be an h-set. Let $-\underline{s}(\boldsymbol{h}) > n-2$. Then*

$$\mathrm{id}^\Gamma : L_2(\Gamma) \to H^{-1}(\Omega). \tag{3.13}$$

4 The fractal Laplacian

In this section we present the fractal Laplacian and we collect some results on this operator which will be applied to construct a solution to a fractal Dirichlet problem.

Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$. As usual

$$-\Delta := -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$$

is the Laplacian in Ω. We use the notation *Dirichlet Laplacian* always with the understanding that the vanishing boundary data at $\partial\Omega$ are incorporated in the definition of the domain of $-\Delta$ in the function spaces considered, as can be seen in the next proposition. The result presented there follows from [26, Remark 1, Sect. 5.7.1], [24, Sects. 4.3.3, 4.3.4] and [22, Sect. 3.5.2, p. 130].

Proposition 4.1 *Let Ω be a bounded C^∞ domain. Then $-\Delta$ maps*

$$\{g \in H^1(\Omega) : \mathrm{tr}_{\partial\Omega}\, g = 0\} = \mathring{H}^1(\Omega) \quad \textit{isomorphically onto } H^{-1}(\Omega).$$

We write $(-\Delta)^{-1}$ to denote the inverse of the Dirichlet Laplacian. Let Γ be an h-set such that $\Gamma \subset \Omega$. We define the fractal Laplacian by

$$B := (-\Delta)^{-1} \circ \mathrm{tr}^\Gamma \tag{4.1}$$

and in this paper we consider this operator acting in $\mathring{H}^1(\Omega)$.

The Green's functions, presented in the next proposition, will be applied to get a explicit representation of Bf, for all $f \in \mathring{H}^1(\Omega)$.

Proposition 4.2 *Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \ge 2$. There is a function G,*

$$G : \overline{\Omega} \times \Omega \to \mathbb{R},$$

usually called Green's function such that

(i) *for all* $x^0 \in \Omega$ *and* $\varepsilon > 0$,

$$G(x^0, \cdot) \in C^\infty(\Omega \setminus \overline{B(x^0, \varepsilon)});$$

(ii) *for all* $x, y \in \Omega$, *with* $x \neq y$, $G(x, y) = G(y, x)$;

(iii) *if* $n \geq 3$

$$0 < G(x, y) \lesssim |x - y|^{2-n}, \quad x, y \in \Omega, \ x \neq y,$$

and, if $n = 2$,

$$0 < G(x, y) \lesssim \max_{z \in \partial\Omega} \ln|z - x| - \ln|x - y|, \quad x, y \in \Omega, \ x \neq y;$$

(iv) $G(x, y) = 0$, *for all* $x \in \partial\Omega$ *and* $y \in \Omega$;

(v) *for all* $\varphi \in \mathcal{D}(\Omega)$,

$$(-\Delta)^{-1}\varphi(x) = \int_\Omega G(x, y)\varphi(y)dy, \quad x \in \Omega. \tag{4.2}$$

Therefore, for all $f \in H^{-1}(\Omega)$,

$$(-\Delta)^{-1} f = \lim_{j \to \infty} \int_\Omega G(\cdot, y)\varphi_j(y)dy \quad in\ \mathring{H}^1(\Omega),$$

where

$$(\varphi_j)_{j \in \mathbb{N}_0} \subset \mathcal{D}(\Omega) \quad with\ f = \lim_{j \to \infty} \varphi_j\ in\ H^{-1}(\Omega). \tag{4.3}$$

Remark 4.3 In (4.3) we used the fact that $\mathcal{D}(\Omega)$ is dense in $H^{-1}(\Omega)$, which can be justified as follows: the restriction of $\mathcal{S}(\mathbb{R}^n)$ to Ω is dense in $H^{-1}(\Omega)$. Any function in the restriction of $\mathcal{S}(\mathbb{R}^n)$ to Ω can be approximated in $L_2(\Omega)$ by functions belonging to $\mathcal{D}(\Omega)$. But this is also an approximation in $H^{-1}(\Omega)$.

For the results in the previous proposition we refer to [28, p. 299], [12, pp. 160–163, 273].

In connection with the Green's function we also refer to [25, pp. 145, 194–196], [28, pp. 243–244], [29, p. 301] and [1, pp. 10–13].

The proof of the next theorem, which extends the results in [28, Theorem 19.7] and [20, Theorem 4.1.7], can be found in [6].

Theorem 4.4 *Let* $h \in \mathbb{H}$ *be a strictly increasing function. Let* Ω *be a bounded* C^∞ *domain in* $\mathbb{R}^n$, $n \geq 2$, *and* Γ *be an* h*-set such that* $\Gamma \subset \Omega$ *and*

$$n - 2 < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) < n. \tag{4.4}$$

Then B *is a non-negative compact self-adjoint operator in* $\mathring{H}^1(\Omega)$ *with null-space*

$$N(B) = \{f \in \mathring{H}^1(\Omega) : \mathrm{tr}_\Gamma f = 0\}. \tag{4.5}$$

Moreover, B is generated by the sesquilinear form

$$(Bf, g)_{\mathring{H}^1(\Omega)} = \int_\Gamma (\text{tr}_\Gamma f)(\gamma)\overline{(\text{tr}_\Gamma g)(\gamma)}d\mu(\gamma), \quad f, g \in \mathring{H}^1(\Omega), \tag{4.6}$$

with (2.1) as the scalar product in $\mathring{H}^1(\Omega)$. Furthermore, B is given by

$$Bf = \int_\Gamma G(\cdot, \gamma)(\text{tr}_\Gamma f)(\gamma)d\mu(\gamma), \quad f \in \mathring{H}^1(\Omega), \tag{4.7}$$

where G is the Green's function referred in Proposition 4.2.

Let ρ_k denote the positive eigenvalues of B repeated according to multiplicity and ordered by decreasing order of their magnitude and u_k denote related eigenfunctions

$$Bu_k = \rho_k u_k, \quad k \in \mathbb{N}.$$

(i) *The largest eigenvalue is simple, i.e.,*

$$\rho_1 > \rho_2 \geq \rho_3 \cdots$$

and, for all $k \in \mathbb{N}$,

$$\rho_k \sim k^{-1} H(k^{-1})^{2-n}, \tag{4.8}$$

where H denotes the inverse function of h.

(ii) *The eigenfunctions u_k are (classical) harmonic functions in $\Omega \setminus \Gamma$,*

$$\Delta u_k(x) = 0 \quad \textit{if } x \in \Omega \setminus \Gamma. \tag{4.9}$$

(iii) *The eigenfunctions $u_1(x)$ have no zeros in Ω*

$$u_1(x) = cu(x) \quad \textit{with } c \in \mathbb{C} \quad \textit{and} \quad u(x) > 0 \quad \textit{if } x \in \Omega.$$

Remark 4.5 In [6] there are some more results on the eigenfunctions associated to the operator B, which are not presented here. It is presented a class of Besov spaces of generalised smoothness where the eigenfunctions do and do not belong to.

5 The fractal Dirichlet Laplacian—constructing a solution

In this section we study a fractal Dirichlet Laplacian in the context of h-sets. This problem in the context of d-sets was considered by Triebel in [28, Sect. 20].

In this section we will assume that conditions in Theorem 4.4 are satisfied.

Assumption 5.1 *Let $h \in \mathbb{H}$ be a strictly increasing function. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$, and Γ be an h-set such that $\Gamma \subset \Omega$ and*

$$n - 2 < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) < n.$$

We will denote by B the operator given by (4.1) considered acting in $\mathring{H}^1(\Omega)$, as in Theorem 4.4.

5.1 The spaces $\mathbb{H}^\sigma(\Gamma)$: a scalar product and the dual space

In the result presented next we establish the existence of an isometry between $\mathbb{H}^\sigma(\Gamma)$ and a subspace of $\mathring{H}^1(\Omega)$. The existence of such an operator gives us the possibility to define a scalar product on these function spaces on fractals.

Proposition 5.2 *The operator*

$$\mathrm{tr}_\Gamma : N(B)^\perp \to \mathbb{H}^\sigma(\Gamma) \tag{5.1}$$

is an isometry.

Proof Step 1: The proof of injectivity and surjectivity relies in (4.5). Let us present the proof of surjectivity. Let $f \in \mathbb{H}^\sigma(\Gamma)$. Then there is $g \in \mathring{H}^1(\Omega)$ such that $\mathrm{tr}_\Gamma\, g = f$. As $g \in \mathring{H}^1(\Omega)$, there exist $u \in N(B)$ and $v \in N(B)^\perp$ for which $g = u + v$. By (4.5), $\mathrm{tr}_\Gamma\, u = 0$ and so $\mathrm{tr}_\Gamma\, v = \mathrm{tr}_\Gamma\, g = f$.

Step 2: Let us prove that

$$\| \mathrm{tr}_\Gamma\, u \mid \mathbb{H}^\sigma(\Gamma)\| = \|u \mid \mathring{H}^1(\Omega)\|,$$

for $u \in N(B)^\perp$.

$$\begin{aligned}\| \mathrm{tr}_\Gamma\, u \mid \mathbb{H}^\sigma(\Gamma)\| &= \inf\{\|v \mid \mathring{H}^1(\Omega)\| : v \in \mathring{H}^1(\Omega), \mathrm{tr}_\Gamma\, v = \mathrm{tr}_\Gamma\, u\}\\ &= \inf\{\|v \mid \mathring{H}^1(\Omega)\| : v = u + \omega, \omega \in N(B)\}\\ &= \|u \mid \mathring{H}^1(\Omega)\|,\end{aligned}$$

where we used the fact that for $v = u + \omega$, with $u \in N(B)^\perp$ and $\omega \in N(B)$,

$$\|v \mid \mathring{H}^1(\Omega)\|^2 = \|u \mid \mathring{H}^1(\Omega)\|^2 + \|\omega \mid \mathring{H}^1(\Omega)\|^2 \geq \|u \mid \mathring{H}^1(\Omega)\|^2.$$

So, in (5.1) we have an isometry. □

Definition 5.3 We define a *scalar product in* $\mathbb{H}^\sigma(\Gamma)$ as follows

$$(f, g)_{\mathbb{H}^\sigma(\Gamma)} := (\mathrm{tr}_\Gamma^{-1} f, \mathrm{tr}_\Gamma^{-1} g)_{\mathring{H}^1(\Omega)}, \quad f, g \in \mathbb{H}^\sigma(\Gamma),$$

where tr_Γ is the bijective operator referred in Proposition 5.2.

Next we define the dual space of $\mathbb{H}^\sigma(\Gamma)$ and a scalar product for this space.

Definition 5.4

(i) Let

$$D(\Gamma) := \mathrm{tr}_\Gamma\, \mathcal{S}(\mathbb{R}^n) \quad \text{and} \quad D'(\Gamma) := (D(\Gamma))'.$$

(ii) We define

$$\mathbb{H}^{\sigma^{-1}}(\Gamma) := (\mathbb{H}^\sigma(\Gamma))'$$

with respect to the dual paring $(D(\Gamma), D'(\Gamma))$.

By the Riesz representation theorem there is an isomorphic map

$$\mathrm{id}_H : \mathbb{H}^{\sigma^{-1}}(\Gamma) \to \mathbb{H}^{\sigma}(\Gamma) \tag{5.2}$$

such that, for all $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$,

$$\begin{aligned} &\langle F, g\rangle = (g, \mathrm{id}_H\, F)_{\mathbb{H}^{\sigma}(\Gamma)}, \quad g \in \mathbb{H}^{\sigma}(\Gamma), \quad \text{and} \\ &\| F \mid \mathbb{H}^{\sigma^{-1}}(\Gamma)\| = \| \mathrm{id}_H\, F \mid \mathbb{H}^{\sigma}(\Gamma)\|. \end{aligned} \tag{5.3}$$

Based on this, we define a scalar product in $\mathbb{H}^{\sigma^{-1}}(\Gamma)$.

Definition 5.5 We define a *scalar product in* $\mathbb{H}^{\sigma^{-1}}(\Gamma)$ as follows

$$(F, G)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} := (\mathrm{id}_H\, F, \mathrm{id}_H\, G)_{\mathbb{H}^{\sigma}(\Gamma)}, \quad F, G \in \mathbb{H}^{\sigma^{-1}}(\Gamma),$$

where id_H is as in (5.2)–(5.3).

Remark 5.6 By definition it is clear that $\mathbb{H}^{\sigma}(\Gamma) \subset L_2(\Gamma)$. Let $f \in L_2(\Gamma)$. Then f can be interpreted as a functional in $L_2(\Gamma)$:

$$\langle f, g\rangle = \int_\Gamma g(\gamma)\overline{f(\gamma)}d\mu(\gamma), \quad g \in L_2(\Gamma).$$

As $\mathbb{H}^{\sigma}(\Gamma) \subset L_2(\Gamma)$, then $f \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$. So in this sense $\mathbb{H}^{\sigma}(\Gamma) \subset L_2(\Gamma) \subset \mathbb{H}^{\sigma^{-1}}(\Gamma)$.

5.2 The operator B^Γ and complete o.n. systems

Proposition 5.7 *Consider that conditions in Assumption 5.1 are satisfied. Let*

$$B^\Gamma := \mathrm{tr}_\Gamma \circ B \circ (\mathrm{tr}_\Gamma)^{-1}. \tag{5.4}$$

Then B^Γ is a non-negative compact self-adjoint operator in $\mathbb{H}^{\sigma}(\Gamma)$ with null-space $N(B^\Gamma) = \{0\}$. Furthermore B^Γ is generated by the sesquilinear form

$$(B^\Gamma f, g)_{\mathbb{H}^{\sigma}(\Gamma)} = \int_\Gamma f(\gamma)\overline{g(\gamma)}d\mu(\gamma), \quad f, g \in \mathbb{H}^{\sigma}(\Gamma). \tag{5.5}$$

The positive eigenvalues of B and B^Γ coincide. Let $(\rho_k)_{k\in\mathbb{N}}$ denote the sequence of the positive eigenvalues of B^Γ repeated according to multiplicity and ordered by decreasing order of their magnitude, as in Theorem 4.4. Then there is complete orthonormal system in $\mathbb{H}^{\sigma}(\Gamma)$, $(u_k)_{k\in\mathbb{N}}$, such that

$$B^\Gamma u_k = \rho_k u_k, \quad k \in \mathbb{N}. \tag{5.6}$$

Proof Step 1: Let us prove that the image of the operator B is a subset of $N(B)^\perp$. Let $f \in \mathring{H}^1(\Omega)$ arbitrary. Let us prove that $Bf \in N(B)^\perp$. Consider $g \in N(B)$ arbitrary. Then, by Theorem 4.4, $\mathrm{tr}_\Gamma\, g = 0$ and $(Bf, g)_{\mathring{H}^1(\Omega)} = 0$.

Step 2: By Theorem 4.4, B^Γ is compact operator. Let us prove that it is generated by the sesquilinear form (5.5) and, consequently, a non-negative self-adjoint operator. Let $f, g \in \mathbb{H}^\sigma(\Gamma)$. Then, by (4.6) and by the previous step,

$$\begin{aligned}(B^\Gamma f, g)_{\mathbb{H}^\sigma(\Gamma)} &= (\mathrm{tr}_\Gamma^{-1} \mathrm{tr}_\Gamma (B\, \mathrm{tr}_\Gamma^{-1} f), \mathrm{tr}_\Gamma^{-1} g)_{\mathring{H}^1(\Omega)} \\ &= (B\, \mathrm{tr}_\Gamma^{-1} f, \mathrm{tr}_\Gamma^{-1} g)_{\mathring{H}^1(\Omega)} \\ &= \int_\Gamma f(\gamma)\overline{g(\gamma)}d\mu(\gamma).\end{aligned}$$

Step 3: Let us prove that $N(B^\Gamma) = \{0\}$. Let $f \in \mathbb{H}^\sigma(\Gamma)$ be such that $B^\Gamma f = 0$. Then $\mathrm{tr}_\Gamma(B\, \mathrm{tr}_\Gamma^{-1} f) = 0$. By Step 1 and Proposition 5.2, $B\, \mathrm{tr}_\Gamma^{-1} f = 0$. By (4.5), $\mathrm{tr}_\Gamma\, \mathrm{tr}_\Gamma^{-1} f = 0$, i.e., $f = 0$.

Step 4: Let us prove that all the positive eigenvalues for B are eigenvalues for B^Γ. The prove of the reverse implication is analogous. Let $\rho > 0$ and $u \in \mathring{H}^1(\Omega)$ be an eigenvalue and an associated eigenfunction, respectively, for B. Then, by Step 1, $u \in N(B)^\perp$. So, for $v = \mathrm{tr}_\Gamma u \in \mathbb{H}^\sigma(\Gamma)$,

$$B^\Gamma v = \mathrm{tr}_\Gamma(Bu) = \rho\, \mathrm{tr}_\Gamma u = \rho v.$$

By Proposition 5.2, as $u \neq 0$ and $u \in N(B)^\perp$, $v \neq 0$ and so v is an eigenfunction associated to ρ for the operator B^Γ.

Step 5: Let us prove the last assertion of the proposition. It is a well-known result from spectral theory that there is an orthonormal sequence of associated eigenfunctions $(u_k)_{k\in\mathbb{N}} \subset \mathbb{H}^\sigma(\Gamma)$. By [23, p. 357, Theorem 4.4], $N(B^\Gamma)$ coincides with the orthogonal complement of the subspace generated by $(u_k)_k$, say $M^\perp$. As we have already proved, $N(B^\Gamma) = \{0\}$ and, so, $M = \mathbb{H}^\sigma(\Gamma)$, i.e., $(u_k)_k$ generates $\mathbb{H}^\sigma(\Gamma)$. □

Proposition 5.8 *Let $(u_k)_{k\in\mathbb{N}}$ be as in Proposition 5.7 and*

$$v_k = \frac{u_k}{\sqrt{\rho_k}}, \quad k \in \mathbb{N}.$$

Then $(v_k)_{k\in\mathbb{N}}$ is a complete orthonormal system in $L_2(\Gamma)$.

Proof Let us prove that $\{u_k\}_k$ is a complete system in $L_2(\Gamma)$. One can prove, analogously to what was done in [27, p. 6, Theorem 3.8], that $D(\Gamma)$ is dense in $L_2(\Gamma)$. Let $f \in L_2(\Gamma)$ and $\varepsilon > 0$. There is $\varphi \in D(\Gamma)$ such that

$$\| f - \varphi \mid L_2(\Gamma)\| < \varepsilon/2. \tag{5.7}$$

As $\varphi \in D(\Gamma) \subset \mathbb{H}^\sigma(\Gamma)$ and $\{u_k\}_k$ generates $\mathbb{H}^\sigma(\Gamma)$, there are $k_0 \in \mathbb{N}$ and $(\alpha_k)_{k=1}^{k_0}$ such that

$$\left\| \varphi - \sum_{k=1}^{k_0} \alpha_k u_k \mid \mathbb{H}^\sigma(\Gamma) \right\| < \varepsilon/2. \tag{5.8}$$

Hence, as $\mathbb{H}^\sigma(\Gamma) \subset L_2(\Gamma)$, it follows from (5.7) and (5.8) that

$$\left\| f - \sum_{k=1}^{k_0} \alpha_k u_k \mid L_2(\Gamma) \right\| \lesssim \varepsilon.$$

So $\{u_k\}_k$ is a complete system in $L_2(\Gamma)$. It follows from (5.5) and (5.6) that

$$(u_j, u_k)_{L_2(\Gamma)} = (B^\Gamma u_j, u_k)_{\mathbb{H}^\sigma(\Gamma)} = \rho_j (u_j, u_k)_{\mathbb{H}^\sigma(\Gamma)}$$

and so $(v_k)_{k\in\mathbb{N}}$ is orthonormal in $L_2(\Gamma)$. □

Proposition 5.9 *Let $(u_k)_{k\in\mathbb{N}}$ be as in Proposition 5.7 and*

$$\omega_k = \frac{u_k}{\rho_k}, \quad k \in \mathbb{N}.$$

Then $(\omega_k)_{k\in\mathbb{N}}$ is a complete orthonormal system in $\mathbb{H}^{\sigma^{-1}}(\Gamma)$.

Proof By Remark 5.6, $u_k \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$ and, for all $g \in \mathbb{H}^\sigma(\Gamma)$,

$$\langle u_k, g\rangle = \int_\Gamma g(\gamma)\overline{u_k(\gamma)}d\mu(\gamma).$$

Applying (5.5) we obtain

$$\langle u_k, g\rangle = (g, B^\Gamma u_k)_{\mathbb{H}^\sigma(\Gamma)} = (g, \rho_k u_k)_{\mathbb{H}^\sigma(\Gamma)}, \quad g \in \mathbb{H}^\sigma(\Gamma). \tag{5.9}$$

So, by (5.2)–(5.3) and (5.9), we conclude that

$$\mathrm{id}_H\, u_k = \rho_k u_k. \tag{5.10}$$

Let $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$. For all $\delta > 0$ the exist $k_0 \in \mathbb{N}$ and $(\alpha_k)_{k=1}^{k_0}$ such that

$$\left\| \mathrm{id}_H\, F - \sum_{k=1}^{k_0} \alpha_k \rho_k u_k \mid \mathbb{H}^\sigma(\Gamma) \right\| < \delta.$$

Hence, by (5.2)–(5.3) and (5.10),

$$\left\| F - \sum_{k=1}^{k_0} \alpha_k u_k \mid \mathbb{H}^{\sigma^{-1}}(\Gamma) \right\| = \left\| \mathrm{id}_H\, F - \sum_{k=1}^{k_0} \alpha_k\, \mathrm{id}_H\, u_k \mid \mathbb{H}^\sigma(\Gamma) \right\| < \delta.$$

Furthermore

$$(u_k, u_j)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} = (\mathrm{id}_H\, u_k, \mathrm{id}_H\, u_j)_{\mathbb{H}^\sigma(\Gamma)} = \rho_k \rho_j (u_k, u_j)_{\mathbb{H}^\sigma(\Gamma)}.$$

Therefore $(u_k/\rho_k)_k$ is a complete o.n. system in $\mathbb{H}^{\sigma^{-1}}(\Gamma)$. □

5.3 An extension of B^Γ

By Proposition 5.8, for all $f \in L_2(\Gamma)$,

$$f = \sum_{k=1}^{\infty} (f, v_k)_{L_2(\Gamma)} v_k \quad \text{in } L_2(\Gamma) \text{ with } \|f \mid L_2(\Gamma)\|^2 = \sum_{k=1}^{\infty} |(f, v_k)_{L_2(\Gamma)}|^2. \tag{5.11}$$

We define

$$\widetilde{\sqrt{B^\Gamma}} f := \sum_{k=1}^{\infty} \sqrt{\rho_k} (f, v_k)_{L_2(\Gamma)} v_k = \sum_{k=1}^{\infty} (f, v_k)_{L_2(\Gamma)} u_k, \quad \text{convergence in } \mathbb{H}^\sigma(\Gamma), \tag{5.12}$$

which converges by (5.11), because $(u_k)_k$ is a o.n. system in $\mathbb{H}^\sigma(\Gamma)$. The operator

$$\widetilde{\sqrt{B^\Gamma}} : L_2(\Gamma) \to \mathbb{H}^\sigma(\Gamma)$$

is well-defined and it is an extension of $\sqrt{B^\Gamma}$. Furthermore, it is an isomorphism. Let us prove that it is surjective. Let $g \in \mathbb{H}^\sigma(\Gamma)$. Then, as $(u_k)_k$ is a complete o.n. system in $\mathbb{H}^\sigma(\Gamma)$,

$$g = \sum_{k=1}^{\infty} (g, u_k)_{\mathbb{H}^\sigma(\Gamma)} u_k, \quad \text{convergence in } \mathbb{H}^\sigma(\Gamma) \quad \text{and}$$
$$\|g \mid \mathbb{H}^\sigma(\Gamma)\|^2 = \sum_{k=1}^{\infty} |(g, u_k)_{\mathbb{H}^\sigma(\Gamma)}|^2. \tag{5.13}$$

As $(v_k)_k$ is a complete o.n. system in $L_2(\Gamma)$,

$$\sum_{k=1}^{\infty} (g, u_k)_{\mathbb{H}^\sigma(\Gamma)} v_k, \quad \text{converges in } L_2(\Gamma) \text{ to, say, } f \quad \text{and}$$
$$\|f \mid L_2(\Gamma)\|^2 = \sum_{k=1}^{\infty} |(g, u_k)_{\mathbb{H}^\sigma(\Gamma)}|^2. \tag{5.14}$$

From (5.11), (5.12), (5.13) and (5.14) one concludes that $\widetilde{\sqrt{B^\Gamma}} f = g$.

It follows immediately as a consequence of (5.11)–(5.12)

$$\|\widetilde{\sqrt{B^\Gamma}} f \mid \mathbb{H}^\sigma(\Gamma)\| = \|f \mid L_2(\Gamma)\|, \quad f \in L_2(\Gamma).$$

Let $\widetilde{\sqrt{B^\Gamma}}'$ denote the dual operator of $\widetilde{\sqrt{B^\Gamma}}$. Hence

$$\widetilde{\sqrt{B^\Gamma}}' : \mathbb{H}^{\sigma^{-1}}(\Gamma) \to L_2(\Gamma)$$

is an isomorphic map.

Let us determine an expression for $\widetilde{\sqrt{B^\Gamma}}'F$, with $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$. By Proposition 5.9,

$$F = \sum_{j=1}^{\infty} \left(F, \frac{u_j}{\rho_j}\right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j} \quad \text{in } \mathbb{H}^{\sigma^{-1}}(\Gamma) \quad \text{and}$$
$$\|F \mid \mathbb{H}^{\sigma^{-1}}(\Gamma)\|^2 = \sum_{j=1}^{\infty} \left| \left(F, \frac{u_j}{\rho_j}\right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \right|^2. \tag{5.15}$$

Let $g \in L_2(\Gamma)$ be represented by (5.11) with g instead of f. Then

$$\begin{aligned}
\langle \widetilde{\sqrt{B^\Gamma}}'F, g\rangle &= \langle F, \widetilde{\sqrt{B^\Gamma}} g\rangle \\
&= \left\langle \sum_{j=1}^{\infty} \left(F, \frac{u_j}{\rho_j}\right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}, \sum_{k=1}^{\infty} \sqrt{\rho_k}(g, v_k)_{L_2(\Gamma)} v_k \right\rangle \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left(F, \frac{u_j}{\rho_j}\right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \sqrt{\rho_k}(g, v_k)_{L_2(\Gamma)} \left\langle \frac{u_j}{\rho_j}, v_k \right\rangle \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left(F, \frac{u_j}{\rho_j}\right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \sqrt{\rho_k}(g, v_k)_{L_2(\Gamma)} \frac{1}{\sqrt{\rho_j}} \langle v_j, v_k\rangle.
\end{aligned}$$

As

$$\langle v_j, v_k\rangle = \int_\Gamma v_k(\gamma)\overline{v_j(\gamma)}d\mu(\gamma) = (v_j, v_k)_{L_2(\Gamma)},$$

we obtain

$$\begin{aligned}
\langle \widetilde{\sqrt{B^\Gamma}}'F, g\rangle &= \sum_{j=1}^{\infty} \left(F, \frac{u_j}{\rho_j}\right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} (g, v_j)_{L_2(\Gamma)} \\
&= \left\langle \sum_{j=1}^{\infty} \sqrt{\rho_j}\left(F, \frac{u_j}{\rho_j}\right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}, \sum_{k=1}^{\infty} (g, v_k)_{L_2(\Gamma)} v_k \right\rangle.
\end{aligned}$$

So

$$\widetilde{\sqrt{B^\Gamma}}'F = \sum_{j=1}^{\infty} \sqrt{\rho_j}\left(F, \frac{u_j}{\rho_j}\right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}. \tag{5.16}$$

In particular, if $F \in L_2(\Gamma)$, we obtain, by (5.16) and Proposition 5.8,

$$\begin{aligned}
\widetilde{\sqrt{B^\Gamma}}' F &= \sum_{j=1}^{\infty} \sqrt{\rho_j} \left(F, \frac{u_j}{\rho_j} \right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j} \\
&= \sum_{j=1}^{\infty} \sqrt{\rho_j} \left(\sum_{k=1}^{\infty} (F, v_k)_{L_2(\Gamma)} v_k, \frac{u_j}{\rho_j} \right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j} \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sqrt{\rho_j} (F, v_k)_{L_2(\Gamma)} \sqrt{\rho_k} (\omega_k, \omega_j)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j} \\
&= \sum_{j=1}^{\infty} \sqrt{\rho_j} (F, v_j)_{L_2(\Gamma)} \sqrt{\rho_j} \frac{u_j}{\rho_j} \\
&= \sum_{j=1}^{\infty} \sqrt{\rho_j} (F, v_j)_{L_2(\Gamma)} v_j,
\end{aligned}$$

which is, by (5.12), $\widetilde{\sqrt{B^\Gamma}} F$.

Therefore

$$\widetilde{B^\Gamma} := \widetilde{\sqrt{B^\Gamma}} \circ \widetilde{\sqrt{B^\Gamma}}' : \mathbb{H}^{\sigma^{-1}}(\Gamma) \to \mathbb{H}^{\sigma}(\Gamma) \tag{5.17}$$

is an isomorphic map and it is an extension of B^Γ.

In the next proposition we state what we have just proved on this extension of B^Γ and we relate it with the initial definition of B^Γ as

$$B^\Gamma = \mathrm{tr}_\Gamma \circ (-\Delta)^{-1} \circ \mathrm{id}^\Gamma. \tag{5.18}$$

Proposition 5.10 *Consider that the conditions in Assumption 5.1 are satisfied. Let B^Γ be as in (5.4) and $\widetilde{B^\Gamma}$ be as in (5.17). Then*

$$\widetilde{B^\Gamma} : \mathbb{H}^{\sigma^{-1}}(\Gamma) \to \mathbb{H}^{\sigma}(\Gamma) \tag{5.19}$$

is an isomorphic map and it is an extension of B^Γ. Moreover, for all $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$, there is an element of $H^{-1}(\Omega)$, say $\widetilde{\mathrm{id}^\Gamma} F$, such that

$$\widetilde{B^\Gamma} F = \mathrm{tr}_\Gamma \circ (-\Delta)^{-1} \circ \widetilde{\mathrm{id}^\Gamma} F. \tag{5.20}$$

Proof The first part of the proposition was already proved. Let us prove the second part, relating (5.17) and (5.18). Consider $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$ as in (5.15). Let

$$F_N := \sum_{j=1}^{N} \left(F, \frac{u_j}{\rho_j} \right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}, \quad N \in \mathbb{N}.$$

So

$$F = \lim_{N\to\infty} F_N \quad \text{in } \mathbb{H}^{\sigma^{-1}}(\Gamma) \quad \text{and} \quad \widetilde{B^\Gamma} F = \lim_{N\to\infty} B^\Gamma F_N \quad \text{in } \mathbb{H}^{\sigma}(\Gamma).$$

Hence the sequence $(\mathrm{tr}_\Gamma \circ (-\Delta)^{-1} \circ \mathrm{id}^\Gamma F_N)_{N\in\mathbb{N}}$ is convergent in $\mathbb{H}^\sigma(\Gamma)$. One can easily see that, for all $N \in \mathbb{N}$, $(-\Delta)^{-1} \circ \mathrm{id}^\Gamma F_N$ belongs to the image of the operator B considered acting in $\mathring{H}^1(\Omega)$. So, according to what was done in Step 1 of the proof of Proposition 5.7, $(-\Delta)^{-1} \circ \mathrm{id}^\Gamma F_N$ belongs to $N(B)^\perp$, for all $N \in \mathbb{N}$. Thus, by Proposition 5.2, the sequence $((-\Delta)^{-1} \circ \mathrm{id}^\Gamma F_N)_{N\in\mathbb{N}}$ is convergent in $\mathring{H}^1(\Omega)$. Applying Theorem 4.1 one concludes that the sequence $(\mathrm{id}^\Gamma F_N)_{N\in\mathbb{N}}$ converges in $H^{-1}(\Omega)$ to, say, $\widetilde{\mathrm{id}^\Gamma} F$. So (5.20) is satisfied. □

5.4 The main result

In the next theorem we state the existence of solution for a fractal Dirichlet problem in the context of h-sets. For a corresponding result for d-sets see [28, Theorem 20.7].

Theorem 5.11 *Let $h \in \mathbb{H}$ be a strictly increasing function. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$, $n \geq 2$ and Γ be an h-set with $\Gamma \subset \Omega$. Suppose that*

$$n - 2 < -\overline{s}(\boldsymbol{h}) \leq -\underline{s}(\boldsymbol{h}) < n.$$

Let $g \in \mathbb{H}^\sigma(\Gamma)$, where σ is as in (3.8). Then the Dirichlet problem

$$u \in H^1(\Omega), \quad \Delta u(x) = 0 \quad \text{in } \Omega \setminus \Gamma, \tag{5.21}$$

$$\mathrm{tr}_{\partial\Omega}\, u = 0, \qquad \mathrm{tr}_\Gamma\, u = g, \tag{5.22}$$

has at least one solution, which can be represented by

$$u = \lim_{N\to\infty} \int_\Gamma G(\cdot, \gamma) F_N(\gamma) d\mu(\gamma), \quad \text{in } \mathring{H}^1(\Omega), \tag{5.23}$$

where

$$F_N := \sum_{j=1}^{N} \left(F, \frac{u_j}{\rho_j}\right)_{\mathbb{H}^{\sigma^{-1}}(\Gamma)} \frac{u_j}{\rho_j}, \quad N \in \mathbb{N}, \text{ for } F = \widetilde{B^\Gamma}^{-1} g \tag{5.24}$$

and for $(\rho_k)_k$ and $(u_k)_k$ as in Proposition 5.7.

Proof Let us prove that the Dirichlet problem (5.21)–(5.22) has a solution. By Proposition 5.10, particularly (5.19), as $g \in \mathbb{H}^\sigma(\Gamma)$, then there is $F \in \mathbb{H}^{\sigma^{-1}}(\Gamma)$ such that $\widetilde{B^\Gamma} F = g$. Let

$$u := (-\Delta)^{-1} \circ \widetilde{\mathrm{id}^\Gamma} F, \tag{5.25}$$

where $\widetilde{\mathrm{id}^\Gamma} F$ is as in the proof of Proposition 5.10.

By Propositions 4.1 and 5.10, $u \in \mathring{H}^1(\Omega)$. In particular, $u \in H^1(\Omega)$ and $\operatorname{tr}_{\partial\Omega} u = 0$. Moreover, by (5.25),

$$\operatorname{tr}_\Gamma u = \widetilde{B^\Gamma} F = g$$

and

$$\Delta u = \widetilde{\mathrm{id}^\Gamma} F$$

and so supp $\Delta u \subset \Gamma$. Therefore

$$\Delta u(x) = 0 \quad \text{in } \Omega \setminus \Gamma.$$

The representation in (5.23), which can be also obtained in terms of g as

$$F_N = \sum_{j=1}^{N} \big(g, u_j\big)_{\mathbb{H}^\sigma(\Gamma)} \frac{u_j}{\rho_j}, \qquad N \in \mathbb{N},$$

follows from (4.7) and (5.25).

□

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