On the structure of a triangulation

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Abstract We characterize the notion of a triangulation, in which every vertex has a star-neighbourhood, in terms of an internal categorical structure. This structure is important since it allows to have more efficient and robust algorithms that can be used in additive manufacturing.

0.1 Introduction

The structure of a triangulation generalizes the one of a directed graph. If $\mathbb{C}$ is a category then an internal directed graph in $\mathbb{C}$ consists of a pair of objects (the object of vertices and the object of edges) together with two parallel morphisms between them, called the domain and codomain morphisms. A triangulation, in our sense, consists of two objects and three parallel morphisms between them, as displayed

$$\begin{tikzcd} T \arrow[r, shift left=2, "a"] \arrow[r, "b" description] \arrow[r, shift right=2, "c"'] & V, \end{tikzcd} \tag{1}$$

with an element $t \in T$, say in the category of sets and maps, being interpreted as a triangle in the following manner

$$\begin{tikzcd} & c(t) \arrow[ddl] & \\ & t & b(t). \arrow[ul] \\ a(t) \arrow[urr] & & \end{tikzcd}$$

In practice we are concerned with triangulated surfaces. Suppose we are given a triangulation

$$\begin{tikzcd} T \arrow[r, shift left=2, "a"] \arrow[r, "b" description] \arrow[r, shift right=2, "c"'] & \mathbb{R}^3, \end{tikzcd} \tag{2}$$

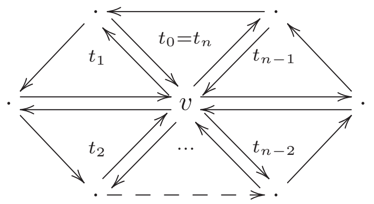
in the category `Sub64`, whose objects are all sets and the morphisms are those maps whose domain is either infinite (in which case they are required to be bijections) or else have a cardinality not greater than 2^{64}. This particular category can be used as a model for computational systems such as `Matlab` or `Julia`.

0.2 The problem

The following problem arises in the area of additive manufacturing [1] which is concerned with the production of 3-D real world objects using the so-called layer by layer fabrication.

How can we distinguish the triangulations (2) which are enclosing a physical real object in the 3-D euclidean space from the ones who don't?

The answer is fairly simple and well-known: each vertex $v \in V$ should admit a star-neighbourhood. Or, in other words, the set of all triangles in T, that are incident with the vertex $v \in V$, admits a cyclic order that is compatible with the adjacency of the edges, as illustrated.



(3)

This well-known characterization, however, has some draw-backs with respect to computational implementations and algorithms that operate on it [1]. Hence the need for finding an alternative solution to this problem. We have thus found an alternative which is equivalent and more appropriate to our needs.

First we replace $\mathbb{R}^3$ with the Cayley algebra of quaternions $\mathbb{H}$, see e.g. [3], this is only a technical issue and in practice even the algebra of octonions is of great use since it allows to take into account other physical properties such as color, type of material, density, etc.

Secondly, instead of triangulations of the form

$$\begin{tikzcd} T \arrow[r, shift left=2, "a"] \arrow[r, "b" description] \arrow[r, shift right=2, "c"'] & \mathbb{H} \end{tikzcd} \tag{4}$$

in `Sub64`, we consider the structures of the form

$$\begin{tikzcd} A \arrow[loop left, "\theta{,}\varphi"] \arrow[r, "g"] & \mathbb{H} \end{tikzcd} \tag{5}$$

such that

$$\theta^3 = 1_A \tag{6}$$

$$\theta^2 = \varphi\theta\varphi \tag{7}$$

$$g\varphi = g \tag{8}$$

and we can show that the *star-neighbourhood* property displayed in (3) is equivalent to the requirement that φ is an isomorphism.

0.3 The solution

We are now going to give a quick explanation on how to transform (4) into (5) and vice-versa.

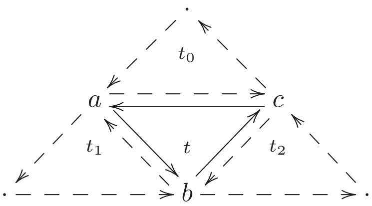
Having a structure such as (4) with the star-neighbourhood property for each of its vertices, in order to obtain a structure such as (5), we define:

$$
\begin{aligned}
A &= T \times \{0,1,2\} \\
\theta(t,i) &= (t, i+1 \mod 3) \\
g(t,i) &= \begin{cases} a(t) & \text{if} \quad i=0 \\ b(t) & \text{if} \quad i=1 \\ c(t) & \text{if} \quad i=2 \end{cases}
\end{aligned}
$$

and

$$\varphi(t,i) = (t_i, j(t,i))$$

with t_i, $i = 0,1,2$ as illustrated



and $j(t,i)$ given by

$$
\begin{aligned}
j(t,0) &= \begin{cases} 0 & \text{if} \quad a(t_0) = a(t) \\ 1 & \text{if} \quad b(t_0) = a(t) \\ 2 & \text{if} \quad c(t_0) = a(t) \end{cases} \\
j(t,1) &= \begin{cases} 0 & \text{if} \quad a(t_1) = b(t) \\ 1 & \text{if} \quad b(t_1) = b(t) \\ 2 & \text{if} \quad c(t_1) = b(t) \end{cases} \\
j(t,2) &= \begin{cases} 0 & \text{if} \quad a(t_2) = c(t) \\ 1 & \text{if} \quad b(t_2) = c(t) \\ 2 & \text{if} \quad c(t_2) = c(t) \end{cases}
\end{aligned}
$$

Clearly, each vertex has a star-neighbourhood, if and only if φ is a bijection. Note that we are working on the category `Sub64`, which in particular has the property that every regular epimorphism is a split epimorphism. Also note that, in order to construct φ it is sufficient that each triangle in (4) to have three adjacent triangles, one for each edge, which is a weaker condition that the star-neighbourhood property.

Conversely, if having a structure such as (5) with its three conditions, then we define a triangulation as follows. The triangles are obtained by identifying the orbits of θ, via the coequalizer $p = coeq(1_A, \theta)\colon A \to T$ (note that this particular coequalizer exists in `Subd64`, even though its kernel relation may fail to exist); by identifying the orbits of φ, say with $q = coeq(1_A, \varphi)\colon A \to V$, we obtain unique labels for the vertices in the triangulation. The complete structure, with $a = qs$, $b = q\theta s$ and $c = q\theta^2 s$ (where s is any section for p) is displayed as

$$\begin{tikzcd}
T \arrow[r, shift left=2, "a"] \arrow[r, "v" description] \arrow[r, shift right=2, "c"'] & V \arrow[r, "m"] & \mathbb{H}
\end{tikzcd}$$

with m the unique map such that $g = mq$.

0.4 Conclusion

The structure (A, θ, φ, g), with conditions (6)–(8) and the extra requirement that $varphi$ is an isomorphism, when internal to the category `Sub64`, is equivalent to the structure of a triangulation (T, V, a, b, c) satisfying the star-neighbourhood property. The advantage of this new structure is that it allows the development and implementation of several new algorithms and other geometrical constructions in a more robust and efficient way. An important example is the slicing algorithm which decomposes the triangulation into several disjoint contour levels and permits it fabrication with a layer by layer technique.

As a final remark we observe that a somewhat different but related problem is the one of characterizing those triangulations (1) that can appear as the multiplicative structure of an internal groupoid, see [5] for more details: a triangulation such as (T, V, a, b, c) displayed in (1) is the multiplicative structure of an internal groupoid if and only if the two directed graphs (T, V, a, b) and (T, V, b, c) centralize each other and moreover the pushout of the span (a, c) is a pullback square.

This work was presented in the conference [6]

References

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