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Reconstructing Classical Algebras via Ternary Operations

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Abstract: Although algebraic structures are frequently analyzed using unary and binary operations, they can also be effectively defined and unified using ternary operations. In this context, we introduce structures that contain two constants and a ternary operation. We demonstrate that these structures are isomorphic to various significant algebraic systems, including Boolean algebras, de Morgan algebras, MV-algebras, and (near-)rings of characteristic two. Our work highlights the versatility of ternary operations in describing and connecting diverse algebraic structures.

Keywords: Boolean algebras; MV-algebras; de Morgan algebras; ternary operations; rings and near-rings of characteristic two

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1. Introduction

Ternary Boolean algebras [1] were introduced by Grau in 1947 to axiomatize Boolean algebras by means of the ternary operation $xy \circ yz \circ zx$ (see also [2]). In the same year, this operation was used independently by Birkhoff and Kiss [3] to characterize distributive lattices. Both approaches are particular cases of median algebras [4–6]. Although the set of axioms is distinct in each case, complete commutativity is a common feature. In 1948, Church [7] shows that it is possible to axiomatize Boolean algebras in terms of the conditioned disjunction $\bar{y}x \circ yz$, which is not completely commutative (see also [8]).

In this article, we explore the axiomatization of Boolean algebras, de Morgan algebras, MV-algebras, and rings or near-rings of characteristic two using ternary structures. By a ternary structure, we mean an algebraic system consisting of a set A with two constants, 0 and 1, and a ternary operation $p(x, y, z) \in A$. This ternary operation gives rise to derived unary and binary operations, and each formula specifying p in terms of these operations corresponds to a new axiom, leading to subvarieties of the original structure. Interestingly, each classical structure considered here has a characteristic expression that determines the ternary operation p using its derived binary operations. The purpose of this paper is to examine these expressions and their implications for the unification of classical algebras within the framework of ternary structures.

For example, within the set of axioms **(T1)**–**(T4)** introduced in Section 2, Boolean algebras form a subvariety if and only if $p(x, y, z)$ is interpreted as Church's conditioned disjunction, as proved in Section 3. Section 4 gives the formula for p that turns the **(T1)**–**(T4)** structure into a de Morgan algebra, while Section 5 discusses the cases of rings and near-rings of characteristic 2. In Section 6, it is shown how to modify the axioms **(T1)**–**(T4)** so that MV-algebras can be characterized with a ternary structure, and the corresponding ternary operation is given. Finally, conclusions are draw in Section 7, where a table is presented with a comparison between the cases considered here.

Further observations and some more details are presented in the preprints [9,10].

2. Preliminaries

Let us begin by introducing the principal notations used for the unary and binary operations derived from a general structure with one ternary operation p and two constants 0 and 1.

Definition 1. *Let $(A, p, 0, 1)$ be a system consisting of a set A, together with a ternary operation p and two constants $0, 1 \in A$. From p, the following operations are defined:*

$$\bar{a} = p(1, a, 0) \tag{1}$$
$$a \cdot b = p(0, a, b) \tag{2}$$
$$a \circ b = p(a, b, 1) \tag{3}$$
$$a \wedge b = p(b, \bar{a}, 0) \tag{4}$$
$$a \vee b = p(1, \bar{b}, a) \tag{5}$$
$$a + b = p(a, b, \bar{a}). \tag{6}$$

Next, we present an algebraic structure $(A, p, 0, 1)$ satisfying four axioms. Axiom **(T2)** alone defines a Church algebra [11,12], while a Menger algebra of rank 2 (see, e.g., [13]) uses axiom **(T4)**. The axioms **(T1)**, **(T2)**, and **(T4)** have been used to define proposition algebras [14], while axioms **(T2)**, **(T3)**, and **(T4)**, among others, have been used to study spaces with geodesic paths [15–17]. Lemma 1 below displays the basic properties of the structure **(T1)**–**(T4)**.

Lemma 1. *Let $(A, p, 0, 1)$ be a system consisting of a set A, together with a ternary operation p and two constants $0, 1 \in A$ satisfying*

(T1) $p(0, a, 1) = a$
(T2) $p(a, 0, b) = a = p(b, 1, a)$
(T3) $p(a, b, a) = a$
(T4) $p(a, p(b_1, b_2, b_3), c) = p(p(a, b_1, c), b_2, p(a, b_3, c)).$

Then, the following properties hold:

$$\bar{1} = 0 \quad , \quad \bar{0} = 1 \tag{7}$$
$$\bar{\bar{a}} = a \tag{8}$$
$$p(c, b, a) = p(a, \bar{b}, c) \tag{9}$$
$$\overline{p(a, b, c)} = p(\bar{a}, b, \bar{c}) \tag{10}$$
$$\overline{p(a, b, c)} = p(\bar{c}, \bar{b}, \bar{a}) \tag{11}$$
$$a \cdot b = a \wedge b \quad , \quad a \circ b = a \vee b \tag{12}$$
$$\overline{a \cdot b} = \bar{b} \circ \bar{a} \quad , \quad \overline{a \circ b} = \bar{b} \cdot \bar{a} \tag{13}$$
$$(A, \cdot, 1) \text{ and } (A, \circ, 0) \text{ are monoids} \tag{14}$$
$$a \cdot 0 = 0 = 0 \cdot a \tag{15}$$
$$a \circ 1 = 1 = 1 \circ a \tag{16}$$
$$(A, +, 0) \text{ is a monoid} \tag{17}$$
$$a + 1 = \bar{a} = 1 + a \tag{18}$$

Proof. In each step of the proof, the needed property when required is written above the corresponding equality.

$$\overline{1} = p(1,1,0) \overset{\textbf{(T2)}}{=} 0, \quad \overline{0} = p(1,0,0) \overset{\textbf{(T2)}}{=} 1$$
$$\overline{\overline{a}} = p(1,p(1,a,0),0) \overset{\textbf{(T4)}}{=} p(p(1,1,0),a,p(1,0,0)) \overset{\textbf{(T2)}}{=} p(0,a,1) \overset{\textbf{(T1)}}{=} a$$
$$p(a,\overline{b},c) = p(a,p(1,b,0),c) \overset{\textbf{(T4)}}{=} p(p(a,1,c),b,p(a,0,c)) \overset{\textbf{(T2)}}{=} p(c,b,a)$$
$$\overline{p(a,b,c)} = p(1,p(a,b,c),0) \overset{\textbf{(T4)}}{=} p(p(1,a,0),b,p(1,c,0)) = p(\overline{a},b,\overline{c}).$$

Property (11) is just a combination of (9) and (10), whereas (12) is just a particular case of (9). Next is the proof of Properties (13):

$$\overline{a \cdot b} = \overline{p(0,a,b)} \overset{(11)}{=} p(\overline{b},\overline{a},\overline{0}) \overset{(7)}{=} p(\overline{b},\overline{a},1) = \overline{b} \circ \overline{a}$$
$$\overline{a \circ b} = \overline{p(a,b,1)} \overset{(11)}{=} p(\overline{1},\overline{b},\overline{a}) \overset{(7)}{=} p(0,\overline{b},\overline{a}) = \overline{b} \cdot \overline{a}.$$

With respect to (14), we have associativity

$$\begin{aligned} (a \cdot b) \cdot c &= p(0,p(0,a,b),c) \overset{\textbf{(T4)}}{=} p(p(0,0,c),a,p(0,b,c)) \\ &\overset{\textbf{(T2)}}{=} p(0,a,p(0,b,c)) = a \cdot (b \cdot c) \\ a \circ (b \circ c) &= p(a,p(b,c,1),1) \overset{\textbf{(T4)}}{=} p(p(a,b,1),c,p(a,1,1)) \\ &\overset{\textbf{(T2)}}{=} p(p(a,b,1),c,1) = (a \circ b) \circ c, \end{aligned}$$

and identities

$$a \cdot 1 = p(0,a,1) \overset{\textbf{(T1)}}{=} a, \quad 1 \cdot a = p(0,1,a) \overset{\textbf{(T2)}}{=} a$$
$$a \circ 0 = p(a,0,1) \overset{\textbf{(T2)}}{=} a, \quad 0 \circ a = p(0,a,1) \overset{\textbf{(T1)}}{=} a.$$

For properties (15) and (16), the proof is

$$a \cdot 0 = p(0,a,0) \overset{\textbf{(T3)}}{=} 0, \quad 0 \cdot a = p(0,0,a) \overset{\textbf{(T2)}}{=} 0$$
$$a \circ 1 = p(a,1,1) \overset{\textbf{(T2)}}{=} 1, \quad 1 \circ a = p(1,a,1) \overset{\textbf{(T3)}}{=} 1.$$

The structure $(A,+,0)$ is a monoid since

$$a + 0 = p(a,0,\bar{a}) \overset{\textbf{(T2)}}{=} a, \quad 0 + a = p(0,a,1) \overset{\textbf{(T1)}}{=} a$$

$$\begin{aligned} (a+b)+c &= p(p(a,b,\bar{a}),c,\overline{p(a,b,\bar{a})}) \overset{(11)}{=} p(p(a,b,\bar{a}),c,p(a,\bar{b},\bar{a})) \\ &\overset{\textbf{(T4)}}{=} p(a,p(b,c,\bar{b}),\bar{a}) = a + (b+c). \end{aligned}$$

Furthermore, $a + 1 = p(a,1,\bar{a}) \overset{\textbf{(T3)}}{=} \bar{a}$ and $1 + a = p(1,a,0) = \bar{a}$, which proves (18). In particular, $1 + 1 = 0$. □

The following lemmas investigate the consequences of some classic extra conditions on the binary operations besides Axioms **(T1)** to **(T4)**. Note that the de Morgan's laws (13) imply a duality between $\cdot$ and $\circ$. In particular, $\cdot$ is idempotent if and only if $\circ$ is idempotent too.

Lemma 2. *Let $(A, p, 0, 1)$ be a system verifying conditions* **(T1)** *to* **(T4)***. If the operation $\circ$ is idempotent, then we have the following absorption rules:*

$$a \circ (b \cdot a) = a \quad \textit{and} \quad (a \circ b) \cdot a = a. \tag{19}$$

Proof. Idempotency means that

$$p(0, a, a) = a \cdot a = a \quad \text{and} \quad p(a, a, 1) = a \circ a = a, \tag{20}$$

and consequently,

$$\begin{aligned} a \circ (b \cdot a) &= p(a, p(0, b, a), 1) \overset{(\mathbf{T4})}{=} p(p(a, 0, 1), b, p(a, a, 1)) \\ &\overset{(\mathbf{T2})}{=} p(a, b, p(a, a, 1)) \overset{(20)}{=} p(a, b, a) \overset{(\mathbf{T3})}{=} a. \end{aligned}$$

The second equality in (19) follows through the application of (13). □

When commutativity is added, distributivity is obtained.

Lemma 3. *Let $(A, p, 0, 1)$ be a system verifying conditions* **(T1)** *to* **(T4)***. If the operations $\cdot$ and $\circ$ are commutative and idempotent, then they distribute over each other.*

Proof. When $\circ$ is commutative, the first absorption rule in (19) may be written as

$$(b \cdot a) \circ a = p(p(0, b, a), a, 1) = a. \tag{21}$$

Then, we have

$$\begin{aligned} (b \cdot a) \circ (c \cdot a) &= p(p(0, b, a), p(0, c, a), 1) \\ &\overset{(\mathbf{T4})}{=} p(p(p(0, b, a), 0, 1), c, p(p(0, b, a), a, 1)) \\ &\overset{(\mathbf{T2})}{=} p(p(0, b, a), c, p(p(0, b, a), a, 1)) \\ &\overset{(21)}{=} p(p(0, b, a), c, a) \\ &\overset{(\mathbf{T2})}{=} p(p(0, b, a), c, p(0, 1, a)) \\ &\overset{(\mathbf{T4})}{=} p(0, p(b, c, 1), a) = (b \circ c) \cdot a. \end{aligned}$$

Similarly, using $(a \circ b) \cdot a = a$, we obtain $(a \circ b) \cdot (a \circ c) = a \circ (b \cdot c)$. □

Lemma 4. *Let $(A, p, 0, 1)$ be a system verifying conditions* **(T1)** *to* **(T4)***. If $a + a = 0$, then the operation $+$ is commutative, and the operation $\cdot$ is right distributive over $+$:*

$$a + a = 0 \Rightarrow a + b = b + a \textit{ and } (a + b) \cdot c = a \cdot c + b \cdot c. \tag{22}$$

Proof. $a+a=0$ implies, using (17), $(a+b)+(b+a)=0$, and therefore $a+b=b+a$. The right distributivity of $\cdot$ over $+$ can be proven as follows:

$$\begin{aligned} (a\cdot c)+(b\cdot c) &= p(a\cdot c, p(0,b,c), \overline{a\cdot c}) \\ &\overset{(\mathbf{T4}),(\mathbf{T2})}{=} p(a\cdot c, b, p(a\cdot c, c, \overline{a\cdot c})) \\ &= p(a\cdot c, b, (a\cdot c)+c) \\ &= p(a\cdot c, b, c+(a\cdot c)) \\ &= p(a\cdot c, b, p(c, p(0,a,c), \bar{c})) \\ &\overset{(\mathbf{T4}),(\mathbf{T2})}{=} p(a\cdot c, b, p(c, a, p(c,c,\bar{c}))) \\ &= p(p(0,a,c), b, p(c,a,0)) \\ &\overset{(9)}{=} p(p(0,a,c), b, p(0,\bar{a},c)) \\ &\overset{(\mathbf{T4})}{=} p(0, p(a,b,\bar{a}), c) = (a+b)\cdot c. \end{aligned}$$

□

It is worth noting that the left distributivity of $\cdot$ over $+$ is not guaranteed. On the dual side, we have the left distributivity of $\circ$ over the binary operation $a*b=p(\bar{a},b,a)=\bar{a}+b$ when $a*a=1$, and no right distributivity is guaranteed.

In addition to the structure defined by conditions **(T1)**–**(T4)**, in the following lemma, $\bar{a}=p(1,a,0)$ is assumed to be the Boolean complement of $a\in A$. When this is the case, $\cdot$ and $\circ$ are idempotent.

Lemma 5. *Let $(A,p,0,1)$ be a system verifying conditions* **(T1)** *to* **(T4)**. *If every $a\in A$ verifies the relations*

$$\bar{a}\cdot a = p(0,\bar{a},a) = 0 \quad \text{and} \quad \bar{a}\circ a = p(\bar{a},a,1) = 1, \tag{23}$$

then idempotency holds:

$$p(0,a,a) = a\cdot a = a \quad \text{and} \quad p(a,a,1) = a\circ a = a. \tag{24}$$

Proof.

$$\begin{aligned} a\cdot a &= p(0,a,a) \overset{(23),(\mathbf{T2})}{=} p(p(0,\bar{a},a), a, p(0,1,a)) \\ &\overset{(\mathbf{T4})}{=} p(0, p(\bar{a},a,1), a) \overset{(23)}{=} p(0,1,a) \overset{(\mathbf{T2})}{=} a. \end{aligned}$$

The idempotency of $\circ$ is obtained similarly or using (13). □

3. The Boolean Algebra

The next proposition shows how the structure of axioms **(T1)**–**(T4)** can be turned into a Boolean ring. Recall that the notation $a+b$ is being used for $p(a,b,\bar{a})$.

Proposition 1. *If $(A,p,0,1)$ verifies conditions* **(T1)** *to* **(T4)** *and if*

$$p(0,a,b) = p(a,a,b) \tag{25}$$

then,

$$(A,+,\cdot,0,1) \text{ is a Boolean ring.} \tag{26}$$

Proof. Condition (25) implies Boolean complements and $a + a = 0$:

$$\begin{aligned} a \cdot \bar{a} &= p(0, a, \bar{a}) \overset{(9)}{=} p(\bar{a}, \bar{a}, 0) \overset{(25)}{=} p(0, \bar{a}, 0) \overset{(\mathbf{T3})}{=} 0, \\ a + a &= p(a, a, \bar{a}) \overset{(25)}{=} p(0, a, \bar{a}) = a \cdot \bar{a} = 0. \end{aligned} \tag{27}$$

Now, this result and Lemma 4 imply that $+$ is commutative

$$p(a, b, \bar{a}) = p(b, a, \bar{b}) \tag{28}$$

and $\cdot$ is distributed on the right over $+$

$$(a \cdot c) + (b \cdot c) = (a + b) \cdot c. \tag{29}$$

In addition, the following properties hold:

$$a \cdot (a + b) = a + (a \cdot b), \quad b \cdot (a + b) = (b \cdot a) + b. \tag{30}$$

Indeed,

$$\begin{aligned} a \cdot (a + b) &= p(0, a, p(a, b, \bar{a}) \overset{(27)}{=} p(p(a, a, \bar{a}), a, p(a, b, \bar{a})) \\ &\overset{(\mathbf{T4})}{=} p(a, p(a, a, b), \bar{a}) \overset{(25)}{=} p(a, a \cdot b, \bar{a}) = a + (a \cdot b). \end{aligned}$$

The second relation in (30) is a consequence of the commutativity of $+$. We can now prove that $\cdot$ is commutative. From Lemma 5, we already know that under the hypothesis of Proposition 1, $\cdot$ is idempotent, and consequently,

$$\begin{aligned} (a + b) \cdot (a + b) = a + b &\overset{(29)}{\Rightarrow} \big(a \cdot (a + b)\big) + \big(b \cdot (a + b)\big) = a + b \\ &\overset{(30)}{\Rightarrow} \big(a + (a \cdot b)\big) + \big((b \cdot a) + b\big) = a + b \\ &\overset{(17),(28)}{\Rightarrow} (a \cdot b) + (b \cdot a) + a + b = a + b \\ &\overset{(17),(27)}{\Rightarrow} (a \cdot b) + (b \cdot a) = 0 \\ &\overset{(17),(27)}{\Rightarrow} a \cdot b = b \cdot a. \end{aligned}$$

□

The following theorem is a refinement of Grau's ternary Boolean algebra in the sense that it uses Church's operation and a systematization of Hoare's axioms considered in [8].

Theorem 1. *Suppose that* $(A, p, 0, 1)$ *satisfies axioms* **(T1)** *to* **(T4)***. For*

$$\bar{a} = p(1, a, 0), \ a \cdot b = p(0, a, b), \ a \circ b = p(a, b, 1) \ \text{and} \ a + b = p(a, b, \bar{a}),$$

the following conditions are equivalent:

(i) $(A, +, \cdot, 0, 1)$ *is a Boolean ring;*
(ii) $(A, \circ, \cdot, \overline{()}, 0, 1)$ *is a Boolean algebra;*
(iii) $p(a, b, c) = (\bar{b} \cdot a) \circ (b \cdot c)$;
(iv) $p(a, a, b) = a \cdot b$;
(v) $p(a, b, b) = a \circ b$.

Proof. The proof proceeds as follows: (ii) ⇒ (iii) ⇒ ((iv) ⇔ (v)) ⇒ (i) ⇒ (ii). We begin by proving that if $(A, p, 0, 1)$ is a system verifying the hypothesis of Theorem 1, then (ii) implies (iii). It is well known (see, e.g., [18,19]) that in a distributive lattice, if $x \cdot a = x' \cdot a$ and $a \circ x = a \circ x'$ for some given element a in the lattice, then $x = x'$. We show here that if $(A, p, 0, 1)$ verifies **(T1)** to **(T4)** and $(A, \circ, \cdot, \overline{()}, 0, 1)$ is a Boolean algebra, then

$$\begin{cases} p(a,b,c) \cdot c &= ((\overline{b} \cdot a) \circ (b \cdot c)) \cdot c \\ c \circ p(a,b,c) &= c \circ ((\overline{b} \cdot a) \circ (b \cdot c)) \end{cases},$$

which proves (iii). Indeed,

$$\begin{aligned} p(a,b,c) \cdot c &= p(0, p(a,b,c), c) \overset{(\mathbf{T4})}{=} p(p(0,a,c), b, p(0,c,c)) \\ &\overset{(24)}{=} p(p(0,a,c), b, c) \overset{(\mathbf{T2})}{=} p(p(0,a,c), b, p(0,1,c)) \\ &\overset{(\mathbf{T4})}{=} p(0, p(a,b,1), c) = (a \circ b) \cdot c \\ &= ((\overline{b} \cdot a) \circ b) \cdot c = ((\overline{b} \cdot a) \circ (b \cdot c)) \cdot c \end{aligned}$$

$$\begin{aligned} c \circ p(a,b,c) &= p(c, p(a,b,c), 1) \overset{(\mathbf{T4})}{=} p(p(c,a,1)), b, p(c,c,1)) \\ &\overset{(24)}{=} p(p(c,a,1), b, c) \overset{(\mathbf{T2})}{=} p(p(c,a,1), b, p(c,0,1)) \\ &\overset{(\mathbf{T4})}{=} p(c, p(a,b,0), 1) \overset{(9)}{=} p(c, p(0, \overline{b}, a), 1) \\ &= c \circ (\overline{b} \cdot a) = c \circ ((\overline{b} \cdot a) \circ (b \cdot c)). \end{aligned}$$

Next, it is shown that condition (iii) implies condition (iv). Indeed, when (iii) is true, we have

$$p(1,a,1) = (\bar{a} \cdot 1) \circ (a \cdot 1)$$

which means that using **(T3)** and (14), $1 = \bar{a} \circ a$ and, by duality, that $\bar{a} \cdot a = 0$. Therefore, $p(a,a,b) = (\bar{a} \cdot a) \circ (a \cdot b) = a \cdot b$. Conditions (iv) and (v) are equivalent according to duality (13). Proposition 1 proves that (iv) implies (i). It remains to prove (i) ⇒ (ii); that is, if $(A, +, \cdot, 0, 1)$ is a Boolean ring, then $(A, \circ, \cdot, \overline{()}, 0, 1)$ is a Boolean algebra with $\bar{a}$ defined as $a + 1$ and $a \circ b$ defined as $a + b + a \cdot b$. Indeed, firstly, we have

$$a + 1 = p(a, 1, \bar{a}) \overset{(\mathbf{T2})}{=} \bar{a} \tag{31}$$

and consequently,

$$a + (a+1) = a + \bar{a} \overset{(17),(27)}{\Rightarrow} a + \bar{a} = 1. \tag{32}$$

Secondly, we have

$$a + b + a \cdot b \overset{(26)}{=} a + (b \cdot (a+1)) \overset{(31)}{=} a + (b \cdot \bar{a}) \tag{33}$$

$$\begin{aligned} \Rightarrow a + b + a \cdot b &\overset{(33)}{=} p(a, p(0,b,\bar{a}), \bar{a}) \overset{(\mathbf{T4}),(\mathbf{T2})}{=} p(a, b, p(a, \bar{a}, \bar{a})) \\ &= p(a, b, a + \bar{a}) \overset{(32)}{=} p(a,b,1) = a \circ b. \end{aligned}$$

□

It is straightforward to prove that the category of Boolean algebra is isomorphic to the category of structures $(X, p, 0, 1)$ of type (3,1,1) verifying **(T1)**–**(T4)** and $p(a,b,c) = p(p(0, \bar{b}, a), p(0,b,c), 1)$ or any other equivalent relation presented in Theorem 1. A detailed proof of this result is given in the next section for the more general case of de Morgan algebras.

4. De Morgan Algebras

A de Morgan algebra is a structure $(A, \cdot, \circ, \overline{()}, 0, 1)$ consisting of a bounded distributive lattice $(A, \cdot, \circ, 0, 1)$ together with an involution $\overline{()}$ verifying $\overline{a \cdot b} = \bar{b} \circ \bar{a}$ (or $\overline{a \circ b} = \bar{b} \cdot \bar{a}$). Simple examples of de Morgan algebras are the sets of divisors of any given positive integer n with gdc as $\cdot$, lmc as $\circ$, and $\bar{a} = n/a$. Multiple-valued logic [20–22] is also an example of de Morgan algebras. In this section, a characterization of de Morgan algebras in terms of a ternary structure is given.

Theorem 2. *Let $(A, p, 0, 1)$ be a system consisting of a set A, a ternary operation p, and two constants $0, 1 \in A$ satisfying the conditions* **(T1)** *to* **(T4)***. For $a \cdot b = p(0, a, b)$, $a \circ b = p(a, b, 1)$, and $\bar{a} = p(1, a, 0)$, the following conditions are equivalent:*

(i) *The system $(A, \circ, \cdot, \overline{()}, 0, 1)$ is a de Morgan algebra;*
(ii) *The system $(A, \circ, \cdot)$ is a distributive lattice;*
(iii) *$(A, \circ)$ is a join-semilattice;*
(iv) *$(A, \cdot)$ is a meet-semilattice;*
(v) *$(A, \circ)$ is an idempotent and commutative magma;*
(vi) *$(A, \cdot)$ is an idempotent and commutative magma;*
(vii) *$p(a, b, c) = (\bar{b} \cdot a) \circ (a \cdot c) \circ (b \cdot c)$;*
(viii) *$p(a, b, c) = (\bar{b} \circ c) \cdot (b \circ a) \cdot (a \circ c)$.*

Proof. We begin by proving that if $(A, p, 0, 1)$ is a system verifying the conditions **(T1)–(T4)**, then (vi) implies (i). Indeed, $(A, \cdot, 0)$ and $(A, \circ, 1)$ are monoids, as demonstrated in Lemma 1. The commutativity and idempotency of $\cdot$ are precisely what (vi) states. The commutativity and idempotency of $\circ$ follow by duality. The distributivity of $\cdot$ and $\circ$ over each other is guaranteed by Lemma 3. As shown in Lemma 1, the unary operation $\overline{()}$ is an involution and verifies the de Morgan laws (13), which concludes the proof (vi)⇒(i).

We will now prove that the ternary operation (vii) is the only one compatible with (i). It is well known (see, e.g., [18,19]) that in a distributive lattice, if $x \cdot a = x' \cdot a$ and $a \circ x = a \circ x'$ for some given element a in the lattice, then $x = x'$. We show here that if $(A, p, 0, 1)$ verifies **(T1)–(T4)** and $(A, \circ, \cdot, \overline{()}, 0, 1)$ is a de Morgan algebra, then

$$\begin{cases} p(a, b, c) \cdot c &= ((\bar{b} \cdot a) \circ (a \cdot c) \circ (b \cdot c)) \cdot c \\ c \circ p(a, b, c) &= c \circ ((\bar{b} \cdot a) \circ (a \cdot c) \circ (b \cdot c)) \end{cases},$$

which proves (i)→(vii). Indeed,

$$\begin{aligned} p(a, b, c) \cdot c &= p(0, p(a, b, c), c) \overset{(\mathbf{T4})}{=} p(p(0, a, c), b, p(0, c, c)) \\ &\overset{(24)}{=} p(p(0, a, c), b, c) \overset{(\mathbf{T2})}{=} p(p(0, a, c), b, p(0, 1, c)) \\ &\overset{(\mathbf{T4})}{=} p(0, p(a, b, 1), c) = (a \circ b) \cdot c; \end{aligned}$$

$$((\bar{b} \cdot a) \circ (a \cdot c) \circ (b \cdot c)) \cdot c = ((\bar{b} \cdot a) \circ a \circ b) \cdot c = (a \circ b) \cdot c;$$

$$
\begin{aligned}
c \circ p(a,b,c) &= p(c,p(a,b,c),1) \overset{(\mathbf{T4})}{=} p(p(c,a,1)),b,p(c,c,1)) \\
&\overset{(24)}{=} p(p(c,a,1),b,c) \overset{(\mathbf{T2})}{=} p(p(c,a,1),b,p(c,0,1)) \\
&\overset{(\mathbf{T4})}{=} p(c,p(a,b,0),1) \overset{(9)}{=} p(c,p(0,\bar{b},a),1) \\
&= c \circ (\bar{b} \cdot a);
\end{aligned}
$$

$$c \circ (\bar{b} \cdot a) \circ (a \cdot c) \circ (b \cdot c) = c \circ ((a \circ b) \cdot c) \circ (\bar{b} \cdot a) = c \circ (\bar{b} \cdot a).$$

We now turn to the proof that condition (vii) implies condition (vi) in the context of Theorem 2. When (vii) is true, we have in particular that $p(b,1,a) = (\bar{1} \cdot b) \circ (b \cdot a) \circ (1 \cdot a)$ which means, using **(T2)**, (7), (14), (14), and (15), that

$$a = (b \cdot a) \circ a. \tag{34}$$

By duality, we also have the absorption rule

$$a \cdot (a \circ b) = a. \tag{35}$$

The idempotency of $\cdot$ and $\circ$ follows as particular cases:

$$a \overset{(34)}{=} (1 \cdot a) \circ a \overset{(14)}{=} a \circ a, \qquad a \overset{(35)}{=} a \cdot (a \circ 0) \overset{(14)}{=} a \cdot a. \tag{36}$$

When (vii) is true, we also have $p(a,b,a) = (\bar{b} \cdot a) \circ (a \cdot a) \circ (b \cdot a)$, which means, using **(T3)**, (36), and (34), that

$$a = a \circ (b \cdot a). \tag{37}$$

Now, the commutativity of $\circ$ can be proven as follows:

$$a \circ b = p(a,b,1) \overset{(9)}{=} p(1,\bar{b},a) \overset{(iii),(8)}{=} (b \cdot 1) \circ (1 \cdot a) \circ (\bar{b} \cdot a) \overset{(14),(37)}{=} b \circ a.$$

The commutativity of $\cdot$ follows by duality. This proves (vii)⇒(vi) and concludes the proof of Theorem 2 because the other equivalences are trivially verified. □

We show now that the category of de Morgan algebras is isomorphic to the category of systems $(A,p,0,1)$ satisfying conditions **(T1)**–**(T4)** when the operations $\cdot$ or $\circ$ are commutative and idempotent.

Theorem 3. *Let $(A,p,0,1)$ be a system consisting of a set A, together with a ternary operation p and two constants $0,1 \in A$ satisfying conditions* **(T1)** *to* **(T4)** *and*

(T5) $p(0,a,b) = p(0,b,a)$ *and* $p(0,a,a) = a$.

The category of such systems is isomorphic to the category of de Morgan algebras.

Proof. Consider a system $(A,p,0,1)$ verifying **(T1)**–**(T5)** and define

$$a \cdot b = p(0,a,b), \; a \circ b = p(a,b,1) \text{ and } \bar{a} = p(1,a,0). \tag{38}$$

Then, according to Theorem 2, $(A,\cdot,\circ,\overline{()},0,1)$ is a de Morgan algebra. Conversely, consider a de Morgan algebra $(A,\cdot,\circ,\overline{()},0,1)$ and define the ternary operation

$$p(a,b,c) = (\bar{b} \cdot a) \circ (a \cdot c) \circ (b \cdot c) \tag{39}$$

then p verifies axioms **(T1)**–**(T5)**. We will last prove that **(T4)** is verified. For the other axioms,

$$\begin{aligned}
p(0,a,1) &= (\bar{a}\cdot 0)\circ(a\cdot 1)\circ(0\cdot 1) = 0\circ a\circ 0 = a \\
p(a,b,a) &= (\bar{b}\cdot a)\circ(b\cdot a)\circ(a\cdot a) \\
&= (\bar{b}\cdot a)\circ(b\cdot a)\circ a = (\bar{b}\cdot a)\circ a = a \\
p(a,0,b) &= (1\cdot a)\circ(0\cdot b)\circ(a\cdot b) = a\circ(a\cdot b) = a \\
p(a,1,b) &= (0\cdot a)\circ(1\cdot b)\circ(a\cdot b) = b\circ(a\cdot b) = b \\
p(0,a,b) &= (\bar{a}\cdot 0)\circ(0\cdot b)\circ(a\cdot b) = (a\cdot b) = (b\cdot a) = p(0,b,a) \\
p(0,a,a) &= (\bar{a}\cdot 0)\circ(0\cdot a)\circ(a\cdot a) = a.
\end{aligned}$$

Before proving **(T4)**, note that within a de Morgan algebra, (39) can be written as

$$p(a,b,c) = (\bar{b}\circ c)\cdot(b\circ a)\cdot(a\circ c).$$

Using this result and also (10), we have

$$\begin{aligned}
p(a,p(b_1,b_2,b_3),c) &= ([(\bar{b}_2\cdot\bar{b}_1)\circ(b_2\cdot\bar{b}_3)\circ(\bar{b}_1\cdot\bar{b}_3)]\cdot a) \\
&\quad\circ([(\bar{b}_2\cdot b_1)\circ(b_2\cdot b_3)\circ(b_1\cdot b_3)]\cdot c) \\
&\quad\circ(a\cdot c) \\
&= ((\bar{b}_2\cdot\bar{b}_1\cdot a)\circ(b_2\cdot\bar{b}_3\cdot a)\circ(\bar{b}_1\cdot\bar{b}_3\cdot a)) \\
&\quad\circ((\bar{b}_2\cdot b_1\cdot c)\circ(b_2\cdot b_3\cdot c)\circ(b_1\cdot b_3\cdot c)) \\
&\quad\circ(\bar{b}_2\cdot a\cdot c)\circ(b_2\cdot a\cdot c)\circ(a\cdot c) \\
&= (\bar{b}_2\cdot[(\bar{b}_1\cdot a)\circ(b_1\cdot c)\circ(a\cdot c)]) \\
&\quad\circ(b_2\cdot[(\bar{b}_3\cdot a)\circ(b_3\cdot c)\circ(a\cdot c)]) \\
&\quad\circ(\bar{b}_1\cdot\bar{b}_3\cdot a)\circ(b_1\cdot b_3\cdot c)\circ(a\cdot c)
\end{aligned}$$

The last line is equal to

$$\begin{aligned}
p(a,b_1,c)\cdot p(a,b_3,c) &= ((\bar{b}_1\cdot a)\circ(b_1\cdot c)\circ(a\cdot c)) \\
&\quad\cdot((\bar{b}_3\cdot a)\circ(b_3\cdot c)\circ(a\cdot c)) \\
&= (\bar{b}_1\cdot\bar{b}_3\cdot a)\circ(\bar{b}_1\cdot b_3\cdot a\cdot c)\circ(\bar{b}_1\cdot a\cdot c) \\
&\quad\circ(b_1\cdot\bar{b}_3\cdot a\cdot c)\circ(b_1\cdot b_3\cdot c)\circ(b_1\cdot a\cdot c) \\
&\quad\circ(\bar{b}_3\cdot a\cdot c)\circ(b_3\cdot a\cdot c)\circ(a\cdot c) \\
&= (\bar{b}_1\cdot\bar{b}_3\cdot a)\circ(b_1\cdot b_3\cdot c)\circ(a\cdot c),
\end{aligned}$$

and consequently, we have that

$$\begin{aligned}
p(a,p(b_1,b_2,b_3),c) &= (\bar{b}_2\cdot p(a,b_1,c))\circ(b_2\cdot p(a,b_3,c)) \\
&\quad\circ(p(a,b_1,c)\cdot p(a,b_3,c)) \\
&= p(p(a,b_1,c),b_2,p(a,b_3,c)).
\end{aligned}$$

If we start with a de Morgan algebra $(A, \circ, \cdot, \overline{()}, 0, 1)$ and construct a system $(A, p, 0, 1)$ through (39), then the de Morgan algebra $(A, \circ', \cdot', \overline{()}', 0, 1)$ obtained from it is exactly the same as the original one. Indeed,

$$\begin{aligned} a \circ' b = p(a,b,1) &= (\bar{b} \cdot a) \circ (a \cdot 1) \circ (b \cdot 1) \\ &= (\bar{b} \cdot a) \circ a \circ b = a \circ b \\ a \cdot' b = p(0,a,b) &= (\bar{a} \cdot 0) \circ (0 \cdot b) \circ (a \cdot b) = a \cdot b \\ \bar{a}' = p(1,a,0) &= (\bar{a} \cdot 1) \circ (1 \cdot 0) \circ (a \cdot 0) = \bar{a}. \end{aligned}$$

If starting with a system $(A, p, 0, 1)$, verifying **(T1)–(T5)**, and constructing a de Morgan algebra $(A, \circ, \cdot, \overline{()}, 0, 1)$ through (38), then the system $(A, p', 0, 1)$ obtained from it is exactly the same as the original one. Indeed,

$$\begin{aligned} p'(a,b,c) &= (\bar{b} \cdot a) \circ (a \cdot c) \circ (b \cdot c) \\ &= p(p(p(0,\bar{b},a), p(0,a,c), 1), p(0,b,c), 1) \\ &= p(a,b,c). \end{aligned}$$

Morphisms are defined as usual in a de Morgan algebra. On the ternary side, a morphism f verifies the following requirements:

$$f(0) = 0, f(1) = 1, f(p(a,b,c)) = p(f(a), f(b), f(c)).$$

All the morphisms are trivially preserved by the isomorphism. □

Expression (vii) in Theorem 2 implies that in a de Morgan algebra, $p(a,a,b) = (\bar{a} \cdot a) \circ (a \cdot b)$ and $p(a,b,b) = (\bar{b} \circ b) \cdot (a \circ b)$. This means that in a general de Morgan algebra, the binary operation $p(a,a,b)$ is different from $a \cdot b$, and $p(a,b,b)$ is different from $a \circ b$. These operations are equal when the complement is Boolean, in which case the de Morgan algebra is a Boolean algebra. It is worth noting too that in a de Morgan algebra, the notion of a sum still remains from the ternary structure through $a + b = p(a,b,\bar{a})$ and that $(A, +, 0)$ is a monoid.

5. Rings and Near-Rings of Characteristic Two

Each new interpretation of the ternary operation p satisfying axioms **(T1)–(T4)** in terms of its derived operations is equivalent to adding a new axiom and gives rise to a new subvariety, as Theorems 4 and 5 illustrate. An example of a ternary operation obtained from a unitary Abelian near-ring [23], which is not necessarily determined by its derived operations, is presented in the next proposition. The algebraic model of the unit interval considered in [15,24] is another example.

Proposition 2. *If $(A, +, \cdot, 0, 1)$ is a unitary Abelian (right) near-ring, in which $a \cdot 0 = 0$, then $(A, p, 0, 1)$ with $p(a,b,c) = a + b(c - a)$ satisfies axioms* **(T1)** *to* **(T4)**.

Proof. The proof is straightforward. □

When $a + b(c - a)$ is changed to $(1 - b)a + bc$ and $1 - b$ is replaced by $1 + b$, the formula for p, presented in Theorem 4 below, is obtained. Recall that a ring (or a near-ring) of characteristic 2 is such that $b + b = 0$ for all b, so that $1 - b = 1 + b$. Moreover, rewriting $(1 + b)a + bc$ as $a + b(a + c)$ gives the formula used in Theorem 5.

Theorem 4. *Suppose that* $(A, p, 0, 1)$ *satisfies axioms* **(T1)** *to* **(T4)**. *For*

$$\bar{a} = p(1, a, 0), \ a \cdot b = p(0, a, b) \ \text{and} \ a + b = p(a, b, \bar{a}),$$

the following conditions are equivalent:

(i) $(A, +, \cdot, 0, 1)$ *is a unitary ring of characteristic* 2*;*
(ii) $p(a, b, c) = (\bar{b} \cdot a) + (b \cdot c)$*;*
(iii) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Proof. It is clear that (i) implies (iii). According to (18), (iii) implies that $a + a = a \cdot (1 + 1) = a \cdot 0 = 0$ and hence, considering (14), (17), and Lemma 4, (iii) implies (i). Moreover, when $a + a = 0$,

$$a + p(a, b, c) = p(a, p(a, b, c), \bar{a}) \ = \ p(p(a, a, \bar{a}), b, p(a, c, \bar{a})) = b \cdot (a + c). \tag{40}$$

Consequently, (iii) implies (ii) as $p(a, b, c) = a + ba + bc = (1 + b)a + bc$. It remains to prove that (ii) implies (iii). According to **(T3)**, (17), and (18), (ii) implies $1 = p(1, a, 1) = \bar{a} + a = 1 + a + a$, i.e., $a + a = 0$. Then, (40) and Lemma 4 imply left distributivity:

$$a \cdot (b + c) = b + p(b, a, c) = b + (1 + a)b + ac = ab + ac.$$

□

The unique non-commutative ring of order 8, say consisting of all upper triangular binary 2-by-2 matrices, illustrates Theorem 4. Note that the addition is the Boolean symmetric difference, as in a Boolean ring.

Theorem 5. *Suppose that* $(A, p, 0, 1)$ *satisfies axioms* **(T1)** *to* **(T4)**. *For*

$$\bar{a} = p(1, a, 0), \ a \cdot b = p(0, a, b) \ \text{and} \ a + b = p(a, b, \bar{a}),$$

the following conditions are equivalent:

(i) $(A, +, \cdot, 0, 1)$ *is a unitary (right) near ring of characteristic* 2*;*
(ii) $p(a, b, c) = a + (b \cdot (a + c))$*;*
(iii) $a + a = 0$*;*
(iv) $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$.

Proof. It is clear that (i) implies (iii), and considering Lemma 4 and the associativity of $+$, (iii) implies (i). Using (40), condition (iii) implies (ii):

$$a + p(a, b, c) = b \cdot (a + c) = a + (a + b \cdot (a + c)).$$

Using (17) and **(T2)**, condition (ii) implies (iii): $0 = p(a, 1, 0) = a + a$. Properties (18) and Lemma 4 show that (iii) and (iv) are equivalent. □

Several examples of unitary (right) near-rings of characteristic 2 with four elements can be found. The following example illustrates Theorem 5. Multiplication is neither commutative nor idempotent, and addition is the same as Boolean symmetric difference. Note that if the formula $(1+y)x+yz$ is used as the ternary operation p instead of $x+y(x+z)$, then **(T4)** is not satisfied.

$\cdot$	0	u	v	1
0	0	0	0	0
u	0	0	0	u
v	0	u	v	v
1	0	u	v	1

$+$	0	u	v	1
0	0	u	v	1
u	u	0	1	v
v	v	1	0	u
1	1	v	u	0

6. MV-Algebra

An MV-algebra is a structure of type $(2,1,0)$ that can be defined as follows.

Definition 2. *A MV-algebra is a system* $(X,\circ,\overline{()},0)$ *such that*

(M1) $x\circ(y\circ z)=(x\circ y)\circ z$;
(M2) $0\circ x=x$;
(M3) $\overline{\overline{x}}=x$;
(M4) $\overline{0}\circ x=\overline{0}$;
(M5) $x\circ\overline{x\circ\overline{y}}=y\circ\overline{y\circ\overline{x}}$.

Usually, the commutativity of the binary operation $\circ$ is included as an axiom of MV-algebras. Nevertheless, Kolařík [25] proved that the commutativity of $\circ$ is a consequence of the other axioms. The canonical example of a MV-algebra consist of the set $X=[0,1]$ and the operations $\bar{x}=1-x$ and $x\circ y=min(x+y,1)$.

Next, some well-known properties of MV-algebras are presented.

Proposition 3. *Let* $(X,\circ,\overline{()},0)$ *be an MV-algebra, and consider the following notations:*

$$1=\bar{0}$$
$$x\cdot y=\overline{\bar{y}\circ\bar{x}}$$
$$x\vee y=x\circ(y\cdot\bar{x})$$
$$x\wedge y=(\bar{y}\circ x)\cdot y.$$

Then, the following properties hold:

- $x\circ y=y\circ x$ *and* $x\cdot y=y\cdot x$;
- $x\circ\bar{x}=1$ *and* $x\cdot\bar{x}=0$;
- $x\vee y=\overline{\bar{y}\wedge\bar{x}}$;
- $(X,\vee,\wedge,0,1)$ *is a distributive lattice;*
- $(X,\circ,\overline{()},0)$ *is a Boolean algebra* $\Leftrightarrow x\circ y=x\vee y\Leftrightarrow x\circ x=x$;
- $x\circ(y\wedge z)=(x\circ y)\wedge(x\circ z)$ *and* $x\cdot(y\vee z)=(x\cdot y)\vee(x\cdot z)$.

Proof. See, for instance, [26]. □

Considering these properties and Lemma 5, if an MV-algebra comes from a **(T1)–(T4)** ternary structure, then it is a Boolean algebra. This means that a ternary structure isomorphic to a general MV-algebra cannot contain the full **(T1)–(T4)** structure. We propose here replacing Axiom **(T4)** with some of its consequences, namely particular cases of properties (11) and (14). This implies that properties (9) and (12) will not apply in general, and consequently, the operations $\circ$ and $\vee$ will be different. We call the resulting structure a ternary MV-algebra.

Definition 3. *A ternary MV-algebra is a system* $(A,p,0,1)$ *consisting of a set* A*, together with a ternary operation* $p:A\times A\times A\to A$ *and two constants* $0,1\in A$ *satisfying*

(T1) $p(0,a,1)=a$;

(T2) $p(a,0,b) = a = p(b,1,a)$;
(T3) $p(a,b,a) = a$;
(T4-1) $p(a,p(b,c,1),1) = p(p(a,b,1),c,1)$;
(T4-2) $p(0,b,c) = \overline{p(\bar{c},\bar{b},1)}$ *and* $p(1,b,c) = \overline{p(\bar{c},\bar{b},0)}$;
(TMV) $p(a,b,c) = p(p(\bar{b},\bar{a},0),p(0,c,b),1)$.

In the next propositions, it is shown that a ternary MV-algebra is isomorphic to an MV-algebra.

Proposition 4. *Let* $(A,p,0,1)$ *be a ternary MV-algebra. Then, for* $\bar{a} = p(1,a,0)$ *and* $a \circ b = p(a,b,1)$, *the structure* $(A,\circ,\overline{()},0)$ *is an MV-algebra.*

Proof. It is clear that 0 and 1 are still complements of each other:

$$\bar{0} = p(1,0,0) \overset{(\mathbf{T2})}{=} 1,\ \bar{1} = p(1,1,0) \overset{(\mathbf{T2})}{=} 0.$$

Now, we can prove the five axioms of Definition 2:

$$\text{(M1) } x \circ (y \circ z) = p(x,p(y,z,1),1) \overset{(\mathbf{T4\text{-}1})}{=} p(p(x,y,1),z,1) = (x \circ y) \circ z$$
$$\text{(M2) } 0 \circ x = p(0,x,1) \overset{(\mathbf{T1})}{=} x$$
$$\text{(M3) } x \overset{(\mathbf{T1})}{=} p(0,x,1) \overset{(\mathbf{T4\text{-}2})}{=} \overline{p(\bar{1},\bar{x},\bar{0})} \overset{(\mathbf{T2})}{=} \overline{p(0,\bar{x},1)} \overset{(\mathbf{T1})}{=} \bar{\bar{x}}$$
$$\text{(M4) } 1 \circ x = p(1,x,1) \overset{(\mathbf{T3})}{=} 1$$
$$\begin{aligned}\text{(M5) } x \circ \overline{(x \circ \bar{y})} &= p(x,\overline{p(x,\bar{y},1)},1) \overset{(\mathbf{T4\text{-}2})}{=} p(x,p(0,y,\bar{x}),1)\\ &\overset{(\mathbf{T2})}{=} p(p(x,\bar{1},0),p(0,y,\bar{x}),1) \overset{(\mathbf{TMV})}{=} p(1,\bar{x},y) \overset{(\mathbf{T4\text{-}2})}{=} \overline{p(\bar{y},x,0)}\\ &\overset{(\mathbf{TMV})}{=} \overline{p(p(\bar{x},y,0),p(0,0,x),1)} \overset{(\mathbf{T2})}{=} \overline{p(\bar{x},y,0)} \overset{(\mathbf{T4\text{-}2})}{=} p(1,\bar{y},x)\\ &= y \circ \overline{(y \circ \bar{x})}.\end{aligned}$$

□

Proposition 5. *Let* $(A,\circ,\overline{()},0)$ *be an MV-algebra, and consider the usual dual operation* $a \cdot b = \overline{\bar{b} \circ \bar{a}}$. *Then, for*

$$p(a,b,c) = ((\bar{a} \circ \bar{b}) \cdot a) \circ (b \cdot c), \quad (41)$$

the structure $(A,p,0,1)$ *is a ternary MV-algebra.*

Proof. First, from (41) and using Proposition 3, we observe the following correspondences.

$p(1,a,0) = \bar{a}$
$p(0,a,b) = 0 \circ (a \cdot b) = a \cdot b$
$p(b,\bar{a},0) = (\bar{b} \circ a) \cdot b = a \wedge b$
$p(1,\bar{b},a) = b \circ (\bar{b} \cdot a) = b \circ (a \cdot \bar{b}) = b \vee a = a \vee b$
$p(a,b,1) = ((\bar{a} \circ \bar{b}) \cdot a) \circ b = (\bar{b} \wedge a) \circ b = (\bar{b} \circ b) \wedge (a \circ b) = a \circ b.$

Note that these results are compatible with definitions (1) to (5).

Now, we can prove the properties of p included in Definition 3:

$$\begin{aligned}
&\text{(T1) } p(0,a,1) = 0 \circ (a \cdot 1) = a \\
&\text{(T2) } p(a,0,b) = (1 \cdot a) \circ 0 = a \text{ and } p(a,1,b) = (\bar{a} \cdot a) \circ b = 0 \circ b = b \\
&\text{(T3) } p(a,b,a) = ((\bar{a} \circ \bar{b}) \cdot a) \circ (b \cdot a) = (\overline{b \cdot a} \cdot a) \circ (b \cdot a) = (b \cdot a) \vee a \\
&\qquad = (b \cdot a) \vee (1 \cdot a) = (b \vee 1) \cdot a = 1 \cdot a = a \\
&\text{(T4-1) } p(a, p(b,c,1), 1) = a \circ (b \circ c) = (a \circ b) \circ c = p(p(a,b,1),c,1) \\
&\text{(T4-2) } p(0,b,c) = b \cdot c = \overline{\bar{c} \circ \bar{b}} = \overline{p(\bar{c},\bar{b},1)} \\
&\text{(T4-2) } p(1,b,c) = c \vee \bar{b} = \overline{b \wedge \bar{c}} = \overline{p(\bar{c},\bar{b},0)} \\
&\text{(TMV) } p(a,b,c) = ((\bar{a} \circ \bar{b}) \cdot a) \circ (b \cdot c) = (\bar{b} \wedge a) \circ (c \cdot b) = (a \wedge \bar{b}) \circ (c \cdot b) \\
&\qquad = p(p(\bar{b},\bar{a},0), p(0,c,b), 1).
\end{aligned}$$

□

Theorem 6. *Ternary MV-algebras and MV-algebras are isomorphic.*

Proof. Let $(A, p, 0, 1)$ be a ternary MV-algebra, and consider the operations $\bar{a} = p(1,a,0)$ and $a \circ b = p(a,b,1)$. Then, according to Proposition 4, the structure $(A, \circ, \overline{()}, 0)$ is an MV-algebra. Through Proposition 5, a new ternary MV-algebra $(A, p', 0, 1)$ is recovered. We prove now that the new ternary MV-algebra is equal to the original one, i.e., that $p' = p$:

$$\begin{aligned}
p'(a,b,c) &= ((\bar{a} \circ \bar{b}) \cdot a) \circ (b \cdot c) = (\bar{b} \wedge a) \circ (b \cdot c) \\
&= (a \wedge \bar{b}) \circ (c \cdot b) = p(p(\bar{b},\bar{a},0), p(0,c,b), 1) = p(a,b,c).
\end{aligned}$$

Conversely, let us begin with an MV-algebra $(A, \circ, \overline{()}, 0)$. Then, using Proposition 5, a ternary MV-algebra $(A, p, 0, 1)$ is obtained, with $p(a,b,c) = ((\bar{a} \circ \bar{b}) \cdot a) \circ (b \cdot c)$, which, through Proposition 4, gives back an MV-algebra $(A, \circ', \overline{()}', 0)$. We now prove that $\circ' = \circ$ and $\overline{()}' = \overline{()}$:

$$\begin{aligned}
a \circ' b &= p(a,b,1) \\
&= ((\bar{a} \circ \bar{b}) \cdot a) \circ b = (\bar{b} \wedge a) \circ b = (\bar{b} \circ b) \wedge (a \circ b) = a \circ b. \\
\bar{a}' &= p(1,a,0) = ((0 \circ \bar{a}) \cdot 1) \circ (a \cdot 0) = \bar{a}.
\end{aligned}$$

□

To compare between the binary operations observed in Boolean and de Morgan algebras with equalities, let us notice that in a ternary MV-algebra, the following relations hold:

$$a \wedge b = p(a, \bar{b}, b) \quad \text{and} \quad b \vee a = p(a, \bar{a}, b).$$

Note also that in a Ternary MV-algebra, the notion of a sum is still well defined through $a + b = p(a, b, \bar{a})$, with $a + 0 = a = 0 + a$ and $a + 1 = \bar{a} = 1 + a$.

7. Conclusions

We have presented examples of ternary structures that provide a common background for several classical algebras. It has long been recognized that de Morgan algebras with Boolean complements or MV-algebras with idempotency are Boolean algebras. However, other characteristics of Boolean algebras, like the existence of a sum leading to Boolean rings, are not so easily generalized (see [27] for an example). Here, the sum defined as $x + y = p(x, y, \bar{x})$ is well defined in all structures derived from a ternary system and allows for generalization of the notion of a ring. With respect to the algebras considered here, we observe the following, Table 1:

Table 1. List of algebras considered and the corresponding properties.

	*T*1–*T*3	*T*4	BC †	$a \cdot a = a$	LD *
Boolean algebra	✓	✓	✓	✓	✓
De Morgan algebra	✓	✓	×	✓	×
MV-algebra	✓	×	✓	×	×
Ring, char = 2	✓	✓	×	×	✓
Near ring, char = 2	✓	✓	×	×	×

† BC means Boolean complement: $(1 + a) \cdot a = 0$. * LD means the left distributivity of $\cdot$ over $+$: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$. Much more can be pursued in the study of structures involving the operation $+$.

In Section 6, we explained how the initial ternary structure **(T1)–(T4)** is generalized to allow for MV-algebras. Of course, the structure can be modified in other ways given other type of structures. With only **(T1)–(T3)**, the unary operation $\bar{x} = p(1, x, 0)$ is not an involution, which could be a starting point for studying a ternary version of Heyting algebras, for example.

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