

RESEARCH ARTICLE | SEPTEMBER 01 2023

Uniformly distributed-order wave equation in higher dimensions

N. Vieira ; M. M. Rodrigues; M. Ferreira

 Check for updates

AIP Conference Proceedings 2849, 380002 (2023)

<https://doi.org/10.1063/5.0162287>


View
Online


Export
Citation

CrossMark

Articles You May Be Interested In

Eigenfunctions of the time-fractional telegraph equation of distributed order

AIP Conference Proceedings (September 2023)

Tauberian theorems for the weighted mean method of summability of integrals

AIP Conference Proceedings (December 2019)

A revisited Tauberian theorem for which slow decrease with respect to a weight function is a tauberian condition for the weighted mean summability of integrals over \mathbb{R}_+

AIP Conference Proceedings (March 2021)

08 September 2023 10:54:44

AIP Advances

Why Publish With Us?

 25 DAYS average time to 1st decision	 740+ DOWNLOADS average per article	 INCLUSIVE scope
---	--	---

[Learn More](#)

 AIP
Publishing

Uniformly Distributed-Order Wave Equation in Higher Dimensions

N. Vieira,^{1, a)} M.M. Rodrigues,^{1, b)} and M. Ferreira^{1,2, c)}

¹⁾*CIDMA - Center for Research and Development in Mathematics and Applications,
Department of Mathematics - University of Aveiro, Campus Universitário de Santiago 3810-193 Aveiro, Portugal.*

²⁾*School of Technology and Management - Polytechnic of Leiria, Campus 2 Morro do Lena – Alto do Vieiro,
P-2411-901 Leiria, Portugal*

^{a)}Corresponding author: nloureirovieira@gmail.com

^{b)}mrodrigues@ua.pt

^{c)}milton.ferreira@ipleiria.pt

Abstract In this short paper, we obtain the eigenfunctions of the uniformly distributed-order wave equation in $\mathbb{R}^n \times \mathbb{R}^+$, as Laplace integral of Fox H-functions. For the particular case of the first fundamental solution, the fractional moment of second order of the fundamental solution is studied using the Tauberian Theorem.

INTRODUCTION

In the last years, fractional partial differential equations with distributed order received increasing attention from researchers on differential equations. One reason of the interest is the relation of these equations with physical processes involving time-scales (see [1, 2] and references therein indicated). More recently, the analysis of fractional differential equations with distributed order has been extended to the case of Hilfer (or composite) fractional derivatives (see [6] for the case of higher dimensions). The Hilfer fractional derivative allows interpolating smoothly between the Riemann-Liouville and the Caputo fractional derivatives. Using Fourier-Laplace transformation techniques, we obtain an integral representation of the eigenfunctions of the uniformly distributed-order wave equation in $\mathbb{R}^n \times \mathbb{R}^+$ with time-fractional Hilfer fractional derivatives. For the particular case of the first fundamental solution, we make use of the Tauberian Theorem to study the second-order moment.

PRELIMINARIES

Let $a, b \in \mathbb{R}$ with $a < b$ and $\alpha > 0$. The left Riemann-Liouville fractional integral $I_{a^+}^\gamma$ of order $\gamma > 0$ is given by (see [1])

$$(I_{a^+}^\gamma f)(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(w)}{(x-w)^{1-\gamma}} dw, \quad x > a.$$

The Hilfer (or composite) fractional derivative ${}_t D_{0^+}^{\gamma, \nu}$ of order $\gamma > 0$ and type $0 \leq \nu \leq 1$ is given by (see [2])

$$({}_t D_{0^+}^{\gamma, \nu} f)(t) = \left(I_{0^+}^{\nu(m-\gamma)} \frac{d}{dt} \left(I_{0^+}^{(1-\nu)(m-\gamma)} f \right) \right) (t), \quad (1)$$

where $m = [\gamma] + 1$ and $[\gamma]$ means the integer part of γ . We observe that in the case when $\nu = 0$ we recover the left Riemann-Liouville fractional derivative and in the case when $\nu = 1$ we have the left Caputo fractional derivative. The Fox H-function $H_{p,q}^{m,n}$ is defined, via a Mellin-Barnes type integral, by (see [3])

$$H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)} z^{-s} ds, \quad (2)$$

where $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}^+$, for $i = 1, \dots, p$ and $j = 1, \dots, q$, and \mathcal{C} is a suitable contour in the complex plane separating the poles of the two factors in the numerator (see [3]). In [6] it is proved the following result (see Corollary 3.5):

Theorem 1 *The solution of the generalized time-fractional wave equation of distributed order in $\mathbb{R}^n \times \mathbb{R}^+$*

$$\int_0^1 \int_1^2 b_2(\beta, \nu) {}_t\partial_{0^+}^{\beta, \nu} u(x, t) d\beta d\nu - c^2 \Delta_x u(x, t) + d^2 u(x, t) = q(x, t) \quad (3)$$

for a given integrable order-density function $b_2(\beta, \nu)$, subject to the following initial and boundary conditions

$$({}_t I_{0^+}^{(1-\nu)(2-\beta)} u)(x, 0^+) = g_1(x), \quad \left[\frac{\partial}{\partial t} ({}_t I_{0^+}^{(1-\nu)(2-\beta)} u) \right] (x, 0^+) = g_2(x), \quad \lim_{|x| \rightarrow +\infty} u(x, t) = 0, \quad \int_0^1 \int_1^2 b_2(\beta, \nu) d\beta d\nu = C_1 \in \mathbb{R}^+,$$

where $(x, t) \in \mathbb{R}^n \times \mathbb{R}^+$, $c, d \in \mathbb{R}^+$, Δ_x is the classical Laplace operator in \mathbb{R}^n , and the partial time-fractional derivative of order $\beta \in]1, 2]$ and type $\nu \in]0, 1[$ is in the Hilfer sense given by (1), is given, in terms of convolution integrals, by

$$u(x, t) = \int_{\mathbb{R}^n} g_1(z) G_2(x-z, t) dz + \int_{\mathbb{R}^n} g_2(z) G_3(x-z, t) dz + \int_0^t \int_{\mathbb{R}^n} q(z, w) G_4(x-z, t-w) dw dz,$$

where G_2, G_3 , and G_4 are given by

$$G_2(x, t) = \frac{-1}{\pi^{\frac{n-1}{2}} (2|x|)^n} \int_0^{+\infty} \frac{re^{-rt}}{\rho \sin(\gamma\pi)} \left[\rho^* \sin(\gamma^*\pi) \mathcal{H} \left(\frac{1}{|x| \sqrt{\rho}} \right) + \frac{d^2}{c^2} \sin(\gamma\pi) \mathcal{H}^* \left(\frac{1}{|x| \sqrt{\rho}} \right) \right] dr,$$

$$G_3(x, t) = \frac{1}{\pi^{\frac{n-1}{2}} (2|x|)^n} \int_0^{+\infty} \frac{e^{-rt}}{\rho \sin(\gamma\pi)} \left[\rho^* \sin(\gamma^*\pi) \mathcal{H} \left(\frac{1}{|x| \sqrt{\rho}} \right) + \frac{d^2}{c^2} \sin(\gamma\pi) \mathcal{H}^* \left(\frac{1}{|x| \sqrt{\rho}} \right) \right] dr,$$

$$G_4(x, t) = \frac{-1}{c^2 \pi^{\frac{n-1}{2}} (2|x|)^n} \int_0^{+\infty} \frac{e^{-rt}}{\rho} \mathcal{H}^* \left(\frac{1}{|x| \sqrt{\rho}} \right) dr$$

with

$$\left\{ \begin{matrix} \rho = |B_2(re^{i\pi})| \\ \gamma = \frac{1}{\pi} \arg(B_2(re^{i\pi})) \end{matrix} \right\}, \quad \left\{ \begin{matrix} \rho^* = |B_2^*(re^{i\pi})| \\ \gamma^* = \frac{1}{\pi} \arg(B_2^*(re^{i\pi})) \end{matrix} \right\}, \quad \left\{ \begin{matrix} B_2(s) = \frac{1}{c^2} \left(\int_0^1 \int_1^2 b_2(\beta, \nu) s^\beta d\beta d\nu + d^2 \right) \\ B_2^*(s) = \frac{1}{c^2} \left(\int_0^1 \int_1^2 b_2(\beta, \nu) s^{-\nu(2-\beta)} d\beta d\nu + d^2 \right) \end{matrix} \right\} \quad (4)$$

and the functions \mathcal{H} and \mathcal{H}^* are expressed in terms of the following Fox H-functions

$$\mathcal{H} \left(\frac{1}{|x| \sqrt{\rho}} \right) = H_{3,2}^{0,2} \left[\frac{1}{|x| \sqrt{\rho}} \left| \begin{matrix} (1-n, 1), \left(1, \frac{1}{2}\right), \left(0, \frac{\gamma}{2}\right) \\ \left(\frac{1-n}{2}, \frac{1}{2}\right), \left(0, \frac{\gamma}{2}\right) \end{matrix} \right. \right], \quad \mathcal{H}^* \left(\frac{1}{|x| \sqrt{\rho}} \right) = H_{3,2}^{0,2} \left[\frac{1}{|x| \sqrt{\rho}} \left| \begin{matrix} (1-n, 1), \left(0, \frac{1}{2}\right), \left(-\gamma, \frac{\gamma}{2}\right) \\ \left(\frac{1-n}{2}, \frac{1}{2}\right), \left(-\gamma, \frac{\gamma}{2}\right) \end{matrix} \right. \right]. \quad (5)$$

Moreover, from the expression (86) in [6], we have that the second-order moment in the Laplace domain for the first fundamental solution of (3) ($g_1(x) = \delta(x)$, $g_2(x) = 0 = q(x, t)$, and $d = 0$) is given by

$$\tilde{M}^2(s) = \frac{2^{1-n} \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}} s B_2^*(s) (B_2(s))^{\frac{n-5}{2}}, \quad n \neq 5 + 2k, k \in \mathbb{N}_0. \quad (6)$$

In order to perform the Tauberian analysis in the last section, let us recall some necessary Laplace inversion formulas that can be found in [4]:

- Formula (2.1.1.1):

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^v} \right\} (t) = \frac{t^{v-1}}{\Gamma(v)}, \quad v > 0, \quad (7)$$

- Formula (2.5.1.12):

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^v} \ln^n(s) \right\} (t) = \left(-\frac{d}{d\mu} \right)^n \left[\frac{t^{\mu-1}}{\Gamma(\mu)} \right] \Bigg|_{\mu=v}, \quad n \in \mathbb{N}. \quad (8)$$

- Formula (2.5.6.5):

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^v} \ln^\mu(as) \right\} (t) = \frac{a^{v-1}}{\Gamma(-\mu)} \int_0^{+\infty} \frac{w^{-\mu-1}}{\Gamma(v+w)} \left(\frac{t}{a} \right)^{w+v-1} dw, \quad \text{Re}(\mu) < 0, a > 0, \text{Re}(s) > 0. \quad (9)$$

UNIFORMLY DISTRIBUTED-ORDER OPERATOR

The aim of this section is to obtain a representation in terms of Laplace integrals of Fox H-functions of the eigenfunctions of the uniformly distributed-order operator. In this sense let us consider the following eigenfunction equation

$$\int_1^2 {}_t\partial_{0^+}^{\beta, \eta} u(x, t) d\beta - c^2 \Delta_x u(x, t) = \lambda u(x, t), \quad (10)$$

subject to the following initial and boundary conditions

$$\left({}_tI_{0^+}^{(1-\eta)(2-\beta)} u \right) (x, 0^+) = \delta(x) = \prod_{i=1}^n \delta(x_i), \quad \left[\frac{\partial}{\partial t} \left({}_tI_{0^+}^{(1-\eta)(2-\beta)} u \right) \right] (x, 0^+) = 0, \quad \lim_{|x| \rightarrow +\infty} u(x, t) = 0. \quad (11)$$

The boundary value problem (10)-(11) is a particular case of Theorem 1, where b_2 is given by

$$b_2(\beta, v) = \delta(v - \eta) p(\beta), \quad \text{with } p(\beta) = 1 \quad \text{and } 0 < \eta < 1, \quad (12)$$

$d = i\sqrt{\lambda}$, $g_1(x) = \delta(x)$, and $g_2(x) = 0 = q(x, t)$. In these conditions, the integral representation of the solution of (10)-(11) is given by

$$u(x, t) = \frac{-1}{\pi^{\frac{n-1}{2}} (2|x|)^n} \int_0^{+\infty} \frac{r e^{-rt}}{\rho \sin(\gamma\pi)} \left[\rho^* \sin(\gamma^* \pi) \mathcal{H} \left(\frac{1}{|x| \sqrt{\rho}} \right) - \frac{\lambda}{c^2} \sin(\gamma\pi) \mathcal{H}^* \left(\frac{1}{|x| \sqrt{\rho}} \right) \right] dr, \quad (13)$$

where ρ , ρ^* , \mathcal{H} , and \mathcal{H}^* , are given, respectively by (4), and (5). Moreover, from (12), we have from (4)

$$B_2(s) = \frac{1}{c^2} \frac{s(s-1)}{\ln(s)}, \quad \text{and} \quad B_2(s) = \frac{1}{c^2 \eta} \frac{1-s^{-\eta}}{\ln(s)}. \quad (14)$$

Remark 2 If we consider $\lambda = 0$ in (10) the solution $u(x, t)$ corresponds to the first fundamental solution.

SECOND ORDER MOMENT OF THE FUNDAMENTAL SOLUTION

In this section, we obtain the expression for second-order moment of the first fundamental solution of (10) in the Laplace domain, and we apply the Tauberian Theorem to study the asymptotic behaviour in the time domain for $t \rightarrow 0^+$ and $t \rightarrow +\infty$. Considering (14) in (6) and making straightforward calculations, we arrive to

$$\tilde{\mathbf{M}}^2(s) = \frac{2^{1-n} \Gamma\left(\frac{5-n}{2}\right) s^{\frac{n-3}{2}} (1-s^{-\eta})(s-1)^{\frac{n-5}{2}}}{\pi^{\frac{n-1}{2}} c^{n-3} \eta (\ln(s))^{\frac{n-3}{2}}}, \quad n \neq 5 + 2k, k \in \mathbb{N}_0. \quad (15)$$

When $s \rightarrow 0^+$ we have the following asymptotic behaviour

$$\tilde{\mathbf{M}}^2(s) = \frac{(-1)^{\frac{n-3}{2}} 2^{1-n} \Gamma\left(\frac{5-n}{2}\right) s^{\frac{n-3}{2}-\eta} (1-s^\eta)(1-s)^{\frac{n-5}{2}}}{\pi^{\frac{n-1}{2}} c^{n-3} \eta (\ln(s))^{\frac{n-3}{2}}} \sim \frac{(-1)^{\frac{n-3}{2}} 2^{1-n} \Gamma\left(\frac{5-n}{2}\right) (\ln(s))^{\frac{3-n}{2}}}{\pi^{\frac{n-1}{2}} c^{n-3} \eta s^{\eta + \frac{3-n}{2}}}.$$

Making use of (8) (for $n = 1$), (7) (for $n = 3$), and (9) (for $n = 4 + 2k, k \in \mathbb{N}_0$) to invert the Laplace transform, and applying the Tauberian Theorem, we obtain for $t \rightarrow +\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \frac{2c^2}{\eta} \frac{t^\eta \ln(t)}{\Gamma(1+\eta)}, & n = 1 \\ \frac{c}{4\eta} \mathcal{L}^{-1} \left\{ \frac{(-\ln(s))^{\frac{1}{2}}}{s^{\eta+\frac{1}{2}}} \right\} (t), & n = 2 \\ \frac{1}{4\pi\eta} \frac{t^{\eta-1}}{\Gamma(\eta)}, & n = 3 \\ \frac{2^{1-n} \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}} c^{n-3} \eta} \frac{(-1)^{\frac{n-3}{2}}}{\Gamma\left(\frac{n-3}{2}\right)} \int_0^{+\infty} \frac{w^{\frac{n-5}{2}} t^{w+\eta+\frac{1-n}{2}}}{\Gamma(w+\eta+\frac{3-n}{2})} dw, & n = 4 + 2k, k \in \mathbb{N}_0 \end{cases}$$

When $s \rightarrow +\infty$ we have the following asymptotic behaviour

$$\tilde{\mathbf{M}}^2(s) \sim \frac{2^{1-n} \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}} c^{n-3} \eta} \frac{(\ln(s))^{\frac{3-n}{2}}}{s^{4-n}}.$$

Making use of (7) (for $n = 1$), (8) (for $n = 3$), and (9) (for $n = 4 + 2k, k \in \mathbb{N}_0$) to invert the Laplace transform, and applying the Tauberian Theorem, we obtain for $t \rightarrow 0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} \frac{c^2}{\eta} t^2 \ln\left(\frac{1}{t}\right), & n = 1 \\ \frac{c}{4\eta} \mathcal{L}^{-1} \left\{ \frac{(\ln(s))^{\frac{1}{2}}}{s^2} \right\} (t), & n = 2 \\ \frac{1}{4\pi\eta}, & n = 3 \\ \frac{2^{1-n} \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}} c^{n-3} \eta} \frac{1}{\Gamma\left(\frac{n-3}{2}\right)} \int_0^{+\infty} \frac{w^{\frac{n-5}{2}} t^{w+3-n}}{\Gamma(w+4-n)} dw, & n = 4 + 2k, k \in \mathbb{N}_0 \end{cases}$$

Remark 3 When $n = 1$, all the results presented in this section correspond to the correspondent ones obtained in [7] (Section 3). Moreover, when $n = 3$ if we consider the limit case of $\eta = 1$ (i.e., the case where the time-fractional partial derivatives are in the Caputo sense), we obtain the results presented in Section 5.1 of [5].

ACKNOWLEDGMENTS

The work of the authors was supported by Portuguese funds through CIDMA-Center for Research and Development in Mathematics and Applications, and FCT-Fundação para a Ciência e a Tecnologia, within projects UIDB/04106/2020 and UIDP/04106/2020. N. Vieira was also supported by FCT via the 2018 FCT program of Stimulus of Scientific Employment - Individual Support (Ref: CEECIND/01131/2018).

REFERENCES

1. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies-Vol.204, Elsevier, Amsterdam, 2006.
2. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
3. A. Kilbas and M. Saigo, *H-transforms. Theory and applications*, Analytical Methods and Special Functions-Vol.9, Chapman & Hall/CRC, Boca Raton, FL, 2004.
4. A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and series. Volume 5: Inverse Laplace transforms*, Gordon and Breach Science Publishers, New York etc., 1992.
5. N. Vieira, M.M. Rodrigues, and M. Ferreira, *Time-fractional telegraph equation of distributed order in higher dimensions*, Comm. Nonlinear Sci. Numer. Simulat., in press.
6. N. Vieira, M.M. Rodrigues, and M. Ferreira, *Time-fractional telegraph equation of distributed order in higher dimensions with Hilfer fractional derivatives*, submitted.
7. Ž. Tomovski and T. Sandev, *Distributed-order wave equations with composite time fractional derivative*, *Int. J. Comput. Math.* **95**(6-7) (2018) 1100–1113.