# The notion of multi-link, its applications and examples

by Nelson Martins-Ferreira

Centre for Rapid and Sustainable Product Development
ESTG, Polytechnic Institute of Leiria
martins.ferreira@ipleiria.pt

**Abstract** In this paper we introduce the notion of a multi-link which is a new mathematical structure that can be used as a tool for the encoding and systematization of new and more efficient algorithms in the aim of 3D-printing. Applications and examples are provided.

## 1 Introduction

From a computer's memory storage point of view, the notion of a 2-dimensional matrix, with lines and columns, does not make much sense. At least it does not make much more sense than an arbitrarily $n$-dimensional matrix. In practice, what is it that it is really stored into a computer's disk memory is simply an array, and the most efficient arrays are the linear ones. In spite of everything that has been said during the last two or three decades about formal file systems, the non-structural ones are still the ones that are preferred. This explains, for example, why the STL file format is still so common nowadays.

In this paper we are proposing a new mathematical structure which on the one hand can be stored as a linear array of information, while, on the other hand, it can be used to encode highly non-trivial structures such as surfaces and their properties. These properties, as we will see, may be decomposed into logical, functional and geometrical information and they cover most of the whole spectrum of processes that are involved in 3D-printing.

The new mathematical structure that we are introducing is an abstract system, called multi-link, and it was motivated by a long series of experiments with mathematical structures and their properties, namely the ones that are related with efficiency and encoding of information. See for example [4, 2, 8, 9] and the references therein.

This work is organized as follows. We first motivate the notion of a link, which can be seen as an abstraction for the notion of a curve (appropriate for computational purposes) and then, by generalizing it into several different ways, we get the notion of multi-link. This basic notion was the result of a long period of maturation and its main characteristic is the fact that it is suitable for the encoding of $n$-dimensional matrices as simple linear arrays. The key ingredient is the observation that the transition maps from $(i, j)$ to $(i + 1, j)$ and $(i, j + 1)$ can be seen as two permutable maps from the set of linearized indexes into itself. We give some details on this passage on section 3.

From section 3 onwards we concentrate our attention on the abstract notion of a multi-link by observing that it has several useful and important particular cases. Indeed, as we will see, each one of which has its purpose and can be applied into a very specific situation for 3D printing.

At he end we give a detailed description on the iso-slice algorithm, as well as an application to the generation of cooling and refrigerating channels in a mould. This will came later on, for the moment let us concentrate our attention on links, first, and then on multi-links, as a mathematical abstract structure.

## 2 The notion of a link and its motivation from a planar curve

A classical planar curve is usually defined as a continuous map from the unit interval $[0, 1]$ into the field of complex numbers. From the point of view of Mathematics this is a perfectly reasonable notion and it naturally extends to curves in the 3D-space. One simply substitute the field of complex numbers by the euclidean three space, and it is then just one more step to move to the $n$-dimensional vector field $\mathbb{R}^n$. However, from the point of view of computation, this is not really a good definition and many attempts have been made to find a better alternative. Several variations can be considered and each one of them has its own advantages and disadvantages. Here we consider one which seems to be good for the purposes of encoding contour level curves, the ones that are obtained from the slicing of triangulated surfaces, and their applications into the area of 3D-printing and direct digital manufacturing. The notion that we are proposing as an abstraction for a curve is called a link. It has arisen by observing that a curve, if approximated by a piecewise-linear sequence of directed edges, is a particular case of a directed graph. A directed graph is a mathematical object consisting in a set of vertices, a set of edges and two parallel maps that assign a vertex to an edge, namely its source and target. It turns out that some directed graphs, namely the ones that are obtained by taking an approximation to a curve, share the characteristic property of having a linking map. This linking map associates to each edge a successor edge along the direction of the curve. In this way we have arrived to the abstract notion of a link. This notion is intended to be a computational model for a classical curve.

A link is a mathematical object which consists of a set, together with an endomap and a map into a geometrical algebra (the notion of a geometrical algebra has a precise meaning in mathematics, however, the reader not familiar with it may safely assume that it is simply a vector space, and for the purpose of this paper, $\mathbb{R}^n$ will be enough), as illustrated

$$\varphi \circlearrowleft A \xrightarrow{g} \mathbb{R}^n .$$

This notion is thus interpreted as a generalized curve in $\mathbb{R}^n$ as follows. The curve is a piecewise linear sequence of segments; each segment in the curve is determined by an indexing element in the set of indexes $A$, and it is geometrically realised as the vector in $\mathbb{R}^n$ whose endpoints are

$$g(a) \longrightarrow g\varphi(a).$$

This means that each segment in the curve is indexed by an element in $A$, in fact we will sometimes pictured the segment as a labelled edge in a directed graph

$$g(a) \xrightarrow{a} g\varphi(a).$$

The set $A$ is called the set of indexes, the map $\varphi$ is called the successor, or transition map (it tells to each index, which is considered as the origin of the edge, what is its successor — the successor is at the same time the endpoint of its predecessor and the starting point of the edge of which it is the index of), this idea can be illustrated as follows

$$g(a) \xrightarrow{a} g\varphi(a) \xrightarrow{\varphi(a)} g\varphi^2(a) \longrightarrow \cdots$$

For further examples we refer the reader to [5].

# 3 Moving from an array to a matrix while keeping it linear

If we try to generalize the notion of a link as an abstraction of a curve into some new abstract entity which would serve as a good model for a surface, we would easily be led to something with the form

$$X \times Y \xrightarrow{\varphi} X \times Y \xrightarrow{g} \mathbb{R}^n .$$

However, as soon as we try to interpret it as a surface we soon realize that $\varphi(x, y) = (\varphi_1(x, y), \varphi_2(x, y))$ should be of the form

$$\begin{array}{ccc} (\varphi_1(x), y) & \longrightarrow & (\varphi_1(x), \varphi_2(y)) \\ \uparrow & & \uparrow \\ (x, y) & \longrightarrow & (x, \varphi_2(y)). \end{array}$$

In other words, it should consist on two independent maps $\varphi_1 \colon X \to X$ and $\varphi_2 \colon Y \to Y$, together with the realization (or geometrical) map

$$X \times Y \xrightarrow{g} \mathbb{R}^n .$$

The role of $\varphi_1$ and $\varphi_2$ is to determine the behaviour of the transitions along the $x$-direction and the $y$-direction. Note that these directions are only abstract and they should not be confused with the directions of $\mathbb{R}^n$.

Let us see a concrete example. Suppose we are interested in modelling the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \quad 0 \le z \le 1\},$$

in this case we could do an approximation, say, with $X = \{1, 2, 3, \ldots, 360\}$, $Y = \{0, 1\}$ and define the maps $\varphi_1 \colon X \to X$, $\varphi_2 \colon Y \to Y$ and $g \colon X \times Y \to \mathbb{R}^3$ as follows: $\varphi_1(x) = x + 1$ if $x < 360$ and $\varphi_1(360) = 1$; $\varphi_2(0) = \varphi_2(1) = 1$; and

$$g(u, v) = \left(\cos\left(\frac{2\pi u}{360}\right), \sin\left(\frac{2\pi u}{360}\right), v\right)$$

As already remarked at the introduction, the crucial point here is to observe that we may exchange the set $X \times Y$ with another set, which is bijective to it, say $A$, and the endomap $\varphi \colon X \times Y \to X \times Y$ with two endomaps $\alpha, \beta \colon A \to A$ that are permutable, i.e., $\alpha\beta = \beta\alpha$. In this way we form squares indexed by the elements of $A$ as illustrated

$$\begin{array}{ccc} \beta(a) & \longrightarrow & \alpha\beta(a) \\ \uparrow & & \uparrow \\ a & \longrightarrow & \alpha(a). \end{array}$$

If the set $X$ has $n_X$ elements, and the set $Y$ has $n_Y$ elements, then we can take the set $A$ to be the set $\{1, 2, 3, \ldots, n_X n_Y\}$ and the well known bijection which transforms pairs of indexes $(i, j)$ into linear indexes $a = in_X + j$.

The notion of a multi-link is a natural generalization of the notion of link and it is motivated by the concrete examples of a square-link (like the one above), a double-link (which is a structure that models arbitrary surfaces) and several other that where designed for more specific purposes, such as contour filling algorithms or generating voxelized porous 3-dimensional physical structures from the real world.

# 4 Multi-link

The notion of multi-link arises thus as a need to encode and organize the whole bunch of information in the form of data and algorithms that are used in the whole process of 3D printing in general and conceptual terms.

A multi-link is a mathematical object consisting in a set, called a set of indexes, say A, a collection of endomaps, say $\alpha_i \colon A \to A$ with $i \in \mathbb{N}$, called the transition maps, a map, say $g$ from the set of indexes into a geometrical algebra (again the reader not familiar with a geometrical algebra may assume that it is simply a finite dimensional vector space), this map is called the geometric realization map, and a collection of surjective maps $p_j\colon A \to B_j$ with $j \in \mathbb{N}$. The maps may be subject to some commutativity conditions written in the form of equations expressed in terms of the composition of maps. The diagram displaying all the information may be pictured as follows.

$$\begin{array}{ccc} \alpha_i \circlearrowright A & \xrightarrow{g} & \mathbb{R}^n \\ \downarrow{\scriptstyle p_j} & & \\ B_j & & \end{array}$$

with $i \in \{1, 2, 3, \ldots, n\}$ and $j \in \{1, 2, 3, \ldots, m\}$.

The family of maps $(\alpha_i)$ is considered to be the logical part of the multi-link (since it deals with the indexes and re-indexation), the map $g$ is considered for obvious reasons to be the geometrical part, while the family of projections $(p_j)$ is considered as the functional part of the structure. This is because in most of the examples the projection maps are simply assigning some functional behaviour to the edges, like color properties or materials or other kind of physical interpretation which in practice gives to a specific edge what its role is, in the sense of what its function is about.

let us now see some concrete examples of the structure. In the following sections we give some details on how the structures can be used as well as from where they were motivated.

## 4.1 Particular cases as examples

### 4.1.1 Coloured link

A coloured link is simply a link with a surjective map into some set $C$, of colors, in other words it is of the form

$$\begin{array}{ccc} \alpha \circlearrowright A & \xrightarrow{g} & \mathbb{R}^n \\ \downarrow{\scriptstyle c} & & \\ C & & \end{array}$$

such that $c\alpha = c$.

It is interpreted as a link in which every edge has a certain color associated to it, and moreover, the edges in the same component (in the sense of orbits of $\alpha$) have the same color, but different components may have different colors.

### 4.1.2 A square-link

A square link was introduced above and it can be seen as a special case of a multi-link with two endomaps and no projections. Moreover the two endomaps, say $\alpha$ and $\beta$, have to be permutable, that is, we should have $\alpha\beta = \beta\alpha$. If we take the example of the cylinder from above and use the bijection

$$\phi\colon X \times Y \to A$$

from the cartesian product of $X = \{1, \ldots, 360\}$ and $Y = \{0, 1\}$ into $A = \{1, \ldots, 720\}$ which is defined by $\phi(i, j) = i + j360$, then, in order to give the structure of a square-link it remains to specify the maps $\alpha, \beta\colon A \to A$ and $g\colon A \to \mathbb{R}^3$. In this case we should put $\alpha(360) = 1$, $\alpha(720) = 361$ and $\alpha(x) = x + 1$ in the other cases. For the map $\beta$ we should have $\beta(x) = 360 + x$ if $x \leq 360$ and $\beta(x) = x$ for all the other values of $x \in A$. The map $g$ is now defined as

$$g(x) = \left(\cos\left(\frac{2\pi x}{360}\right), \sin\left(\frac{2\pi x}{360}\right), 0\right)$$

when $x$ is less or equal to 360 and

$$g(x) = \left(\cos\left(\frac{2\pi(x-360)}{360}\right), \sin\left(\frac{2\pi(x-360)}{360}\right), 1\right)$$

for the cases when $x$ is greater than 360.

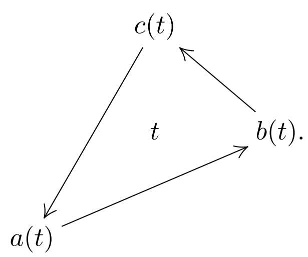
### 4.1.3 A triangulation

The structure of a triangulation has been studied in [6] and it is an important example of a multi-link.

The structure of a triangulation generalizes the one of a directed graph. It consists of two sets (vertices and triangles) and three parallel maps between them, as displayed

$$T \overset{a}{\underset{c}{\overset{\xrightarrow{\quad}}{\underset{\xrightarrow{\quad}}{\xrightarrow{b}}}}} V. \tag{4.1}$$

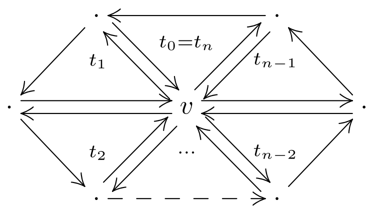
An element $t \in T$ is interpreted as a triangle in the following manner




In practice we are concerned with triangulated surfaces in the three dimensional euclidean space

$$T \overset{a}{\underset{c}{\overset{\xrightarrow{\quad}}{\underset{\xrightarrow{\quad}}{\xrightarrow{b}}}}} \mathbb{R}^3, \tag{4.2}$$

In [6] we explain that those triangulations which are obtained as the boundary of a physical real object in the 3-D euclidean space are the ones with the property that every vertex has a start-neighbourhood of triangles. In other words, these triangulations are the ones for which the collection of triangles that are incident into a given vertex can be cyclically ordered by sharing an adjacent face, as illustrated.




(4.3)

The passage from a triangulation into a multi-link is performed in [6] as a way of efficient encoding the triangulations that have the desired property, namely that all vertices have a star-neighbourhood. For practical reasons we substitute $\mathbb{R}^3$ with the Cayley algebra (or geometric algebra) of quaternions $\mathbb{H}$, see [1]. In [6] it is proved that the triangulations

$$T \overset{a}{\underset{c}{\xrightarrow{\;b\;}}} \mathbb{H}, \tag{4.4}$$

in which every vertex has a star-neighbourhood, are equivalent (has a mathematical structure, in the sense that they contain the same information) to a multi-link of the form

$$\theta,\varphi \circlearrowright A \xrightarrow{g} \mathbb{H} \tag{4.5}$$

such that

$$\theta^3 = 1_A \tag{4.6}$$

$$\theta^2 = \varphi\theta\varphi \tag{4.7}$$

$$g\varphi = g \tag{4.8}$$

and moreover, in order to have the *star-neighbourhood* property displayed in (4.3) one should add the requirement that $\varphi$ is an isomorphism.

#### 4.1.4 A double-link

The notion of a double-link encodes in the most general way the concept of a surface. Here we will only give the definition and the simple example of the tetrahedron. Further examples and the study of its main properties are postponed for future work. The examples of the other platonic solids can be found in [3].

A double-link is an instance of a multi-link which can be displayed as

$$\begin{array}{ccc} & \alpha,\beta \circlearrowright A \xrightarrow{g} \mathbb{R}^4 & \\ & {}^{f}\swarrow \quad \searrow^{v} & \\ F & & V \end{array}$$

and it is such that

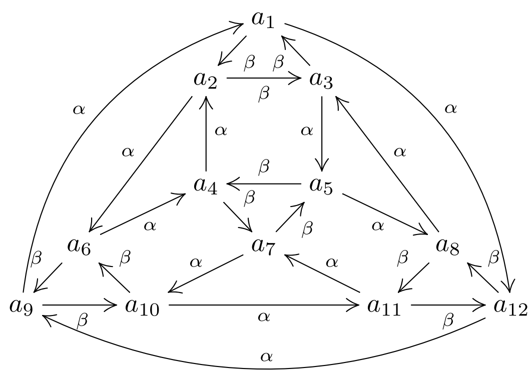
$$\alpha\beta\alpha\beta = 1_A$$

$$\beta\alpha\beta\alpha = 1_A$$

$$f\alpha = f$$

$$v\beta = v$$

The example of a tetrahedron, due to its simplicity, can be used to better illustrate the notion under analysis. From the following picture, which shows the planar graph representation of a tetrahedron (with the usual directed edges replaced by oriented squares that are build up from the labels $\alpha$ and $\beta$), we can deduce a concrete example of a double-link, as follows.




Take $A$ to be the set $\{a_1, \ldots, a_{12}\}$ and let the endomaps $\alpha$ and $\beta$ be defined by the labels indicated in the planar graph representation, that is $\alpha(a_1) = a_{12}$, $\beta(a_1) = a_2$, etc. The projection map $f$ is the quotient over the orbits of $\alpha$, which means it gives us the faces of the tetrahedron. The projection map $v$ is the quotient over the orbits of $\beta$, this gives us the set of vertices. The map $g$ can be any realization of the elements from $A$ into a space and we omit the details of it.

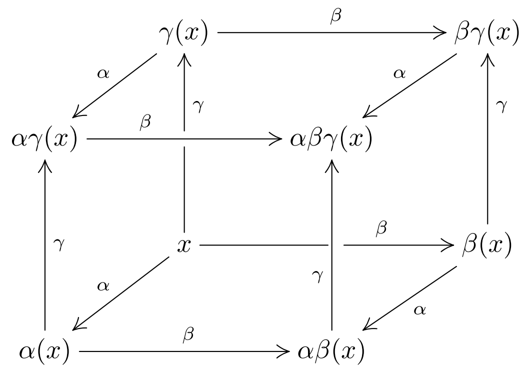
#### 4.1.5 A cubic-link

A cubic link is a straightforward generalization of a square-link and it is useful in modelling volumes with porous structures on it, see for example the Technical Report [10] produced by the Laboratory of Topology and Geometry from CDRSP-IPLeiria.

A cubic link is a structure

$$\alpha,\beta,\gamma \circlearrowright A \xrightarrow{g} \mathbb{H} \simeq \mathbb{R}^4$$

such that $\alpha\beta = \beta\alpha$, $\alpha\gamma = \gamma\alpha$ and $\gamma\beta = \beta\gamma$. It is interpreted as a collection of voxels of a cubic shape. In the same way as a square-link models a surface which is generated by a square patch along two abstract directions, a cubic link can be used to generate a 3d-manifold which is generated by a collection of cubes attached along three different abstract directions, say $\alpha, \beta, \gamma$, as illustrated.




(4.9)

#### 4.1.6 A n-cube-link

Again, a straightforward generalization of a cubic-link is obtained if instead of three abstract directions we consider any finite number $n$ and thus obtaining

$$\alpha_i \circlearrowright A \xrightarrow{g} \mathbb{G}\,, \quad i = 1, 2, \ldots, n$$

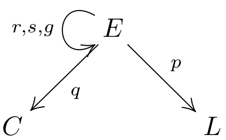
where $\mathbb{G}$ is any geometrical algebra (or more simply a vector space), and the $\alpha_i$ are permutable in the sense that $\alpha_i\alpha_j = \alpha_j\alpha_i$ for any $i, j$ in $\{1, 2, \ldots, n\}$

This structures arises for example in the construction of $n$-dimensional volumes with porous structures on its interior and it has some applications in 3D-printing.

### 4.2 A contour filling curve

The paper [7] describes a procedure on how to generate sweep trajectories for planar regions that are encoded by its boundary and obtained by slicing a three-dimensional body.

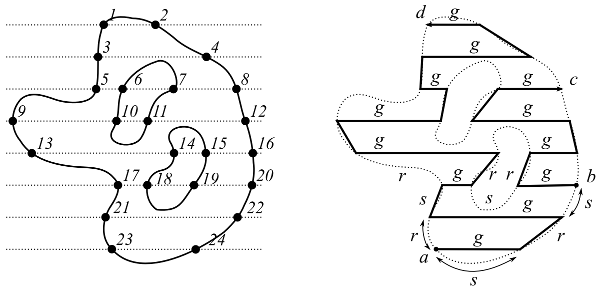
The details that are in the base of the motivation for considering this structure are referred to [7]. Here we recall the structure that is used there, which is an instance of a multi-link. It consists on a diagram of sets and maps, as illustrated,




such that

$$r^2 = s^2 = g^2 = 1_E$$
$$qr = qs = q$$
$$pg = p$$

It may be illustrated as in the following pictures




where the role of the maps $r, s, g$ is visible in the picture on the right. As we have said, our concern here is only to illustrate the structure of a multi-link with one more concrete example and its application, we refer the reader to the paper [7] for further details on this topic.

#### 4.2.1 A square patch

A square patch is an intermediate level between a square-link and a double-link. It is the analogue to a triangulation except that it is made out of squares rather than triangles. The generalization from a triangulation is not difficult to obtain and we omit the details. As an instance of a multi-link it is described as follows.

$$\theta,\varphi \circlearrowright A \xrightarrow{g} \mathbb{H} \tag{4.10}$$

such that

$$\theta^4 = 1_A \tag{4.11}$$
$$\theta^3 = \varphi\theta\varphi \tag{4.12}$$
$$g\varphi = g \tag{4.13}$$

The orbits of $\theta$ are now interpreted as the faces (which are all squares) while the orbits of $\varphi$ are interpreted as the vertices.

## 5 The iso-slice algorithm

In this section we give the necessary details for an implementation of an algorithm that efficiently computes level iso-contours. The contours are the ones obtained by slicing a triangulated surface in the euclidean 3d-space with respect to an iso-surface of a given level.

The algorithm may be decomposed in the following following steps:

(1) Consider a triangulation as input

$$T \overset{a}{\underset{c}{\xrightarrow{b}}} V \overset{x}{\underset{z}{\xrightarrow{y}}} \mathbb{R}$$

(2) Transform the given triangulation into a double-link as explained before (see also [6]). This gives

us a structure of the form

$$\alpha,\beta \circlearrowright A \xrightarrow{g} \mathbb{R}^3, \quad A \xrightarrow{f} F, \quad A \xrightarrow{v} V$$

with the meaning that the orbits of $\alpha$ are the faces (elements in $F$) and the orbits of $\beta$ are the vertices (elements in $V$).

(3) Suppose there is also given a family of iso-surface in the 3d-space, say defined by a map

$$F\colon \mathbb{R}^3 \to \mathbb{R}$$

which intuitively may be thought of as assigning to every point in space a certain hight; the main example is $F(x, y, z) = z$, this gives us planar slices along the $z$-direction, but we may also have $F(x, y, z) = x^2 + y^2$ or $F(x, y, z) = x^2 + y^2 + z^2$ and have, respectively, cylinders and spheres; arbitrary maps are possible as well and depending on each particular case of application they are defined in a specific way with a specific meaning, the algorithm, however, works for an arbitrary map.

(4) Transform the structure of double link from item 2 into the structure of a coloured link by considering $\varphi = \beta\alpha$, $h = Fg$ and forgetting the projection map $v$, this gives us

$$\varphi=\beta\alpha \circlearrowright A \xrightarrow{h=Fg} \mathbb{R}, \quad A \xrightarrow{f} F$$

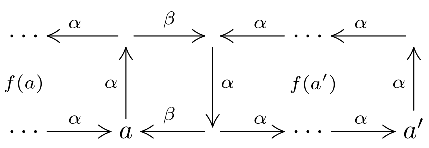
the map $h$ is intuitively the hight of each point in $A$ relative to the iso-surface level $F$ from item 3;

(5) For each contour level $r \in \mathbb{R}$ do:

(a) obtain the subset of $A$ in the coloured link from item 4 defined by:

$$A_r = \{a \in A \mid h(a) \leq r < h\varphi(a)\}$$

(b) consider the directed graph whose edges are the elements in $A_r$ as well as the vertices; the domain map is the identity map and the codomain map is $\varphi = \beta\alpha$, this will produce a picture which may be interpreted as illustrated




where we suppose that both $a$ and $a'$ are in $A_r$. This means that $r \in \mathbb{R}$ is a hight laying between $h(a)$ and $h\beta\alpha(a)$ as well as between $h(a')$ and $h\beta\alpha(a')$. The idea is to link $a$ and $a'$ and to do so it is sufficient to identify the orbits of $\alpha$ via $f$. This procedure creates a directed graph.

(c) construct the directed graph

$$A_r \overset{d}{\underset{c}{\rightrightarrows}} F$$

from the subset $L\colon A_r \hookrightarrow A$ (item 5(a)) to the set of faces $F$ (item 4), with $d = fL$ and $c = f\varphi L = f\beta\alpha L$. This graph is obtained by applying the quotient map $f$ to the graph considered in item 5(b).

(d) Link the digraph from 5(c), that is, find $\varphi_r\colon A_r \to A_r$ such that $d\varphi_r = c$. This is a general process and it can be performed in a unique way, provided the faces are geometrically convex. Indeed, Let $E \overset{d}{\underset{c}{\rightrightarrows}} V$ be an arbitrary directed graph, it has a symmetry, that is, there exists a bijective map $\varphi\colon E \to E$ such that $d\varphi = c$ if and only if the incoming edges are in bijection with the outcoming ones, for every vertex in $V$. In our case, if the faces are convex then they will either not be intersected by the iso-surface level $r$ or they are intersected exactly in two different edges (in the picture displayed at item 5(b) this was assumed to happen at the edges starting at the indexes $a$ and $a'$).

(e) Construct the link structure

$$\varphi_r \circlearrowright A_r \xrightarrow{g_r} \mathbb{R}^3\ ,$$

with $g_r(a) = g(a) + t_r(g\varphi(a) - g(a))$ where

$$t_r = \frac{r - h(a)}{h\varphi(a) - h(a)},$$

recall that $\varphi = \beta\alpha$ and $h = Fg$, come from item 4.

(6) Collect all the links structures $(A_r, \varphi_r, g_r)$ for all the contours $r \in \mathbb{R}$ in which we may be interested in and return this information as output.

# 6 An example of application

An example of application for the iso-slicing algorithm is the following. Suppose we have a solid body object of which we want to produce a mould with refrigeration channels. This means that if $S \subseteq \mathbb{R}^3$ is our solid then we are interested in the region of the space $\bar{S} = \mathbb{R}^3 \setminus S$. Moreover, suppose we wish to make some channels along the surface area of the boundary of $\bar{S}$, while keeping the channels on the interior of the region. Furthermore, in order to simplify the process, let us assume that these channels are generated by planar curves parallel to the xy-plane. To do so we need to determine how the ratio distance between two consecutive layers should be defined so that the cooling temperature is isotropic along the surface metric. One way to do it is to use our procedure, the iso-slice algorithm. The key aspect of it is to define the hight function $F\colon \mathbb{R}^3 \to \mathbb{R}$. In this case the high function is interpreted as the distance measured along the surface between the points in a lower level and the corresponding ones in the upper level immediately above it. We give below some details on how this procedure can be implemented.

Suppose $(T, V, a, b, c, x, y, z)$ is a triangulation such as the one given on item 1 of section 5, which is considered to be an approximation to a surface defined by the boundary of $S = closure(interior(S)) \subseteq \mathbb{R}^3$. We are interested in generating contour levels along the iso-surfaces which are geodesics along a direction perpendicular to the xy-plane. These contour paths will give us the generators for the cooling channels. In order to do that we observe the following steps:

(1) obtain a square-link from the given triangulation:

  (a) find an appropriate set of contours equally spaced that can serve as a good approximation to the given triangulation;

  (b) for each one of the contour levels identified on the previous item, execute the slicing algorithm with $F(x, y, z) = z$;

  (c) resample the number of points obtained in each set of indexes from the final link (as in item 5(2) from section 5) so that they all have the same number of points;

  (d) construct a square-link by letting $A$ to be the union of all $A_r$, assuming that we have chosen say, $r \in \{r_0, r_1, \ldots, r_n\} \subseteq \mathbb{R}$ and that each $A_r$ has, say, $m = 100$ elements. This is done by letting the map $\alpha$ to be given by the collection of $\varphi_r$ and $\beta$ to identify each point in the level $r_i$ with the closest one on the level $r_{i+1}$. This does not necessarily give a structure for a surface which is homeomorphic to the initially given one but it is equivalent from the point of view of the generation of the cooling channels.

(2) having a square link $(A, \alpha, \beta, g)$ as defined in section 4.1.2 we now define the iso-surface family $F\colon A \to \mathbb{R}$ iteratively as follows (suppose $\phi\colon A \to X \times Y$ is a bijection with $X = \{0, 1, \ldots, n\}$ and $Y = \{1, \ldots, m\}$). The base points, that is the ones in the level $r_0$, are all zero $F(x, 0) = 0$; all the points at the same level will have the same value under $F$; suppose we have $F(x, y)$ given, then we define $F(x, y + 1)$ as the formula

$$F(x, y) + \|g\phi^{-1}(x, y) - g\phi^{-1}(x, y + 1)\|$$

(3) we now use the iso-slicing algorithm with the new hight value $F$.

(4) the end result of this procedure gives a family of contour levels parallel to the xy-direction which are isotropic along the geodesic paths measured on the surface.

Having the contour trajectories enveloping the original surface in a way which is isotropic concerning the refrigeration distribution of heat along the geodesic distances on the surface, we can then choose a cross-section for the channels and generate the final structure as a square-patch.

# 7 Conclusion

All this procedures and processes have been implemented in a computer system and proved to be efficient and robust. Indeed, the fact that the data can be modelled as a mathematical structure has the great advantage that it characterizes completely the input data, the output data, as well as the abstract structures which are involved in any intermediate step in the process.

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*University of Cape Town, South Africa*

**Ficha Técnica**
Director: Nelson Martins Ferreira
Proprietário: Instituto Politécnico de Leiria, anotado na ERC
Morada: Rua General Norton de Matos; Apartado 4133; 2411-901 Leiria, Portugal
Director Adjunto: Nuno Alves; Sub Director: Artur Mateus
Editor: Nelson Martins Ferreira; Sede de Redação: Edificio IPL, Rua de Portugal – Zona Industrial 2430-028 – Marinha Grande, Portugal; Contacto: scripta.ingenia@ipleiria.pt
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