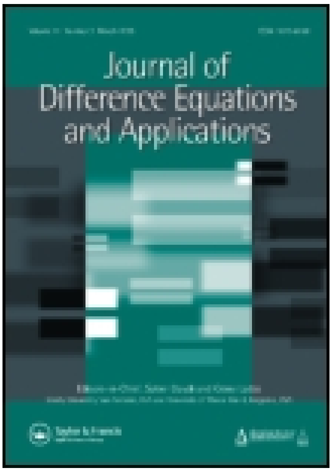

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On the series expansion of the spatial SIS evolution operator

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On the series expansion of the spatial SIS evolution operator

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For the spatial stochastic susceptible–infected–susceptible model, we consider the perturbative series expansion of the gap between the dominant and subdominant eigenvalues of the evolution operator. We compute explicitly the first terms of the series expansion of the gap with difference equations for the calculation of states.

Keywords: SIS model; contact process; evolution operator; series expansion; critical threshold

AMS Subject Classification: 92D30; 30B10

1. Introduction

The characterization of the critical thresholds in epidemic models is probably the most important feature of the mathematical epidemiology research due to the drastic change of the disease spread on the critical threshold. The simple spatial stochastic susceptible–infected–susceptible (SIS) epidemic model, also known as the contact process, has a continuous phase transition from the absorbing state devoid of infected individuals to a non-equilibrium state of infectivity. The phase transition point of the SIS model was recently characterized in pair approximation as particular case of the reinfection SIRI model [8,10], and improves the rough qualitative behaviour in mean field approximation. In the phase transition, the dominant eigenvalue of the evolution operator for the SIS model becomes degenerate that occurs when the gap between the dominant and the subdominant eigenvalues vanishes. To study the gap value, series expansions in terms of the creation rate have been used [1,2,7]. This requires the formulation of the epidemic models in terms of creation and annihilation operators [3–5,9,11] starting from the master equation [6,12]. The critical values follow from the series expansion, based on a perturbation ansatz and using a Padé analysis [1,7].

Here, we consider the spatial stochastic SIS epidemic model formulated via creation and annihilation operators. For the SIS model in 1D lattices, we deduce the perturbative series expansion of the gap between the dominant and subdominant eigenvalues of the evolution operator. The first terms of the series expansion of the gap are computed

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explicitly. Here, we do not assume the translation invariance of the lattice in contrast to the study made in [1].

2. The spatial stochastic SIS model

The stochastic SIS model is one of the best-known epidemic models and one of the simplest. It describes the evolution of an infectious disease through a population of N individuals, which can be either infected or susceptible. This epidemic model is also known as the contact process because it describes an interacting-particle system on a regular lattice, where the particles are annihilated spontaneously and created catalytically. We consider that a particle is annihilated with a rate α, usually $\alpha = 1$, and created with a rate β times the fraction of the nearest neighbours occupied sites. We use the state variables

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1}$$

to represent the site i whenever it is empty or occupied or, in an epidemic context, when the individual i is susceptible or infected. The configuration of the lattice can be represented by

$$|\eta\rangle = |\eta_1 \eta_2 \ldots \eta_N\rangle = |\eta_1\rangle \otimes |\eta_2\rangle \otimes \cdots \otimes |\eta_N\rangle, \tag{2}$$

where $\otimes$ represents the usual tensor product, $|\eta_i\rangle = |0\rangle$ or $|\eta_i\rangle = |1\rangle$ represents the state of each site i and N denotes the total number of the sites.

2.1 *The creation and annihilation operators*

To describe the SIS epidemic model using the creation and annihilation operators, we define the following operators to act on each site of the lattice.

Definition 2.1. *The creation operator c^+ and the annihilation operator c, for a single site are given by*

$$c^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

It is obvious that these local operators applied to the state variables give

$$c^+|0\rangle = |1\rangle \quad \text{and} \quad c|1\rangle = |0\rangle. \tag{4}$$

Let $J_{i,j} \in \{0, 1\}$ be the elements of the $N \times N$ adjacency matrix J that describes the neighbouring structure of the N individuals. We consider that the individuals live on a regular lattice, where each corner has the same number Q of edges. The time evolution of the probability $p(\eta, t) = p(\eta_1, \ldots, \eta_N, t)$ is given by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p(\eta, t) = \sum_{i=1}^{N} w_{\eta_i, 1-\eta_i} p(\eta_1, \ldots, 1 - \eta_i, \ldots, \eta_N, t) - \sum_{i=1}^{N} w_{1-\eta_i, \eta_i} p(\eta_1, \ldots, \eta_i, \ldots, \eta_N, t),$$

for $\eta_i \in \{0, 1\}$ and with the transition rates

$$w_{\eta_i, 1-\eta_i} = \beta\left(\sum_{j=1}^{N} J_{ij}\eta_j\right)\eta_i + \alpha(1-\eta_i) \text{ and } w_{1-\eta_i, \eta_i} = \beta\left(\sum_{j=1}^{N} J_{ij}\eta_j\right)(1-\eta_i) + \alpha\eta_i.$$

Now, we will use the vector representation given by

$$\begin{aligned} |\psi(t)\rangle &= \sum_{\eta_1=0}^{1}\sum_{\eta_2=0}^{1}\cdots\sum_{\eta_N=0}^{1} p(\eta_1, \ldots, \eta_N, t)\left(c_1^+\right)^{\eta_1}\ldots\left(c_N^+\right)^{\eta_N}|O\rangle \\ &= \sum_{\eta} p(\eta, t)\prod_{i=1}^{N}\left(c_i^+\right)^{\eta_i}|O\rangle = \sum_{\eta} p(\eta, t)|\eta\rangle, \end{aligned} \tag{5}$$

where $|O\rangle$ represents the vacuum state and $|\eta\rangle$ represents the configuration of the lattice. Hence, the time evolution of the state vector $|\psi(t)\rangle$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = L|\psi(t)\rangle, \tag{6}$$

where the evolution operator L can be written in terms of the creation and annihilation operators, after some calculations from the master equation, as

$$L = \alpha\sum_{i=1}^{N}(\mathbb{1}_i - c_i^+)c_i + \beta Q\sum_{i=1}^{N}(\mathbb{1}_i - c_i)\left(\frac{1}{Q}\sum_{j=1}^{N} J_{ij}c_j^+c_j\right)c_i^+ = \alpha W_0 + \lambda V, \tag{7}$$

with $\lambda = \beta Q$, $\mathbb{1}_i$ being the identity matrix at site i and zero elsewhere, and W_0 and V abbreviating the single-site and multi-site operator expressions, respectively. Changing the time scale to $\tau = t/\alpha$, the new rates become $\alpha = 1$ and $\lambda = \beta Q/\alpha$, and the evolution operator L is given by

$$L = W_0 + \lambda V. \tag{8}$$

From now on, we will consider only 1D lattices. Hence, each individual has $Q = 2$ neighbours.

Definition 2.2. *Let $\mathbb{1}$ be the 2D identity matrix and let $\hat{B}$, $\hat{Q}$ and $\hat{n}$ be the local operators given by*

$$\hat{B} = (\mathbb{1} - c^+)c, \tag{9}$$

$$\hat{Q} = \frac{1}{2}(\mathbb{1} - c)c^+, \tag{10}$$

$$\hat{n} = c^+c. \tag{11}$$

The operator $\hat{n}$ is called the number operator. In the following lemma, we observe the result of applying these operators to the state variables.

Lemma 2.3. *The local operators presented in Definition* 2.2 *applied to the state variables* $|0\rangle$ *and* $|1\rangle$ *give*

$$\hat{B}|0\rangle = 0 \quad \text{and} \;\; \hat{B}|1\rangle = |0\rangle - |1\rangle, \tag{12}$$

$$\hat{Q}|0\rangle = \frac{1}{2}(|1\rangle - |0\rangle) \quad \text{and} \;\; \hat{Q}|1\rangle = 0, \tag{13}$$

$$\hat{n}|0\rangle = 0 \quad \text{and} \;\; \hat{n}|1\rangle = |1\rangle. \tag{14}$$

The proof of this lemma is trivial and it will not be presented here.

2.2 The σ representation

We start to observe that the operator $\hat{B}$ has eigenvalues 0 and -1, associated with the right eigenvectors $|0\rangle$ and $|1\rangle - |0\rangle$. So, it is convenient to change the coordinated system to these right eigenvectors that we define by

$$|\tilde{0}\rangle = |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \;\; |\tilde{1}\rangle = |1\rangle - |0\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{15}$$

The left eigenvectors of $\hat{B}$ are $\langle\tilde{0}| = \langle 0| + \langle 1|$ and $\langle\tilde{1}| = \langle 1|$, associated with the eigenvalues 0 and -1, respectively.

Lemma 2.4. *The local operators presented in Definition* 2.2 *applied to the state variables* $|\tilde{0}\rangle$ *and* $|\tilde{1}\rangle$ *give*

$$\hat{B}|\tilde{0}\rangle = 0 \quad \text{and} \;\; \hat{B}|\tilde{1}\rangle = -|\tilde{1}\rangle, \tag{16}$$

$$\hat{Q}|\tilde{0}\rangle = \frac{1}{2}|\tilde{1}\rangle \quad \text{and} \;\; \hat{Q}|\tilde{1}\rangle = -\frac{1}{2}|\tilde{1}\rangle, \tag{17}$$

$$\hat{n}|\tilde{0}\rangle = 0 \quad \text{and} \;\; \hat{n}|\tilde{1}\rangle = |\tilde{1}\rangle + |\tilde{0}\rangle. \tag{18}$$

Once again, the proof of this lemma is trivial. Let the state of two sites of the lattice be denoted by e.g.

$$|\tilde{0}\tilde{0}\rangle = |\tilde{0}\rangle \otimes |\tilde{0}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{19}$$

Similarly, we define the two sites states $|\tilde{0}\tilde{1}\rangle$, $|\tilde{1}\tilde{0}\rangle$ and $|\tilde{1}\tilde{1}\rangle$. The operators that act on these two sites are given by, e.g. $B_1 + B_2$, where $B_1 = \hat{B} \otimes \mathbb{1}$ acts on the first site and

$B_2 = \mathbb{1} \otimes \hat{B}$ acts on the second site. Therefore, $B_1 + B_2$ acts, e.g. on the pair $|\tilde{0}\tilde{1}\rangle$ giving

$$\begin{aligned}(B_1 + B_2)|\tilde{0}\tilde{1}\rangle &= (\hat{B}\otimes\mathbb{1})(|\tilde{0}\rangle\otimes|\tilde{1}\rangle) + (\mathbb{1}\otimes\hat{B})(|\tilde{0}\rangle\otimes|\tilde{1}\rangle)\\ &= (\hat{B}|\tilde{0}\rangle)\otimes(\mathbb{1}|\tilde{1}\rangle) + (\mathbb{1}|\tilde{0}\rangle)\otimes(\hat{B}|\tilde{1}\rangle) = 0\otimes|\tilde{1}\rangle + |\tilde{0}\rangle\otimes(-|\tilde{1}\rangle) = -|\tilde{0}\tilde{1}\rangle. \quad (20)\end{aligned}$$

In the same way, we obtain that

$$(B_1 + B_2)|\tilde{0}\tilde{0}\rangle = 0, \quad (21)$$

$$(B_1 + B_2)|\tilde{1}\tilde{0}\rangle = -|\tilde{1}\tilde{0}\rangle, \quad (22)$$

$$(B_1 + B_2)|\tilde{1}\tilde{1}\rangle = -2|\tilde{1}\tilde{1}\rangle. \quad (23)$$

This representation can be generalized for all the sites in the lattice by

$$|\sigma\rangle = |\sigma_1\sigma_2\ldots\sigma_N\rangle, \quad \sigma_i \in \{\tilde{0}, \tilde{1}\}, \quad (24)$$

which we call the σ representation. To act on the $|\sigma\rangle$ vector, we define the operator

$$W_0 = \sum_{i=1}^{N} B_i, \quad (25)$$

where $B_i = \mathbb{1}^{\otimes(i-1)}\otimes\hat{B}\otimes\mathbb{1}^{\otimes(N-i)}$, with $\hat{B}$ being the operator defined in equation (9), Definition 2.2. Hence, generalizing the calculations presented in equation (20), we have that

$$W_0|\sigma\rangle = \sum_{i=1}^{N} B_i|\sigma_1\sigma_2\ldots\sigma_N\rangle = -\sum_{i=1}^{N}\sigma_i|\sigma_1\sigma_2\ldots\sigma_N\rangle. \quad (26)$$

Therefore, the operator $W_0 = \sum_{i=1}^{N} B_i$ has eigenvalues given by

$$\Lambda(\sigma) = -\sum_{i=1}^{N}\sigma_i. \quad (27)$$

To operate on the two sites states, we define the operators

$$Q_1 = \hat{Q}\otimes\mathbb{1}, \quad Q_2 = \mathbb{1}\otimes\hat{Q}, \quad n_1 = \hat{n}\otimes\mathbb{1} \quad \text{and} \quad n_2 = \mathbb{1}\otimes\hat{n}, \quad (28)$$

where $\hat{Q}$ and $\hat{n}$ are the operators defined in equations (10) and (11), Definition 2.2.

THEOREM 2.5. *With the operators defined in equation* (28)*, the following rules are satisfied:*

$$(Q_1 n_2 + n_1 Q_2)|\tilde{0}\tilde{0}\rangle = 0, \quad (29)$$

$$(Q_1 n_2 + n_1 Q_2)|\tilde{0}\tilde{1}\rangle = \frac{1}{2}|\tilde{1}\tilde{0}\rangle + \frac{1}{2}|\tilde{1}\tilde{1}\rangle, \quad (30)$$

$$(Q_1n_2+n_1Q_2)|\tilde{1}\tilde{0}\rangle=\frac{1}{2}|\tilde{0}\tilde{1}\rangle+\frac{1}{2}|\tilde{1}\tilde{1}\rangle, \tag{31}$$

$$(Q_1n_2+n_1Q_2)|\tilde{1}\tilde{1}\rangle=-\frac{1}{2}|\tilde{0}\tilde{1}\rangle-\frac{1}{2}|\tilde{1}\tilde{0}\rangle-|\tilde{1}\tilde{1}\rangle. \tag{32}$$

Proof. Due to the similarity of the calculations, here, we prove only the second rule of the theorem. We start to observe that

$$Q_1n_2=(\hat{Q}\otimes\mathbb{1})(\mathbb{1}\otimes\hat{n})=(\hat{Q}\mathbb{1})\otimes(\mathbb{1}\hat{n})=\hat{Q}\otimes\hat{n}, \tag{33}$$

and, in the same way, we have $n_1Q_2=\hat{n}\otimes\hat{Q}$. Hence, applying Lemma 2.4, we find that

$$Q_1n_2|\tilde{0}\tilde{1}\rangle=(\hat{Q}\otimes\hat{n})(|\tilde{0}\rangle\otimes|\tilde{1}\rangle)=(\hat{Q}|\tilde{0}\rangle)\otimes(\hat{n}|\tilde{1}\rangle)=\frac{1}{2}|\tilde{1}\rangle\otimes(|\tilde{1}\rangle+|\tilde{0}\rangle)=\frac{1}{2}|\tilde{1}\tilde{1}\rangle+\frac{1}{2}|\tilde{1}\tilde{0}\rangle \tag{34}$$

and

$$n_1Q_2|\tilde{0}\tilde{1}\rangle=(\hat{n}\otimes\hat{Q})(|\tilde{0}\rangle\otimes|\tilde{1}\rangle)=(\hat{n}|\tilde{0}\rangle)\otimes(\hat{Q}|\tilde{1}\rangle)=0\otimes\left(-\frac{1}{2}|\tilde{1}\rangle\right)=0. \tag{35}$$

Therefore, when summing equations (34) and (35), it follows immediately that we obtain $(Q_1n_2+n_1Q_2)|\tilde{0}\tilde{1}\rangle=\frac{1}{2}|\tilde{1}\tilde{0}\rangle+\frac{1}{2}|\tilde{1}\tilde{1}\rangle$. □

3. Series expansion

We observe that the annihilation and creation operators that appear in the evolution operator L presented in equation (8) are given (see [4,11]) by the expressions

$$W_0=\sum_{i=1}^{N}B_i \quad \text{and} \quad V=\sum_{i=1}^{N}Q_i(n_{i-1}+n_{i+1}), \tag{36}$$

where the V operator can be reorganized and written in the form

$$V=\sum_{i=1}^{N}(Q_in_{i+1}+n_iQ_{i+1}). \tag{37}$$

From equation (26), we observe that

$$\sum_{i=1}^{N}B_i|O\rangle=\sum_{i=1}^{N}B_i|\tilde{0}\ldots\tilde{0}\rangle=-\sum_{i=1}^{N}0|\tilde{0}\ldots\tilde{0}\rangle=0, \tag{38}$$

and from equation (29), we also observe that

$$\sum_{i=1}^{N}(Q_in_{i+1}+n_iQ_{i+1})|O\rangle=0. \tag{39}$$

Therefore, we conclude that $|O\rangle=|\tilde{0}\ldots\tilde{0}\rangle$ is an eigenvector of L for the zero eigenvalue $L|O\rangle=\alpha 0+\lambda 0=0$. From equation (26), we also observe that the subdominant eigenvalue of W_0 is -1 and the correspondent eigenvector is $|\psi_0\rangle = |.\tilde{1}.\rangle$, where the two dots mean that all sites at the right and the left of $\tilde{1}$ are $\tilde{0}$,

$$W_0|\psi_0\rangle = \sum_{i=1}^{N} B_i|.\tilde{1}.\rangle = -|.\tilde{1}.\rangle = -|\psi_0\rangle. \tag{40}$$

Now, we are interested in determining the subdominant eigenvalue of the evolution operator L. Let $|\psi\rangle$ denote the subdominant eigenvector and A the correspondent eigenvalue

$$L|\psi\rangle = A|\psi\rangle. \tag{41}$$

Hence, the gap between the dominant and the subdominant eigenvalues of L is given by

$$\Gamma = 0 - A = -A. \tag{42}$$

We assume that $|\psi\rangle$ and A can be expanded in powers of the creation rate λ

$$|\psi\rangle = |\psi_0\rangle + \lambda|\psi_1\rangle + \lambda^2|\psi_2\rangle + \cdots = \sum_{n=0}^{\infty} \lambda^n|\psi_n\rangle, \tag{43}$$

$$A = A_0 + \lambda A_1 + \lambda^2 A_2 + \cdots = \sum_{n=0}^{\infty} A_n\lambda^n, \tag{44}$$

where $|\psi_0\rangle = |.\tilde{1}.\rangle$ and $A_0 = -1$. Therefore, the expansion of the gap between the dominant and the subdominant eigenvalues Γ is given by

$$\Gamma = 1 - \lambda A_1 - \lambda^2 A_2 - \cdots. \tag{45}$$

We choose the vectors $|\psi_n\rangle$ to be orthogonal to the vector $|\psi_0\rangle$

$$\langle\psi_0|\psi_n\rangle = 0, \quad \forall\, n \geq 1. \tag{46}$$

We show how to obtain A_n and $|\psi_n\rangle$ in the following computations. Inserting the expanded expressions of $|\psi\rangle$ and A in equation (41), we obtain

$$\begin{aligned}
&(W_0 + \lambda V)\sum_{n=0}^{\infty} \lambda^n|\psi_n\rangle = \left(\sum_{m=0}^{\infty} A_m\lambda^m\right)\left(\sum_{n=0}^{\infty} \lambda^n|\psi_n\rangle\right)\\
&\Leftrightarrow \sum_{n=0}^{\infty} W_0|\psi_n\rangle\lambda^n + \sum_{n=0}^{\infty} V|\psi_n\rangle\lambda^{n+1} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_m|\psi_n\rangle\lambda^{m+n}\\
&\Leftrightarrow \sum_{n=0}^{\infty} W_0|\psi_n\rangle\lambda^n + \sum_{n=1}^{\infty} V|\psi_{n-1}\rangle\lambda^n = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_m|\psi_{n-m}\rangle\lambda^n\\
&\Leftrightarrow W_0|\psi_0\rangle + \sum_{n=1}^{\infty} (W_0|\psi_n\rangle + V|\psi_{n-1}\rangle)\lambda^n = A_0|\psi_0\rangle + \sum_{n=1}^{\infty}\sum_{m=0}^{n} A_m|\psi_{n-m}\rangle\lambda^n.
\end{aligned} \tag{47}$$

Comparing coefficients of the same powers of λ in both sides of equation (47), we verify that

$$W_0|\psi_0\rangle = A_0|\psi_0\rangle, \tag{48}$$

for the term independent of λ and for the other terms

$$W_0|\psi_n\rangle + V|\psi_{n-1}\rangle = \sum_{m=0}^{n} A_m|\psi_{n-m}\rangle. \tag{49}$$

Multiplying by $\langle\psi_0|$ and using the orthogonality $\langle\psi_0|\psi_n\rangle$, $\forall\, n \neq 0$, we obtain

$$\begin{aligned}\langle\psi_0|W_0|\psi_n\rangle + \langle\psi_0|V|\psi_{n-1}\rangle &= \sum_{m=0}^{n} A_m\langle\psi_0|\psi_{n-m}\rangle\\ \Leftrightarrow A_0\langle\psi_0|\psi_n\rangle + \langle\psi_0|V|\psi_{n-1}\rangle &= \sum_{m=0}^{n} A_m\delta_{0,n-m} \Leftrightarrow A_0 0 + \langle\psi_0|V|\psi_{n-1}\rangle = A_n.\end{aligned} \tag{50}$$

Hence, the coefficients A_n of the expansion of the subdominant eigenvalue of L can be computed recursively by the formula

$$A_n = \langle\psi_0|V|\psi_{n-1}\rangle, \forall\ n \geq 1, \quad \text{and } A_0 = -1. \tag{51}$$

Now, we have to determine the vectors $|\psi_n\rangle$. Since W_0 has eigenvalues given by $\Lambda(\sigma) = -\sum_{i=1}^{N}\sigma_i$ (see equations (26) and (27)), this operator can be written by the expression

$$W_0 = \underbrace{\sum_{\sigma_1=\tilde{0}}^{\tilde{1}} \ldots \sum_{\sigma_N=\tilde{0}}^{\tilde{1}}}_{\sigma_1+\cdots+\sigma_N\neq 0} |\sigma_1 \ldots \sigma_N\rangle\Lambda(\sigma_1 \ldots \sigma_N)\langle\sigma_1 \ldots \sigma_N| = \sum_{\Lambda(\sigma)\neq 0} |\sigma\rangle\Lambda(\sigma)\langle\sigma|. \tag{52}$$

Let R be the operator given by

$$\begin{aligned}R &= \underbrace{\sum_{\sigma_1=\tilde{0}}^{\tilde{1}} \cdots \sum_{\sigma_N=\tilde{0}}^{\tilde{1}}}_{\sigma_1+\cdots+\sigma_N\notin\{0,-1\}} |\sigma_1 \ldots \sigma_N\rangle \frac{1}{\Lambda(\sigma_1 \ldots \sigma_N) - A_0}\langle\sigma_1 \ldots \sigma_N|\\ &= \sum_{\Lambda(\sigma)\notin\{0,-1\}} |\sigma\rangle\frac{1}{\Lambda(\sigma) - A_0}\langle\sigma|.\end{aligned} \tag{53}$$

From equation (49), we observe that

$$\begin{aligned}W_0|\psi_n\rangle + V|\psi_{n-1}\rangle &= A_0|\psi_n\rangle + \sum_{m=1}^{n} A_m|\psi_{n-m}\rangle\\ \Leftrightarrow (W_0 - A_0)|\psi_n\rangle &= -V|\psi_{n-1}\rangle + \sum_{m=1}^{n} A_m|\psi_{n-m}\rangle,\end{aligned} \tag{54}$$

and applying R, we obtain

$$R(W_0 - A_0)|\psi_n\rangle = -RV|\psi_{n-1}\rangle + \sum_{m=1}^{n} A_m R|\psi_{n-m}\rangle. \tag{55}$$

But

$$\begin{aligned} R(W_0 - A_0) &= \sum_{\Lambda(\sigma)\notin\{0,-1\}} |\sigma\rangle \frac{1}{\Lambda(\sigma) - A_0} \langle\sigma|(W_0 - A_0) \\ &= \sum_{\Lambda(\sigma)\notin\{0,-1\}} |\sigma\rangle \frac{1}{\Lambda(\sigma) - A_0} (\Lambda(\sigma)\langle\sigma| - A_0\langle\sigma|) \\ &= \sum_{\Lambda(\sigma)\notin\{0,-1\}} |\sigma\rangle\langle\sigma|, \end{aligned} \tag{56}$$

and joining to this sum the terms for the cases $\sigma_1 + \cdots + \sigma_N = 0$ and $\sigma_1 + \cdots + \sigma_N = -1$, we complete the eigenbasis $\sum_\sigma |\sigma\rangle\langle\sigma| = \mathbb{1}$. Hence,

$$R(W_0 - A_0) = \sum_\sigma |\sigma\rangle\langle\sigma| - |O\rangle\langle O| - |\psi_0\rangle\langle\psi_0|, \tag{57}$$

and therefore,

$$R(W_0 - A_0)|\psi_n\rangle = \mathbb{1}|\psi_n\rangle - |O\rangle\langle O|\psi_n\rangle - |\psi_0\rangle\langle\psi_0|\psi_n\rangle = |\psi_n\rangle. \tag{58}$$

Hence, from equation (55), we obtain for the calculation of the state $|\psi_n\rangle$ the difference equation

$$|\psi_n\rangle = -RV|\psi_{n-1}\rangle + \sum_{m=1}^{n-1} A_m R|\psi_{n-m}\rangle, \forall n \geq 2. \tag{59}$$

3.1 Explicit calculation of the series expansion

We will now compute explicitly some coefficients of the expansion of the gap between the dominant and the subdominant eigenvectors of the evolution operator for the SIS model. Here, we do not consider the translation invariance of the lattice, in contrast to de Oliveira [1]. For the initial state

$$|\psi_0\rangle = |\tilde{0}\tilde{1}\tilde{0}\rangle \tag{60}$$

and unperturbed eigenvalue $A_0 = -1$, we obtain

$$A_1 = \langle\psi_0|V|\psi_0\rangle = 0. \tag{61}$$

This value results from

$$\begin{aligned} V|\psi_0\rangle &= ((Q_1 n_2 + n_1 Q_2) + (Q_2 n_3 + n_2 Q_3) + (Q_3 n_1 + n_3 Q_1))|\tilde{0}\tilde{1}\tilde{0}\rangle \\ &= \frac{1}{2}(|\tilde{1}\tilde{0}\tilde{0}\rangle + |\tilde{1}\tilde{1}\tilde{0}\rangle + |\tilde{0}\tilde{0}\tilde{1}\rangle + |\tilde{0}\tilde{1}\tilde{1}\rangle) \end{aligned} \tag{62}$$

and therefore,

$$A_1 = \langle \tilde{0}\tilde{1}\tilde{0}|V|\psi_0\rangle = 0 \tag{63}$$

since $\langle \tilde{0}\tilde{1}\tilde{0}|\tilde{1}\tilde{1}\tilde{0}\rangle = 0$, etc., due to the orthonormality of the states. The state vector of the first order in the series expansion is given by

$$|\psi_1\rangle = -RV|\psi_0\rangle = -\sum_{\Lambda(\sigma)\notin\{0,-1\}} |\sigma\rangle \frac{1}{\Lambda(\sigma) - A_0} \langle\sigma| \cdot V|\psi_0\rangle. \tag{64}$$

Since

$$\begin{aligned}\sum_{\Lambda(\sigma)\notin\{0,-1\}} |\sigma\rangle \frac{1}{\Lambda(\sigma) - A_0}\langle\sigma| &= |\tilde{1}\tilde{1}\tilde{0}\rangle \frac{1}{-2+1}\langle \tilde{1}\tilde{1}\tilde{0}| + |\tilde{1}\tilde{0}\tilde{1}\rangle \frac{1}{-2+1}\langle \tilde{1}\tilde{0}\tilde{1}| \\ &\quad + |\tilde{0}\tilde{1}\tilde{1}\rangle \frac{1}{-2+1}\langle \tilde{0}\tilde{1}\tilde{1}| + |\tilde{1}\tilde{1}\tilde{1}\rangle \frac{1}{-3+1}\langle \tilde{1}\tilde{1}\tilde{1}|,\end{aligned}$$

we find from equation (64)

$$|\psi_1\rangle = -\frac{1}{2}\left(\frac{1}{-2+1}|\tilde{1}\tilde{1}\tilde{0}\rangle + \frac{1}{-2+1}|\tilde{0}\tilde{1}\tilde{1}\rangle\right) = \frac{1}{2}(|\tilde{1}\tilde{1}\tilde{0}\rangle + |\tilde{0}\tilde{1}\tilde{1}\rangle). \tag{65}$$

For the next terms in the expansion, we have to start with system size $N = 5$, hence starting with the state

$$|\psi_0\rangle = |\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle \tag{66}$$

then, because of $(\hat{Q}_i \hat{n}_{i+1} + \hat{n}_i \hat{Q}_{i+1})|\tilde{0}\tilde{0}\rangle = 0$, we obtain as before

$$A_1 = 0, \quad |\psi_1\rangle = \frac{1}{2}(|\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle + |\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\rangle), \tag{67}$$

and computing $V|\psi_1\rangle$ as in equation (62), we obtain now

$$A_2 = \langle\psi_0|V|\psi_1\rangle = -\frac{1}{2}. \tag{68}$$

The second state $|\psi_2\rangle$ of the series expansion

$$|\psi_2\rangle = -RV|\psi_1\rangle + A_1 R|\psi_0\rangle, \tag{69}$$

gives, with $A_1 = 0$, explicitly

$$\begin{aligned}|\psi_2\rangle &= \frac{1}{4}|\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{8}|\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle - \frac{1}{2}|\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{2}|\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\rangle + \frac{1}{4}|\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\rangle - \frac{1}{2}|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\rangle \\ &\quad + \frac{1}{4}|\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\rangle + \frac{1}{8}|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\rangle,\end{aligned} \tag{70}$$

and the third coefficient gives

$$A_3 = \langle \psi_0 | V | \psi_2 \rangle = \frac{1}{2}. \tag{71}$$

With the previous coefficients A_i and the vectors $|\psi_i\rangle$, we obtain for

$$|\psi_3\rangle = -RV|\psi_2\rangle + A_2 R|\psi_1\rangle, \tag{72}$$

the explicit expression given by

$$\begin{aligned}
|\psi_3\rangle = {} & \frac{1}{4}|\tilde{1}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle + \frac{1}{16}|\tilde{1}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle + \frac{15}{16}|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle - \frac{1}{8}|\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle + \frac{1}{16}|\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\tilde{0}\rangle \\
& + \frac{3}{8}|\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{8}|\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{32}|\tilde{1}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle + \frac{1}{48}|\tilde{1}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle - \frac{3}{8}|\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{0}\rangle \\
& + \frac{5}{32}|\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{1}{16}|\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle - \frac{3}{4}|\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\rangle - \frac{1}{4}|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle + \frac{15}{16}|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{0}\rangle \\
& + \frac{3}{8}|\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\rangle + \frac{5}{32}|\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\rangle + \frac{1}{8}|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\rangle + \frac{1}{16}|\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\rangle - \frac{3}{8}|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{0}\rangle \\
& - \frac{1}{8}|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{0}\rangle + \frac{1}{16}|\tilde{0}\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\rangle + \frac{1}{8}|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{0}\tilde{1}\rangle + \frac{1}{16}|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{0}\tilde{1}\tilde{1}\rangle + \frac{1}{32}|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{0}\tilde{1}\rangle \\
& + \frac{1}{48}|\tilde{0}\tilde{0}\tilde{0}\tilde{1}\tilde{1}\tilde{1}\tilde{1}\rangle.
\end{aligned}$$

Then, for the A_4 coefficient, we obtain

$$A_4 = \langle \psi_0 | V | \psi_3 \rangle = -\frac{15}{16}, \tag{73}$$

which can serve as a test value for numeric programs.

In Table 1, we present a few more coefficients A_n computed in an elementary computer. We observe that the computation of the coefficient c_n involves a square matrix of size 2^{2n+1}.

Table 1. The coefficients A_n of the expansion of the gap $\Gamma = \sum_{n=0}^{\infty} - A_n \lambda^n$ are given here for the first nine terms numerically. Of special interest is the fourth term, for which $-15/16$ could be obtained analytically.

n	A_n
0	-1
1	0
2	-0.5
3	0.5
4	-0.9375
5	1.8125
6	-3.94010416666667
7	8.79687500000000
8	-20.45668764467593
9	48.49340518904322

Hence, the next coefficients can be computed only on more sophisticated computers, with a higher memory capacity. Using many more coefficients, we could apply a Padé analysis and obtain probably better critical threshold and critical exponent values of the SIS model than previously achieved. There have been still systematical numerical discrepancies reported between series expansions and direct stochastic simulations [1]. In summary, we have demonstrated an interesting application of difference equations to an epidemiological problem on the front line of research. Future extensions to more complicated epidemiological models are possible [11], but with even more computational effort.

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