The Higher Moments Dynamic on SIS Model

Alberto Pinto*, José Martins† and Nico Stollenwerk**

*Departamento de Matemática, Escola de Ciências da Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal
†Dep. de Matemática, Escola Superior de Tecnologia e Gestão, Instituto Politécnico de Leiria, Apartado 4163, 2411-901 Leiria, Portugal
**Centro de Matemática e Aplicações Fundamentais, Faculdade de Ciências, Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

Abstract. The basic contact process or the SIS model is a well known epidemic process and have been studied for a wide class of people. In an epidemiological context, many authors worked on the SIS model considering only the dynamic of the first moments of infecteds, i.e., the mean value and the variance of the infected individuals. In this work, we study not only the dynamic of the first moments of infecteds but also on the dynamic of the higher moments. Recursively, we consider the dynamic equations for all the moments of infecteds and, applying the moment closure approximation, we obtain the stationary states of the state variables. We observe that the stationary states of the SIS model, in the moment closure approximation, can be used to obtain good approximations of the quasi-stationary states of the SIS model.

Keywords: Contact process; Quasi-stationary distribution; Moment closure approximation.
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INTRODUCTION

One of the best known models is the SIS (Susceptible, Infected and Susceptible) model, also known as the contact process. It describes the evolution of an infectious disease in a population of N individuals, which can be either infected or susceptible [2, 3]. The stochastic version of the SIS model was introduced by Weiss & Dishon [15]. This model is a special case of more complex epidemiological models, like the reinfection SIRI model [13, 7]. The state variables are $I(t)$ ans $S(t)$ that correspond respectively to the number of infected and susceptible individuals at time t. Hence, the stochastic SIS model can be understood as a birth-and-death process with state space given by the number of infected individuals $I(t) \in \{0,1,2,...,N\}$, where N denotes the population size that is assumed to be constant. The number of susceptible individuals follows immediately by the formula $S(t) = N - I(t)$. For the SIS model the state $I(t) = 0$ is the only one absorbing and all the orders are transient. Hence, for a finite time, even if very high, the state zero will be achived and no more changed. Therefore, the stationary distribution of the SIS model is degenerated with probability one at $I(t) = 0$. However, the time to reach the equilibrium $I(t) = 0$ can be so long that the stationary distribution is non informative. Hence, the importance goes to the quasi-stationary distribution [4] of the SIS model [6, 9]. We know that the quasi-stationary distribution of the stochastic SIS model does not have explicit form and can only be computed by iterative methods. With this fact, the computation of explicit approximations of the quasi-stationary distribution of the SIS model gains relevance. This problem was first studied by Kryscio and Lefevre [6] and by Nåsell [9, 10], where two possible approximations were computed. They shown that these two approximations can be determined explicitly and give good approximations of the quasi-stationary distribution of the SIS model when the infection rate β is distinctly smaller or grater than the recovery rate α and the population size N is relatively large [10]. In [8] we consider the dynamic equations of the higher moments of infecteds, derived recursively from the master equation of the SIS model, and we apply the moment closure approximation in order to compute the stationary states. We observe that these stationary states can be used to obtain a good approximation of the quasi-stationary mean value of infecteds in the SIS model. We also observe that the quality of this approximation improves when we consider more and more moments.

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THE SIS MODEL

The stochastic SIS (Susceptible-Infected-Susceptible) model describes the evolution of an infectious disease through a population of N individuals. Let $S(t)$ be quantity of susceptible individuals at time t and $I(t)$ be the quantity of the infected individuals at time t and therefore we have that $S(t)+I(t)=N$. Let β denotes the birth rate and α the death rate. Therefore, the spreading of the infectious disease in the SIS model can be illustrated by

$$S+I \xrightarrow{\beta} I+I \quad , \quad I \xrightarrow{\alpha} S \, .$$

The time evolution of $p\,(I,t)$, the probability of having I infecteds at time t, is given by the master equation [1, 14, 5] of the SIS model

$$\begin{aligned} \frac{d}{dt} p\,(I,t) &= \beta\, \frac{N-(I-1)}{N} (I-1)\; p\,(I-1,t) \\ &+ \alpha\; (I+1)\; p\,(I+1,t) \\ &- \left(\beta\, \frac{N-I}{N} I + \alpha\, I\right)\; p\,(I,t) \quad , \end{aligned} \tag{1}$$

with $I \in \{0,1,...,N\}$.

The SIS quasi-stationary distribution

Since the state $I(t)=0$ is the only one absorbing and is attained for a finite time, the stationary distribution of the stochastic SIS model is degenerated with probability one at the origin

$$p(I^{*}=k) = \left\{ \begin{array}{ll} 1, & if\, k=0 \\ 0, & if\, k\neq 0 \end{array} \right. \, .$$

However, the time to reach the equilibrium $I(t)=0$ can be so long that the stationary distribution is non informative. Instead of the stationary distribution, our interest goes to the quasi-stationary distribution. This quasi-stationary distribution is conditioned to the non-extinction of infecteds, and therefore supported on the set of the transient states, $I(t) \in \{1,2,...,N\}$. Let $\tilde{q}_i(t)$ the probability of having i infecteds in the conditioned process, at time t, we obtain

$$\tilde{q}_i(t) = p(I(t)=i|I(t)>0),\; i=1,2,...,N \quad .$$

The quasi-stationary distribution is the solution of the equation $\frac{d}{dt}\tilde{q}_i(t)=0$. Let $q_i(t)$ denote these solution probabilities. In [9], it is shown that q_i satisfies the relation

$$q_i = \gamma(i)\alpha(i)R_0^{i-1} q_1 \, , \quad i=1,2,...,N \quad , \tag{2}$$

where $R_0=\beta/\alpha$,

$$\gamma(i) = \frac{1}{i} \sum_{k=1}^{i} \frac{1-\sum_{l=1}^{k-1} q_l}{\alpha(k)R_0^{k-1}} \quad , \quad \alpha(i) = \frac{N!}{(N-i)!N^i} \quad , \tag{3}$$

and

$$q_1 = \frac{1}{\sum_{i=1}^{N} \gamma(i)\alpha(i)R_0^{i-1}} \quad . \tag{4}$$

Since the quasi-stationary distribution of the stochastic SIS model does not have an explicit form, it is useful to approximate the model in order to obtain explicit approximations of the quasi-stationary distribution.

THE HIGHER MOMENTS DYNAMIC

Let $\langle I^n\rangle(t) = \sum_{I=0}^{N} I^n p(I,t)$ denotes the n^{th} moment of the state variable I. We observe [8] that the dynamic ordinary differential equation (ODE) of any moment of infecteds $\langle I^n\rangle$, derived from the master equation, is given by

$$\begin{aligned}\frac{d}{dt}\langle I^n\rangle &= \sum_{j=1}^{n}\binom{n}{j}\left(\beta+(-1)^j\alpha\right)\langle I^{n+1-j}\rangle - \frac{\beta}{N}\sum_{j=2}^{n}\binom{n}{j}\langle I^{n+2-j}\rangle \\ &\quad -\frac{\beta}{N}n\langle I^{n+1}\rangle \quad . \qquad (5)\end{aligned}$$

The m^{th} moment closure [16] approximation consists in closing the dynamic equation system of the m first moments of infecteds, under the assumption that the m^{th} cumulante is zero $\langle\langle I^{m+1}\rangle\rangle = 0$. Using the relation between cumulants and moments [12] given by

$$\langle\langle I^{n+1}\rangle\rangle = \langle I^{n+1}\rangle - \sum_{j=1}^{n}\binom{n}{j}\langle I^j\rangle\langle\langle I^{n+1-j}\rangle\rangle, \quad n \geq 1 \quad , \qquad (6)$$

where the first cumulant is equal to the mean value of the infecteds $\langle\langle I\rangle\rangle = \langle I\rangle$, the m^{th} moment closure approximation can be interpreted as substituting the $\langle I^{m+1}\rangle$ moment in the $\langle I^m\rangle$ ODE by lower moments.

We observe [8] that for the m moment closure ODE system, the stationary value of infected individuals $\langle I\rangle^*_{m,\beta}$ is a root of a polynomial function that can be computed recursively.

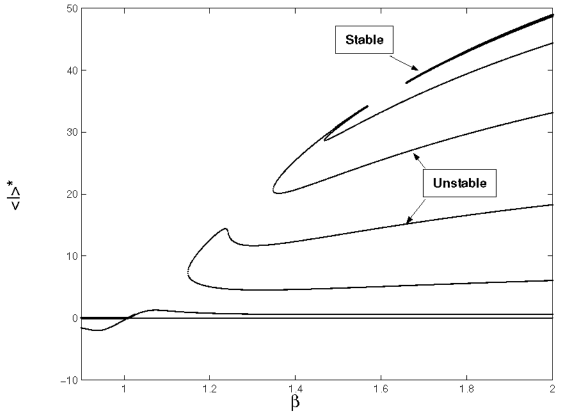

FIGURE 1. *The stationary mean value of infecteds for the $m = 7$ moment closure ODE's. Hence, we consider the dynamic equations of seven moments of infecteds. In the large lines we have the stable equilibria and in the others the unstable. We also consider $\alpha = 1$ and $N = 100$.*

In Fig. 1, we present the real zeros of the polynomial function obtained when are considered the dynamic of the first 7 moments of infecteds, i.e., the $\langle I\rangle^*_{m,\beta}$ for the $m = 7$ moment closure approximation and different infection rate values β. The values used for the parameters α and N are $\alpha = 1$ and $N = 100$. There are multiple equilibria and we present in thick lines the stable equilibria and in thin lines the unstable ones.

An approximating to the quasi-stationary states

In [11] we observe that the stable equilibria of the 1 to 3 moment closure ODE's can be used to give a good approximation of the quasi-stationary mean value of infecteds $\langle I\rangle_{QS,\beta}$ for high values of the population size N. In [8], we observe that this approximation improves when we consider more and more moments, i.e, using higher moment closure ODE's. In Fig. 2, we present the distance $|\langle I\rangle^*_{m,\beta} - \langle I\rangle_{QS,\beta}|$ between the first moment of infecteds obtained by the successive m moment closure ODE's $\langle I\rangle^*_{m,\beta}$ and the quasi-stationary mean value of infecteds $\langle I\rangle_{QS,\beta}$. We consider the infection rate values $\beta = 1.75$, $\beta = 2$ and $\beta = 2.25$, $\alpha = 1$ and $N = 100$. We observe that the distance between

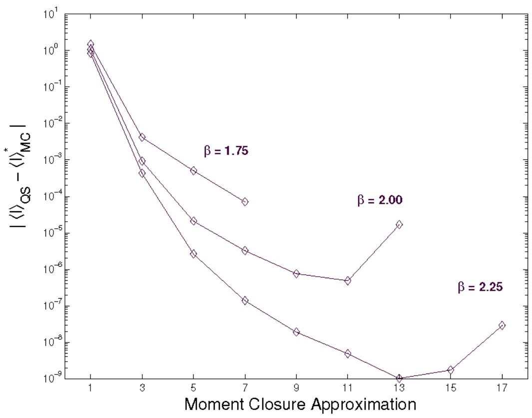

FIGURE 2. *Distance between the first moment of infecteds obtained by the successive m moment closure ODE's* $\langle I \rangle^*_{m,\beta}$ *and the quasi-stationary mean value* $\langle I \rangle_{QS}$*, for the infection rates* $\beta = 1.75$*,* $\beta = 2$ *and* $\beta = 2.25$*. We also consider* $\alpha = 1$ *and* $N = 100$.

the first moment of infecteds in moment closure and the quasi-stationary mean value of infecteds decreases when the number of the moments m increases up to a certain m^{th} moment closure approximation that can even occur before the break down of the stable equilibria for the value of the infection rate β under consideration.

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