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REGULAR ARTICLE

Dynamics of Epidemiological Models

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Abstract We study the SIS and SIRI epidemic models discussing different approaches to compute the thresholds that determine the appearance of an epidemic disease. The stochastic SIS model is a well known mathematical model, studied in several contexts. Here, we present recursively derivations of the dynamic equations

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for all the moments and we derive the stationary states of the state variables using the moment closure method. We observe that the steady states give a good approximation of the quasi-stationary states of the SIS model. We present the relation between the SIS stochastic model and the contact process introducing creation and annihilation operators. For the spatial stochastic epidemic reinfection model SIRI, where susceptibles S can become infected I, then recover and remain only partial immune against reinfection R, we present the phase transition lines using the mean field and the pair approximation for the moments. We use a scaling argument that allow us to determine analytically an explicit formula for the phase transition lines in pair approximation.

Keywords Epidemic models · Quasi-stationary states · Pair approximation

1 Introduction

One of the simplest and best studied epidemiological models is the stochastic SIS model. Many authors worked on the SIS model considering, only, the dynamical evolution of the mean value and the variance of the infected individuals. In this survey, we present recursively derivations of the dynamic equations for all the moments, and we derive the stationary states of the state variables using the moment closure method. The stationary states of the state variables give, surprisingly, good approximated values not of the stationary states but of the quasi-stationary states of the SIS master equation (see Martins et al. 2010b; Pinto et al. 2009). We present the relation between the SIS stochastic model and the contact process introducing creation and annihilation operators. This relation can lead to the characterization of critical thresholds also for more complex epidemiological models (see de Oliveira 2006; Martins et al. 2010a; Stollenwerk and Aguiar 2008). Examples of complex models appear to study reinfection processes in epidemiology and have recently attracted interest, especially for a first description of multi-strain epidemics, where after an initial infection immunity against one strain only gives partial immunity against a genetically close mutant strain. Transitions between no-growth, compact growth and annular growth have been observed (see Grassberger et al. 1997; Stollenwerk et al. 2010). For the spatial stochastic epidemic model SIRI we investigate, in the pair approximation scheme, its phase transition lines. We present the analytic expression for the phase transition line between no-growth and nontrivial stationary equilibria (see Martins et al. 2009; Stollenwerk et al. 2007).

2 The Stochastic SIS Model Dynamics

We consider the stochastic SIS (Susceptible-Infected-Susceptible) model that describes the evolution of an infectious disease in a population of N individuals. Denoting the global quantity of susceptible individuals at time t by $S(t)$ and the infected quantity by $I(t)$, we obtain that $S(t) + I(t) = N$. Let β denotes the birth rate

and α the death rate and therefore the spreading of the epidemic can be illustrated by $S + I \xrightarrow{\beta} I + I$ and $I \xrightarrow{\alpha} S$.

The time evolution of the probability $p(I, t)$ of having I infecteds at time t is given by the master equation of the SIS model

$$\begin{aligned}\frac{d}{dt}p(I,t) = \beta\,\frac{N-(I-1)}{N}(I-1)\,p(I-1,t)\\ +\alpha\,(I+1)\,p(I+1,t) - \left(\beta\,\frac{N-I}{N}I+\alpha\,I\right)p(I,t),\end{aligned} \tag{1.1}$$

with $I \in \{0, 1, \ldots, N\}$.

2.1 A Quasi-Stationary Approach

Since the state $I(t) = 0$ is the only one absorbing and all the orders are transient then $I(t) = 0$ will be attained for a finite time, even if very high, and no more changed. Hence, the stationary distribution is degenerated with probability one at the origin.

The quasi-stationary distribution of the SIS model is the stationary distribution of the stochastic process conditioned to the non-extinction of the infected individuals

$$\{I(t) = i | I(t) > 0\}, \quad i = 1, 2, \ldots, N,$$

and therefore supported on the set of the transient states. In Nåsell (1996), it is shown that the quasi-stationary probabilities q_i, of having i infected individuals given that $I(t) > 0$, satisfies the relation

$$q_i = \gamma(i)\alpha(i)R_0^{i-1}q_1, \quad i = 1, 2, \ldots, N, \tag{1.2}$$

where $R_0 = \beta/\alpha$,

$$\gamma(i) = \frac{1}{i}\sum_{k=1}^{i}\frac{1-\sum_{l=1}^{k-1}q_l}{\alpha(k)R_0^{k-1}}, \quad \alpha(i) = \frac{N!}{(N-i)!N^i},$$

and $q_1 = 1/\sum_{i=1}^{N}\gamma(i)\alpha(i)R_0^{i-1}$. Since Eq. (1.2) does not define the quasi-stationary distribution explicitly, it is useful to approximate the model in order to obtain explicit approximations of the quasi-stationary distribution. Two possible approximations were studied in Kryscio and Lefevre (1989) and Nåsell (1999). One is given by the SIS model with the recovery rate equal to zero when there exists only one infected individual. The stationary distribution of this process can be determined explicitly and gives a very good approximation of the real quasi-stationary distribution, when β is distinctly grater than α and $N \to \infty$.

Let $\langle I^n\rangle = \sum_{I=0}^{N} I^n p(I, t)$ denote the nth moment of the state variable I.

We observe that the ordinary differential equation (ODE) of the nth moment of infecteds, derived from the master equation, is given by

$$\frac{d}{dt}\langle I^n\rangle = f_n\big(\langle I\rangle, \langle I^2\rangle, \ldots, \langle I^n\rangle\big) - \frac{\beta}{N}n\langle I^{n+1}\rangle, \tag{1.3}$$

where

$$f_n(\langle I\rangle, \ldots, \langle I^n\rangle) = \sum_{j=1}^{n}\binom{n}{j}\big(\beta + (-1)^j\alpha\big)\langle I^{n+1-j}\rangle - \frac{\beta}{N}\sum_{j=2}^{n}\binom{n}{j}\langle I^{n+2-j}\rangle.$$

Hence, the ODE's for the n first moments of infecteds are not closed. To close these equations we apply the moment closure technique that consists in vanishing the $(n+1)$th cumulant $\langle\langle I^{n+1}\rangle\rangle = 0$. For the closed ODE system, we observe that the stationary value of infected individuals $\langle I\rangle^*_{n,\beta}$ is a zero of a $(n+1)$th order polynomial function that can be constructed recursively. Hence, for a fixed infection rate β, we compute the stationary states of the SIS model in the successive moment closure approximations.

In Martins et al. (2010b), the values of the stable equilibria $\langle I\rangle^*_{n,\beta}$ obtained in the moment closure of order n are compared with the mean values of the quasi-stationary distribution. In Fig. 1, we present the distances $|\langle I\rangle^*_{n,\beta} - \langle I\rangle_{QS,\beta}|$ between infecteds $\langle I\rangle^*_{n,\beta}$ obtained by the successive moment closure approximations, and the mean value $\langle I\rangle_{QS,\beta}$ for the explicit approximation of the quasi-stationary distribution. The comparison is made taking $\alpha = 1$, $N = 100$ and different infection rates $\beta = 1.75$, $\beta = 2$, $\beta = 2.25$ and $\beta = 5$.

We conclude that the distance $|\langle I\rangle^*_{n,\beta} - \langle I\rangle_{QS,\beta}|$ decreases with n up to some moment closure approximation. In Martins et al. (2010b), it is observed empirically that the approximation is already good for values of β relatively close to the critical values, like $\beta = 1.75$, and for relatively small size populations, like $N = 100$.

2.2 Creation and Annihilation Operators in the SIS Model

We consider now the SIS stochastic epidemic model describing the evolution of a disease throughout a population in a regular lattice. In this case, this epidemic model is also known as the contact process because it describes an interacting-particle

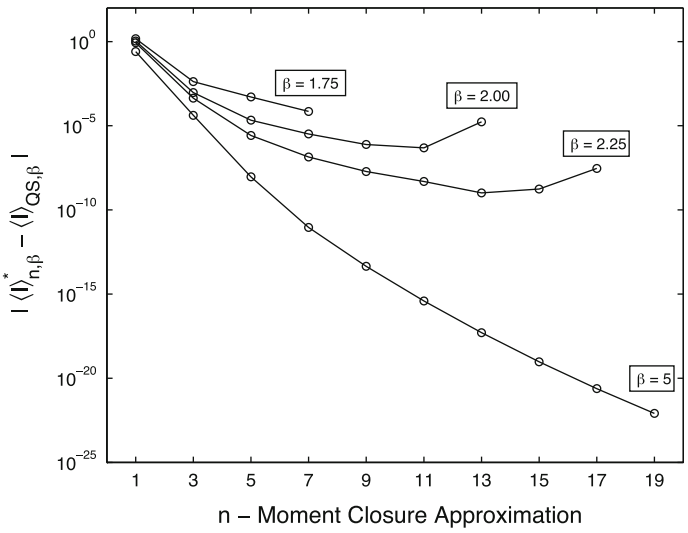

Fig. 1 Distances between the quasi-stationary mean value of infecteds $\langle I\rangle_{QS,\beta}$ and the mean values of infecteds for the nth moment closure approximation $\langle I\rangle^*_{n,\beta}$ for the successive moment closure approximations and for the infection rate values $\beta = 1.75$, $\beta = 2$, $\beta = 2.25$ and $\beta = 5$

system. The particles are annihilated spontaneously and created catalytically. We use the state variables

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.4}$$

to represent the site i whenever is empty or occupied. In an epidemic context, it corresponds to the individual i whenever is susceptible or infected.

The creation operator, c_i^+, and the annihilation operator, c_i, are given by

$$c_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad c_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.5}$$

and therefore we obtain

$$c_i^+|0\rangle = |1\rangle \quad \text{and} \quad c_i|1\rangle = |0\rangle. \tag{1.6}$$

Now we use the vector representation given by

$$\begin{aligned} |\psi(t)\rangle &= \sum_{\eta_1=0}^{1}, \ldots, \sum_{\eta_N=0}^{1} p(\eta_1, \ldots, \eta_N, t)\left(c_1^+\right)^{\eta_1}, \ldots, \left(c_N^+\right)^{\eta_N}|O\rangle \\ &= \sum_{\eta} p(\eta, t) \prod_{i=1}^{N} \left(c_i^+\right)^{\eta_i}|O\rangle, \end{aligned} \tag{1.7}$$

where $|O\rangle$ represents the vacuum state and $|\eta\rangle = |\eta_1, \ldots, \eta_N\rangle = |\eta_1\rangle\otimes, \ldots, \otimes|\eta_N\rangle$ represents the configuration of the lattice. The time evolution of the state vector $|\psi(t)\rangle$ is given by

$$\frac{d}{dt}|\psi(t)\rangle = L|\psi(t)\rangle, \tag{1.8}$$

for a Liouville operator L to be calculated from the master equation of the spatial SIS epidemic model. The Liouville operator is after some calculation given by

$$L = \sum_{i=1}^{N}(1 - c_i)\beta\left(\sum_{j=1}^{N} J_{ij}c_j^+ c_j\right)c_i^+ + \sum_{i=1}^{N}(1 - c_i^+)\alpha c_i. \tag{1.9}$$

The Liouville operator can be given in the form of a perturbation on the single sites operators with an easily diagonalizable free operator W_0, acting only on single sites, and an interaction operator V contributing with strength $\lambda = \beta Q$ to the interaction (see for the 1 dimensional contact process e.g. de Oliveira (2006)

$$L = W_0 + \lambda V, \tag{1.10}$$

where $W_0 = \sum_{i=1}^{N} \hat{B}_i$, with $\hat{B}_i = (1 - c_i^+)c_i$ and without loss of generality $\alpha = 1$, and

$$V = \sum_{i=1}^{N} \hat{Q}_i(\hat{n}_{i-1} + \hat{n}_{i+1}), \tag{1.11}$$

where $\hat{Q}_i = \frac{1}{Q}(1 - c_i)c_i^+$ with Q the number of neighbours of one individual and $\hat{n}_i = c_i^+ c_i$ is the number operator. This completes Eq. (1.10) for the Liouville operator given in Eq. (1.9). The critical threshold and the critical exponents can be

calculated very accurately via a scaling argument, e.g. for the time correlation function using the spectral gap and Padé approximation (de Oliveira 2006). For the SIS model, it is computed explicitly in Martins et al. (2010a) the first coefficients of the series expansion of the gap between the dominant and subdominant eigenvalues of the evolution operator. In Stollenwerk and Aguiar (2008), these ideas are extended to the SIRI model.

3 Phase Transition Lines in Pair Approximation for the SIRI Model

We consider the following transitions between host classes for N individuals being either susceptible S, infected I by a disease or recovered R

$$\begin{aligned} S + I &\xrightarrow{\beta} I + I \\ I &\xrightarrow{\gamma} R \\ R + I &\xrightarrow{\tilde{\beta}} I + I \\ R &\xrightarrow{\alpha} S \end{aligned}$$

resulting in the master equation for the SIRI model (Stollenwerk et al. 2007).

To calculate the dynamics of the moments we apply the master equation into the time derivative of the moments obtaining for the mean total number of susceptible, infected and recovered hosts the following ODE's (see Martins et al. 2009; Stollenwerk et al. 2007)

$$\begin{aligned} \frac{d}{dt}\langle S\rangle &= \alpha\langle R\rangle - \beta\,\langle SI\rangle \\ \frac{d}{dt}\langle I\rangle &= \beta\,\langle SI\rangle - \gamma\langle I\rangle + \tilde{\beta}\,\langle RI\rangle \\ \frac{d}{dt}\langle R\rangle &= \gamma\langle I\rangle - \alpha\langle R\rangle - \tilde{\beta}\,\langle RI\rangle \end{aligned} \tag{1.12}$$

which include the pairs, like

$$\langle SI\rangle(t) = \sum_{S_1=0}^{1}\sum_{I_1=0}^{1}\sum_{R_1=0}^{1},\ldots,\sum_{R_N=0}^{1}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}S_iI_j\right)p(S_1,I_1,R_1,\ldots,R_N,t). \tag{1.13}$$

In the ODE's for the second moments $\langle SS\rangle$, $\langle SI\rangle$, etc., the triples will appear.

Hence, either we have to continue to calculate equations for the triples, which will involve even higher clusters, or we can approximate the higher moments by lower ones. The simplest scheme is the mean field approximation. Here we go one step beyond by approximating the triples into pairs. There is a vast literature on pair approximation, for a summary see e.g. Rand (1999).

To obtain approximate expressions for the triples we consider only the true triples, denoted with a tilde, where the last site is not identical to the first. We have e.g.

$$\langle ISI\rangle = \langle \widetilde{ISI}\rangle + \langle SI\rangle \tag{1.14}$$

when the local variable at site k, here I_k, is of the same type as the one in i, here I_i and simply

$$\langle SIR\rangle = \langle \widetilde{SIR}\rangle \tag{1.15}$$

when the local variable at site k, now R_k, is different from the one in i, now S_i.

From now on we will only consider regular lattices. Hence we can assume that all individuals have the same number of neighbours Q. The pair approximation yields

$$\langle \widetilde{SIR}\rangle \approx \frac{Q-1}{Q} \cdot \frac{\langle SI\rangle \cdot \langle IR\rangle}{\langle I\rangle} \tag{1.16}$$

obtained from an analog for the Bayesian formula for conditional probabilities applied to the local expectation values and a spatial homogeneity argument, e.g. $\langle I_i\rangle \approx \langle I\rangle / N$.

Using the balance equations, e.g. $\langle S\rangle + \langle I\rangle + \langle R\rangle = N$, one can reduce the ODE system for total expectation values and for pair expectation values to five independent variables $\langle I\rangle$, $\langle R\rangle$, $\langle SI\rangle$, $\langle RI\rangle$ and $\langle SR\rangle$ (see Martins et al. 2009; Stollenwerk et al. 2007):

$$\begin{aligned}
\frac{d}{dt}\langle I\rangle &= \beta\,\langle SI\rangle - \gamma\langle I\rangle + \tilde{\beta}\,\langle RI\rangle \\
\frac{d}{dt}\langle R\rangle &= \gamma\langle I\rangle - \alpha\langle R\rangle - \tilde{\beta}\,\langle RI\rangle \\
\frac{d}{dt}\langle SI\rangle &= \alpha\langle RI\rangle - (\gamma+\beta)\,\langle SI\rangle + \beta(Q-1)\,\langle SI\rangle \\
&\quad - \beta\,\frac{Q-1}{Q}\,\frac{(2\langle SI\rangle + \langle SR\rangle)\cdot\langle SI\rangle}{N-\langle I\rangle-\langle R\rangle} + \tilde{\beta}\,\frac{Q-1}{Q}\,\frac{\langle SR\rangle\,\langle RI\rangle}{\langle R\rangle} \\
\frac{d}{dt}\langle RI\rangle &= \gamma\,(Q\langle I\rangle - \langle SI\rangle) - (\alpha+2\gamma+\tilde{\beta})\,\langle RI\rangle + \beta\,\frac{Q-1}{Q}\,\frac{\langle SR\rangle\quad\langle SI\rangle}{N-\langle I\rangle-\langle R\rangle} \\
&\quad + \tilde{\beta}\,\frac{Q-1}{Q}\,\frac{(Q\langle R\rangle - \langle SR\rangle - 2\langle RI\rangle)\cdot\langle RI\rangle}{\langle R\rangle} \\
\frac{d}{dt}\langle SR\rangle &= \gamma\langle SI\rangle + \alpha\,(Q\langle R\rangle - 2\langle SR\rangle - \langle RI\rangle) \\
&\quad - \beta\,\frac{Q-1}{Q}\,\frac{\langle SR\rangle\quad\langle SI\rangle}{N-\langle I\rangle-\langle R\rangle} - \tilde{\beta}\,\frac{Q-1}{Q}\,\frac{\langle RI\rangle\,\langle SR\rangle}{\langle R\rangle}
\end{aligned} \tag{1.17}$$

The full SIRI system cannot be solved analytically in stationarity. After some simplifications, expressing $\langle RI\rangle^*$, $\langle SI\rangle^*$ and $\langle SR\rangle^*$ as functions of the the variables $\langle I\rangle^*$ and $\langle R\rangle^*$ only, we are left with two implicit equations for the remaining variables $\langle I\rangle^*$ and $\langle R\rangle^*$. But from

$$0 = \frac{d}{dt}\langle RI\rangle^* = f(\langle I\rangle^*, \langle R\rangle^*) \tag{1.18}$$

and

$$0 = \frac{d}{dt}\langle SI\rangle^* = g(\langle I\rangle^*, \langle R\rangle^*) \tag{1.19}$$

we can only get implicit equations for the variables $\langle I\rangle^*$ and $\langle R\rangle^*$.

Considering the special cases for reinfection rate equal to first infection rate (the SIS limit of the SIRI model), vanishing the reinfection rate (the SIR limit of the SIRI model) and the limit of vanishing transition from recovered to susceptible α, the above system can be solved analytically. In these cases, we can give the stationary values $\langle I\rangle^*$ etc. as well as the critical parameters. For the general case, no general solution for the total number of infected etc. in stationarity can be given.

In Martins et al. (2009) and Stollenwerk et al. (2007), it is investigated the Eqs. (1.18, 1.19) further, using the information that when $\langle I\rangle^*$ goes to zero, so does $\langle R\rangle^*$, but the quotient stays finite

$$\lim_{\langle I\rangle^*\to 0} \frac{\langle R\rangle^*}{\langle I\rangle^*} = \frac{\gamma \cdot B}{2\alpha \cdot E} \tag{1.20}$$

with B and E given below. Only later, it is obtained the following solution for the critical curve $\beta(\tilde{\beta})$, for general γ and also non-vanishing α, as

$$\beta(\tilde{\beta}) = \frac{C(\tilde{\beta})}{D(\tilde{\beta})} \tag{1.21}$$

with numerator given by

$$C = Q\gamma\tilde{\beta} \cdot B^2, \tag{1.22}$$

and denominator

$$\begin{aligned} D = {} & (Q\gamma - (Q-1)\alpha) \cdot B^2 \\ & + 2(Q(Q-1)\tilde{\beta} + 2(Q-1)\alpha - Q\gamma) \cdot E \cdot B - 4(Q-1)\alpha \cdot E^2, \end{aligned} \tag{1.23}$$

with the expressions

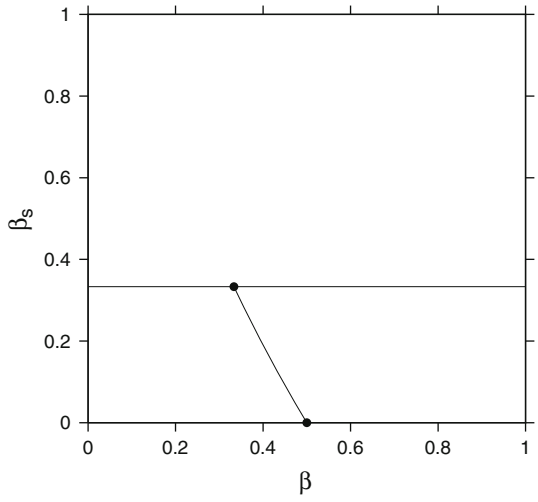

Fig. 2 The phase transition line between no-growth and ring-growth determined from the analytic solution for the $\alpha = 0$ case which is explicitly given in Eq. (1.26). In addition, we also present the phase transition points for the SIS and SIR limiting cases of the SIRI model

$$B = \alpha + Q\gamma - (Q-1)(\alpha + \tilde{\beta}) + A, \qquad E = \alpha + \tilde{\beta} + Q\gamma, \tag{1.24}$$

where

$$A = \sqrt{(Q-1)^2\tilde{\beta}^2 + 2Q(Q-1)(\alpha-\gamma)\tilde{\beta} + Q^2(\alpha+\gamma)^2}. \tag{1.25}$$

This completes the expression for the critical curve $\beta(\tilde{\beta})$ for the general α and γ case.

In the limit of $\alpha \to 0$, one obtains a rather simple expression

$$\beta(\tilde{\beta}) = \frac{\gamma^2 Q - \gamma\tilde{\beta}(Q-1)}{\gamma Q(Q-2) + \tilde{\beta}(Q-1)}. \tag{1.26}$$

shown graphically in Fig. 2, for $\gamma = 1$ and $Q = 4$ appropriated for two dimensional square lattices (see Martins et al. 2009; Stollenwerk et al. 2007).

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