Time-fractional diffusion equation with ψ-Hilfer derivative*

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Abstract

In this work, we consider the multidimensional time-fractional diffusion equation with the ψ-Hilfer derivative. This fractional derivative enables the interpolation between Riemann-Liouville and Caputo fractional derivatives and its kernel depends on an arbitrary positive monotone increasing function ψ thus encompassing several fractional derivatives in the literature. This allows us to obtain general results for different families of problems that depend on the function ψ selected. By employing techniques of Fourier, ψ-Laplace, and Mellin transforms, we obtain a solution representation in terms of convolutions involving Fox H-functions for the Cauchy problem associated with our equation. Series representations of the first fundamental solution are explicitly obtained for any dimension as well as the fractional moments of arbitrary positive order. For the one-dimensional case, we show that the series representation reduces to a Wright function and we prove that it corresponds to a probability density function for any admissible ψ. Finally, some plots of the fundamental solution are presented for particular choices of the function ψ and the order of differentiation.

Keywords: Time-fractional diffusion equation; ψ-Hilfer fractional derivative; ψ-Laplace transform; Fundamental solution; Fractional moments.

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1 Introduction

Since the beginning of fractional calculus, several definitions of fractional integrals and fractional derivatives have been introduced in the literature. The main difference between them lies in their kernel and this makes the number of definitions wide. This diversity allows certain problems to be treated with specific fractional operators. In [16, 26] was proposed a fractional integral operator with respect to another function ψ, obtaining a general operator, in the sense that it is enough to choose a function ψ with certain properties to obtain most of the existing fractional integral operators. Attempting to incorporate a large number of definitions of fractional derivatives into one formulation, the concept of the fractional derivative of a function with respect to another function was recently introduced. In 2017, Almeida [2] proposed a new fractional derivative called ψ-Caputo with respect to a function ψ that generalizes a class of fractional derivatives in the Caputo sense. The same idea can be adapted to define the ψ-Riemmann-Liouville fractional derivative. In 2018, Sousa and Oliveira [29] unified

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both definitions using Hilfer's idea and introduced the so-called ψ-Hilfer fractional derivative. The advantage here is the freedom to choose the type of differentiation, since Hilfer's definition interpolates between fractional derivatives of Riemann-Liouville and Caputo types, and also the freedom to choose the function ψ which makes it possible to obtain as special cases some well-known fractional derivatives such as Caputo, Riemann-Liouville, Hadamard, Katugampola, Chen, Jumarie, Prabhakar, Erdélyi-Kober, Weyl, among others (see [29, Sec. 5]). Regarding the ψ-Hilfer derivative, there are already some works dealing with its mathematical properties in connection with the ψ-Laplace transform [27, 28, 30]. The concept of the ψ-Laplace transform is necessary to solve and discuss properties of fractional differential equations involving the ψ-Hilfer derivative. As for applications, some authors have studied the existence and regularity of weak solutions for fractional boundary problems with ψ-Hilfer derivatives [30], stability of Ulam-Hyers type for fractional differential equations with ψ-Hilfer derivatives with impulses and delay [18], and the time-fractional diffusion equation with time-dependent diffusion coefficient [7].

During the last three decades, diffusion-wave equations with space and/or time-fractional derivatives have been studied by several authors, in unidimensional and multidimensional cases (see [3–6, 15, 19, 20, 24]). Due to the non-local nature of these fractional operators, this type of fractional partial differential equations allows the theoretical and practical analysis of different physical processes. Furthermore, fractional derivatives give us a more accurate interpretation of anomalous diffusion phenomena when compared to the corresponding integer derivatives. Our motivation is to continue our previous work related to this subject (see [8–11,31]) and to present a unified approach to the time-fractional diffusion equation using the ψ-Hilfer derivative. As a by-product, we obtain the solution of the Cauchy problem associated with our equation in terms of convolution integrals involving Fox H-functions. We were able to obtain integral and series representations for the first fundamental solution and prove that in the one-dimensional case, the first fundamental solution can be interpreted as a probability density function, for each positive and monotone increasing function ψ. The key point to obtain this result is the combination of Bernstein's Theorem with the asymptotic behaviour of the Wright function and the complete monotonicity of the two-parameter Mittag-Leffler function. The results derived here are of general nature, as our approach allows us to simultaneously treat several fractional derivatives that appear in the literature and also to consider new derivatives that have not yet been studied or considered.

The structure of the article is as follows: in the Preliminaries, we recall some basic notions about fractional derivatives, special functions, and integral transforms that are necessary for the development of this work. In Section 3, we formulate the problem for the time-fractional diffusion equation in higher dimensions with the ψ-Hilfer fractional derivative. We use a combination of ψ-Laplace, Fourier, and Mellin transforms to obtain a representation of the solution of our equation via convolution integrals involving Fox H functions. The key points to obtain our main result are the interpretation of the inverse ψ-Laplace transform in terms of the classical inverse Laplace transform, and the use of the Mellin transform to invert the Fourier transform. Fractional moments of arbitrary order $\gamma > 0$ are calculated in Section 4 for the first fundamental solution of our equation. In Section 5, we obtain a series representation of the first fundamental solution, for even and odd dimensions. Furthermore, we prove that in the one-dimensional case the fundamental solution can be interpreted as a probability density function, for any admissible function ψ. In the last section, we exhibit plots of the fundamental solution in the one-dimensional case for different choices of the function ψ and the order of differentiation.

2 Preliminaries

We start this section by presenting some concepts related to fractional integrals and derivatives of a function f with respect to another function ψ (for more details see [29] and the references indicated therein).

Definition 2.1 *(cf. [29, Def. 4]) Let (a, b) be a finite or infinite interval on the real line $\mathbb{R}$ and $\alpha > 0$. Also let ψ be an increasing and positive monote function on (a, b). The left Riemann-Liouville fractional integral of a function f with respect to another function ψ on $[a, b]$ is defined by*

$$\left(I_{a^+}^{\alpha;\psi} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(w)\, \left(\psi(t) - \psi(w)\right)^{\alpha-1} f(w)\, dw, \qquad t > a. \tag{1}$$

Now, we introduce the definition of the so-called ψ-Hilfer fractional derivative of a function f with respect to another function.

Definition 2.2 *(cf. [29, Def. 7]) Let $m-1<\alpha<m$ with $n\in\mathbb{N}$, $I=[a,b]$ be a finite or infinite interval on the real line and $f,\psi\in C^{n}[a,b]$ two functions such that ψ is a positive monotone increasing function and $\psi'(t)\neq 0$, for all $t\in I$. The ψ-Hilfer left fractional derivative ${}^{H}\mathbb{D}_{t,a^+}^{\alpha,\mu;\psi}$ of order α and type $\mu\in[0,1]$ is defined by*

$$\left({}^{H}\mathbb{D}_{a^+}^{\alpha,\mu;\psi}f\right)(t)=I_{a^+}^{\mu(m-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{m}I_{a^+}^{(1-\mu)(m-\alpha);\psi}f(t). \tag{2}$$

We observe that when $\mu=0$ we recover the left Riemann-Liouville fractional derivative of a function with respect to ψ (see [29, Def. 5]) and when $\mu=1$ we obtain the left Caputo fractional derivative of a function with respect to ψ (see [29, Def. 6]). In Section 5 of [29] is presented a list of several fractional integrals and fractional derivatives that can be obtained from (1) and (2), respectively, for different choices of μ and $\psi(t)$. The previous definitions of fractional integrals and derivatives can be naturally extended to $\mathbb{R}^n$ considering partial fractional integrals and derivatives (see Chapter 5 in [26]).

Now, we recall some special functions used in this article and some of their main properties. The Gamma function (see [1]) is defined by

$$\Gamma(z)=\int_0^{+\infty}t^{z-1}\,e^{-t}\,dt,\qquad \mathrm{Re}(z)>0,$$

and admits an analytic continuation to the whole complex plane as a meromorphic function with simple poles at the negative integers and zero. The residues of the poles of the Gamma function are given by:

$$\mathrm{res}_{s=-k}\Gamma(s)=\frac{(-1)^k}{k!},\quad k\in\mathbb{Z}_0^+. \tag{3}$$

The one, two, and three parameter Mittag-Leffler functions of a complex variable are defined in terms of power series by (see [12])

$$E_{\beta_1}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta_1 n+1)},\quad \beta_1\in\mathbb{C},\quad z\in\mathbb{C},$$

$$E_{\beta_1,\beta_2}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta_1 n+\beta_2)},\quad \mathrm{Re}(\beta_1)>0,\quad \beta_2\in\mathbb{C},\quad z\in\mathbb{C},$$

$$E_{\beta_1,\beta_2}^{\beta_3}(z)=\sum_{n=0}^{\infty}\frac{(\beta_3)_n\,z^n}{n!\,\Gamma(\beta_1 n+\beta_2)},\quad \mathrm{Re}(\beta_1)>0,\quad \mathrm{Re}(\beta_2)>0,\quad \beta_3>0,\quad z\in\mathbb{C}.$$

In [25] it was proved that $E_{\beta_1,\beta_2}(-x)$, with $x\in\mathbb{R}_0^+$, is a completely monotone function for all $0\leq\beta_1\leq 1$ and $\beta_2\geq\beta_1$. The Wright function $W_{\beta_1,\beta_2}(z)$ is defined in terms of power series by (see [21])

$$W_{\beta_1,\beta_2}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\beta_1 n+\beta_2)\,n!},\quad \beta_1>-1,\quad \beta_2\in\mathbb{R},\quad z\in\mathbb{C}, \tag{4}$$

and is an entire function for $\beta_1>-1$ (see [13]). When $\beta_1\in]0,1[$ we have the following Laplace pair (see Formula (A.27) in [22])

$$\mathcal{L}\left\{W_{-\beta_1,\beta_2}(-x)\right\}(\mathbf{s})=E_{\beta_1,\beta_1+\beta_2}(-\mathbf{s}),\quad x\in\mathbb{R}^+. \tag{5}$$

We also recall the following result

Theorem 2.3 *(cf. [14, Thm. 2.1.3]) If $\beta_2\in\mathbb{R}$, $-1<\beta_1<0$, and $|\arg(-z)|<\min\left\{\frac{3\pi}{2}(1+\beta_1),\pi\right\}-\epsilon$, with $\epsilon>0$, then*

$$W_{\beta_1,\beta_2}(z)=Y^{\frac{1}{2}-\beta_2}\,e^{-Y}\left\{\sum_{m=0}^{M-1}A_mY^{-m}+O\left(Y^{-M}\right)\right\},\quad Y\to+\infty$$

with

$$Y=(1+\beta_1)\left((-\beta_1)^{-\beta_1}(-z)\right)^{\frac{1}{1+\beta_1}}$$

and the coefficients A_m, $m=0,1,\ldots,M-1$ are defined by the asymptotic expansion

$$\frac{\Gamma\left(1-\beta_2-\beta_1 t\right)}{2\pi\left(-\beta_1\right)^{-\beta_1 t}\left(1+\beta_1\right)^{(1+\beta_1)(1+t)}\,\Gamma\left(1+t\right)}=\sum_{m=0}^{M-1}\frac{\left(-1\right)^m\,A_m}{\Gamma\left(\left(1+\beta_1\right)t+\beta_2+\frac{1}{2}+m\right)}$$
$$+O\left(\frac{1}{\Gamma\left(\left(1+\beta_1\right)t+\beta_2+\frac{1}{2}+M\right)}\right) \tag{6}$$

valid for all $\arg(t)$, $\arg(-\beta_1 t)$, *and* $\arg(1-\beta_2-\beta_1 t)$ *lying between* $-\pi$ *and* π *and* t *tending to infinity.*

The Fox H-function $H^{m,n}_{p,q}(z)$ is defined, by means of a Mellin-Barnes type integral, by

$$H^{m,n}_{p,q}\left[\, z \;\middle|\; \begin{matrix}(a_1,\ \alpha_1),\ldots,(a_p,\ \alpha_p)\\ \\ (b_1,\ \beta_1),\ldots,(b_q,\ \beta_q)\end{matrix}\right]=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j s\right)\ \prod_{i=1}^{n}\Gamma\left(1-a_i-\alpha_i s\right)}{\prod_{i=n+1}^{p}\Gamma\left(a_i+\alpha_i s\right)\ \prod_{j=m+1}^{q}\Gamma\left(1-b_j-\beta_j s\right)}\,z^{-s}\,ds, \tag{7}$$

where $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}^+$, for $i=1,\ldots,p$ and $j=1,\ldots,q$, and $\mathcal{C}$ is a suitable contour in the complex plane separating the poles of the two factors in the numerator (see [17]).

In this work, some integral transforms are used, namely, namely the ψ-Laplace, Fourier, and Mellin transforms. The ψ-Laplace transform of a real-valued function f with respect to ψ is defined by (see [27, Def. 13])

$$\mathcal{L}_\psi\left\{f\left(t\right)\right\}\left(\mathbf{s}\right)=\widetilde{f}_\psi\left(\mathbf{s}\right)=\int_0^{+\infty}e^{-\mathbf{s}\,\psi(t)}\,\psi'\left(t\right)\,f\left(t\right)\,dt,\quad \mathrm{Re}\left(\mathbf{s}\right)\in\mathbb{C},$$

where ψ is a non negative monotone increasing function in $\mathbb{R}^+_0$ and such that $\psi\left(0^+\right)=0$. The ψ-Laplace transform may be written as the following composition operator involving the classical Laplace transform (cf. [27, Thm. 4])

$$\mathcal{L}_\psi=\mathcal{L}\circ Q_{\psi^{-1}},\qquad \text{where }\left(Q_{\psi^{-1}}f\right)(t)=f\left(\psi^{-1}\left(t\right)\right).$$

As a consequence, if f is a function whose classical Laplace transform is $\widetilde{f}$, the ψ-Laplace transform of $f\left(\psi\left(t\right)\right)$ is also $\widetilde{f}\left(\mathbf{s}\right)$ (see [27, Cor. 2])

$$\mathcal{L}\left\{f\left(t\right)\right\}\left(\mathbf{s}\right)=\widetilde{f}\left(\mathbf{s}\right)\quad\Rightarrow\quad\mathcal{L}_\psi\left\{f\left(\psi\left(t\right)\right)\right\}\left(\mathbf{s}\right)=\widetilde{f}\left(\mathbf{s}\right).$$

Concerning the inverse ψ-Laplace transform, it can be written as the following composition operator

$$\mathcal{L}_\psi^{-1}=Q_\psi\circ\mathcal{L}^{-1},\qquad\text{where }\left(Q_\psi f\right)(t)=f\left(\psi\left(t\right)\right),$$

i.e.,

$$\mathcal{L}_\psi^{-1}\left\{\widetilde{f}_\psi\left(\mathbf{s}\right)\right\}(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\mathbf{s}\,\psi(t)}\,\widetilde{f}_\psi\left(\mathbf{s}\right)\,d\mathbf{s},$$

where $\mathrm{Re}\left(\mathbf{s}\right)=c$. We note that the definition of the ψ-Laplace can be adapted to any interval $[a,+\infty[\subseteq\mathbb{R}^+_0$ with ψ satisfying $\psi\left(a^+\right)=0$. This is important in our work so that the ψ-Hilfer derivative covers the largest number of fractional derivatives. When the ψ-Laplace transform is applied to the ψ-Hilfer derivative, we get (see [27, Thm. 6])

$$\mathcal{L}_\psi\left\{{}^H\mathbb{D}^{\alpha,\mu;\psi}_{a^+}f\left(t\right)\right\}\left(\mathbf{s}\right)=\mathbf{s}^\alpha\,\widetilde{f}_\psi\left(\mathbf{s}\right)-\sum_{j=0}^{m-1}\mathbf{s}^{m-\mu(m-\alpha)-1-j}\left(I^{(1-\mu)(m-\alpha)-j;\psi}_{t,a^+}f\right)\left(a^+\right), \tag{8}$$

where $m=[\alpha]+1$ and the initial-value terms $\left(I^{(1-\mu)(m-\alpha)-j,\psi}_{a^+}f\right)\left(a^+\right)$ are evaluated at the limit $t\to a^+$. The ψ-Laplace convolution of two functions is defined by (see [27, Def. 15])

$$\left(f*_\psi g\right)(t)=\int_0^t f\left(\psi^{-1}\left(\psi\left(t\right)-\psi\left(w\right)\right)\right)\,\psi'\left(w\right)\,g\left(w\right)\,dw,\quad t\in\mathbb{R}^+, \tag{9}$$

and the correspondent Convolution Theorem is (see [27, Thm. 8])

$$\mathcal{L}_\psi \left\{ \left(f *_\psi g \right)(t) \right\}(\mathbf{s}) = \mathcal{L}_\psi \left\{ f \right\}(\mathbf{s}) \, \mathcal{L}_\psi \left\{ g \right\}(\mathbf{s}). \tag{10}$$

For the three parameter Mittag-Leffler function we have the ψ-Laplace pair (see Example 5 in [27])

$$\mathcal{L}_\psi \left\{ \left(\psi(t) \right)^{\nu-1} E^{\gamma}_{\mu,\nu} \left(\lambda \left(\psi(t) \right)^{\mu} \right) \right\}(\mathbf{s}) = \frac{\mathbf{s}^{\mu\gamma-\nu}}{\left(\mathbf{s}^{\mu} - \lambda \right)^{\gamma}}, \tag{11}$$

where $\mu \in \mathbb{C}$ such that $\mathrm{Re}(\mu) > 0$ and $\left| \frac{\lambda}{\mathbf{s}^\mu} \right| < 1$.

The n-dimensional Fourier transform of a real-valued integrable function $f(x)$ with $x \in \mathbb{R}^n$ is defined by (see [16])

$$\mathcal{F}\left\{ f(x) \right\}(\kappa) = \widehat{f}(\kappa) = \int_{\mathbb{R}^n} e^{i\kappa\cdot x} f(x) \, dx, \quad \kappa \in \mathbb{R}^n,$$

while the corresponding inverse Fourier transform is formally given by

$$f(x) = \mathcal{F}^{-1}\left\{ \widehat{f}(\kappa) \right\}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix\cdot\kappa} \, \widehat{f}(\kappa) \, d\kappa. \tag{12}$$

The Convolution Theorem associated with the Fourier transform states that

$$\mathcal{F}\left\{ \left(f *_x g \right)(x) \right\}(\kappa) = \mathcal{F}\left\{ f \right\}(\kappa) \, \mathcal{F}\left\{ g \right\}(\kappa), \tag{13}$$

where the convolution $*_x$ is given by

$$\left(f *_x g \right)(x) = \int_{\mathbb{R}^n} f(x - z) \, g(z) \, dz. \tag{14}$$

The n-dimensional Laplace operator $\Delta_x = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ has the Fourier symbol $-|\kappa|^2$, i.e.:

$$\mathcal{F}\left\{ \Delta_x f(x) \right\}(\kappa) = -|\kappa|^2 \, \mathcal{F}\left\{ f(x) \right\}(\kappa). \tag{15}$$

Another important integral transform that we use in this work is the Mellin transform. For f locally integrable on $]0, +\infty[$ it is defined by (see [16])

$$\mathcal{M}\left\{ f(w) \right\}(s) = f^*(s) = \int_0^{+\infty} w^{s-1} f(w) \, dw, \quad s \in \mathbb{C}, \tag{16}$$

and the inverse Mellin transform is given by

$$f(w) = \mathcal{M}^{-1}\left\{ f^*(s) \right\}(w) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} w^{-s} f(s) \, ds, \quad w > 0, \quad c = \mathrm{Re}(s). \tag{17}$$

The condition for the existence of (16) is that $-p < c < -q$ (called the fundamental strip), where p, q, are the order of f at the origin and ∞, respectively. The integration in (17) is performed along the imaginary axis and the result does not depend on the choice of c inside the fundamental strip. More information about this transform and its properties can be found, for example, in [16]. The Mellin convolution between two functions is defined by

$$\left(f *_{\mathcal{M}} g \right)(x) = \int_0^{+\infty} f\left(\frac{x}{u} \right) g(u) \, \frac{du}{u}, \tag{18}$$

and satisfies the Mellin Convolution Theorem (see Formula (1.4.40) in [16])

$$\mathcal{M}\left\{ f *_{\mathcal{M}} g \right\}(s) = \mathcal{M}\left\{ f \right\}(s) \, \mathcal{M}\left\{ g \right\}(s).$$

The following relation holds (see Formula (1.4.30) in [16])

$$\mathcal{M}\left\{ f\left(\frac{1}{x} \right) \right\}(s) = \mathcal{M}\left\{ f \right\}(-s). \tag{19}$$

For the two-parameter Mittag-Leffler function we have (see Formula (4.9.3) in [12])

$$\mathcal{M}\left\{ E_{\beta_1,\beta_2}(-z) \right\}(s) = \frac{\Gamma(s)\,\Gamma(1-s)}{\Gamma(\beta_2 - \beta_1 s)}. \tag{20}$$

Throughout the paper, we assume that all the involved functions are ψ-Laplace, Fourier, and Mellin transformable.

3 Generalized time-fractional diffusion equation with ψ-Hilfer derivative

In this work, we consider the following time-fractional diffusion equation

$$c_1\, {}^H\partial_{t,a^+}^{\alpha,\mu;\psi} u(x,t) - c_2^2\, \Delta_x u(x,t) + d^2\, u(x,t) = q(x,t), \tag{21}$$

subject to the following initial and boundary conditions

$$\lim_{|x|\to+\infty} u(x,t) = 0 \qquad \text{and} \qquad \left(I_{t,a^+}^{(1-\mu)(1-\alpha);\psi} u\right)\left(x,a^+\right) = f(x) \tag{22}$$

where the second condition in (22) is evaluated at the limit $t \to a^+$. Moreover, $c_1, c_2 \in \mathbb{R}^+$, $d \in \mathbb{R}$, $(x,\, t) \in \mathbb{R}^n \times I$, with $I = [a,b]$ being a finite or infinite interval on $\mathbb{R}^+$, Δ_x is the classical Laplace operator in $\mathbb{R}^n$, the partial time-fractional derivative of order $\alpha \in]0,1]$ and type $\mu \in [0,1]$ is the ψ-Hilfer derivative given by (2), ψ is a function under the conditions of Definition 2.2, q belongs to $L_1(\mathbb{R}^n \times I)$, and $f \in L_1(\mathbb{R}^n)$. We look for solutions $u(x,t)$ of our problem in the space $C^2(\mathbb{R}^n) \times C^1((a,b))$ with possible exception in $x = 0$.

To obtain the analytical solution of (21)-(22), we start by applying to (21) the Fourier transform to the space variable x and the ψ-Laplace transform to the variable t, and then we solve the equation in the Fourier-Laplace domain for $\widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s})$. After that, we invert the ψ-Laplace transform, resulting in $\widehat{u}(\kappa,t)$, and then invert the Fourier transform of the result. For the inversion of the ψ-Laplace transform we take into account the operational rules presented in [27], while the inversion of the Fourier transform is performed via the Mellin transform. Let us start by applying to (21) the ψ-Laplace transform with respect to the time variable $t \in I$ and the n-dimensional Fourier transform with respect to the space variable $x \in \mathbb{R}^n$. Taking into account (8), (15), and the initial condition in (22), we obtain

$$c_1\, \mathbf{s}^\alpha\, \widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s}) - c_1\, \widehat{f}(\kappa)\, \mathbf{s}^{-\mu(1-\alpha)} + c_2^2\, |\kappa|^2\, \widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s}) + d^2\, \widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s}) = \widehat{\widetilde{q}}_\psi(\kappa,\mathbf{s}), \tag{23}$$

which is equivalent to

$$\widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s}) = \widehat{f}(\kappa)\, \frac{\mathbf{s}^{-\mu(1-\alpha)}}{\mathbf{s}^\alpha + \frac{c_2^2}{c_1}\, |\kappa|^2 + \frac{d^2}{c_1}} + \frac{1}{c_1}\, \widehat{\widetilde{q}}_\psi(\kappa,\mathbf{s})\, \frac{1}{\mathbf{s}^\alpha + \frac{c_2^2}{c_1}\, |\kappa|^2 + \frac{d^2}{c_1}}, \tag{24}$$

where $\widehat{f}$ is the Fourier transform of the function f. Inverting the ψ-Laplace transform, taking into account (11) and using (10), we have

$$\begin{aligned}\widehat{u}(\kappa,t) = {} & \widehat{f}(\kappa)\, (\psi(t))^{\alpha+\mu(1-\alpha)-1}\, E_{\alpha,\alpha+\mu(1-\alpha)}\left(-\frac{1}{c_1}\left(c_2^2\, |\kappa|^2 + d^2\right)(\psi(t))^\alpha\right) \\ & + \frac{1}{c_1}\, \widehat{q}(\kappa,\psi(t)) *_\psi \left[(\psi(t))^{\alpha-1}\, E_{\alpha,\alpha}\left(-\frac{1}{c_1}\left(c_2^2\, |\kappa|^2 + d^2\right)(\psi(t))^\alpha\right)\right],\end{aligned} \tag{25}$$

where the ψ-convolution is given by (9).

Remark 3.1 *If we consider $\psi(t) = t$ and $\mu = 1$ (Caputo fractional derivative of order α), $f(x) = \delta(x) = \prod_{j=1}^n \delta(x_j)$, $c_1 = 1$, $d = 0$, and $q(x,t) = 0$ in* (21)*, expression* (25) *reduces to*

$$\widehat{u}(\kappa,t) = E_\alpha\left(-c_2^2\, |\kappa|^2\, t^\alpha\right)$$

which coincides with the fundamental solution in the Fourier domain presented in [8] (see Formula (31)).

It follows from (25) and the well-known asymptotic expansion for the Mittag-Leffler function (see formula (4.7.3) in [12])

$$E_{\beta_1,\beta_2}(-x) = -\sum_{k=1}^{p} \frac{(-x)^{-k}}{\Gamma(\beta_2 - \beta_1 k)} + O\left(|x|^{-1-p}\right), \quad p \in \mathbb{N},\, x \to +\infty$$

where $0 < \beta_1 < 2$, $\beta_2 \in \mathbb{C}$, that the Mittag-Leffler functions in (25) belong to the space $L_1(\mathbb{R}^n)$. Therefore, applying the inverse Fourier transform and taking into account (12) and (13), we obtain

$$u(x,t) = (\psi(t))^{\alpha+\mu(1-\alpha)-1}\, f(x) *_x \mathcal{F}^{-1}\left\{E_{\alpha,\alpha+\mu(1-\alpha)}\left(-\frac{1}{c_1}\left(c_2^2|\kappa|^2+d^2\right)(\psi(t))^\alpha\right)\right\}(x,t)$$

$$+\frac{1}{c_1}\, q(x,\psi(t)) *_x *_\psi \left[(\psi(t))^{\alpha-1}\mathcal{F}^{-1}\left\{E_{\alpha,\alpha}\left(-\frac{1}{c_1}\left(c_2^2|\kappa|^2+d^2\right)(\psi(t))^\alpha\right)\right\}(x,t)\right]. \quad (26)$$

Considering the following formula presented in [26] for the inverse Fourier transform of radial functions:

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot\kappa}\,\varphi(|\kappa|)\,d\kappa = \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\varphi(w)\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|\,w)\,dw, \quad (27)$$

which is valid for any function $\varphi \in L_1(\mathbb{R}^n)$ (see Lemma 25.1 in [26]), we obtain the following result.

Theorem 3.2 *The solution of the generalized time-fractional diffusion equation with ψ-Hilfer derivative* (21) *subject to the conditions* (22) *is given, in terms of convolution integrals, by*

$$u(x,t) = (\psi(t))^{\alpha+\mu(1-\alpha)-1}\, f(x) *_x \left[\frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty} E_{\alpha,\alpha+\mu(1-\alpha)}\left(-\frac{1}{c_1}\left(c_2^2w^2+d^2\right)(\psi(t))^\alpha\right) w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|\,w)\,dw\right]$$

$$+\frac{1}{c_1}\, q(x,\psi(t)) *_x *_\psi \left[(\psi(t))^{\alpha-1}\frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty} E_{\alpha,\alpha}\left(-\frac{1}{c_1}\left(c_2^2w^2+d^2\right)(\psi(t))^\alpha\right) w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|\,w)\,dw\right], \quad (28)$$

where the convolutions $*_x$ and $*_\psi$ are defined by (14) *and* (9), *respectively.*

Remark 3.3 *If we put*

$$f(x) = \delta(x) = \prod_{i=1}^n \delta(x_i), \qquad q(x,t)=0, \qquad c_1 = c_2 = 1, \qquad d=\sqrt{\lambda}$$

with $\lambda \in \mathbb{R}^+$ in (21)-(22), *then the solution $u(x,t)$ given by* (28) *corresponds to the eigenfunctions of the time-fractional diffusion equation* (21) *subject to the conditions* (22).

For $d=0$ in (21), the inverse Fourier transforms in (28) can be calculated explicitly. To do this, we first prove the following auxiliary result.

Lemma 3.4 *Let $\beta_1, \beta_2 \in \mathbb{C}$ such that $\mathrm{Re}(\beta_1) > 0$, $\tau \in \mathbb{R}^+$, and $\kappa \in \mathbb{R}^n$. The following multidimensional Fourier-type relation is valid*

$$\mathcal{F}^{-1}\left\{E_{\beta_1,\beta_2}\left(-\tau|\kappa|^2\right)\right\}(x) = \frac{1}{\pi^{\frac{n-1}{2}}(2\pi)^n} H^{0,2}_{2,2}\left[\frac{\sqrt{\tau}}{|x|}\;\middle|\; \begin{matrix} (1-n,1),\left(0,\frac{1}{2}\right) \\ \left(\frac{1-n}{2},\frac{1}{2}\right),\left(1-\beta_2,\frac{\beta_1}{2}\right)\end{matrix}\right], \quad (29)$$

where H is the Fox H-function defined in (7).

Proof: We start the proof by noting that we want to apply the inverse Fourier transform to a radial function in the variable κ. Then by (27) we have

$$\mathcal{F}^{-1}\left\{E_{\beta_1,\beta_2}\left(-\tau|\kappa|^2\right)\right\}(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot\kappa}\,E_{\beta_1,\beta_2}\left(-\tau|\kappa|^2\right)d\kappa$$

$$= \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty} E_{\beta_1,\beta_2}\left(-\tau w^2\right) w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|\,w)\,dw. \quad (30)$$

To explicitly compute the integral on the right-hand side of (30) we will use the Mellin transform. First, we rewrite the integral as a Mellin convolution. In fact, considering

$$h_1(w) = E_{\beta_1,\beta_2}\left(-\tau w^2\right) \qquad \text{and} \qquad h_2(w) = \frac{1}{(2\pi)^{\frac{n}{2}}|x|\,w^{\frac{n}{2}-1}}\,J_{\frac{n}{2}-1}\left(\frac{1}{w}\right), \quad (31)$$

and the definition of the Mellin convolution in (18), we have

$$\begin{aligned}
\mathcal{M}\left\{h_1 *_{\mathcal{M}} h_2\right\}\left(\frac{1}{|x|}\right) &= \int_0^{+\infty} h_1(w)\, h_2\left(\frac{1}{|x|\, w}\right) \frac{dw}{w} \\
&= \int_0^{+\infty} E_{\beta_1,\beta_2}\left(-\tau w^2\right) \frac{w^{\frac{n}{2}+1}\, |x|^{\frac{n}{2}+1}}{(2\pi)^{\frac{n}{2}}\, |x|^n} J_{\frac{n}{2}-1}\left(|x|\, w\right) \frac{dw}{w} \\
&= \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^{+\infty} E_{\beta_1,\beta_2}\left(-\tau w^2\right) w^{\frac{n}{2}}\, J_{\frac{n}{2}-1}\left(|x|\, w\right)\, dw.
\end{aligned}$$

From (19) we have

$$\mathcal{M}\left\{h_1 *_{\mathcal{M}} h_2\right\}(s) = \mathcal{M}\left\{h_1\right\}(-s)\, \mathcal{M}\left\{h_2\right\}(-s)$$

which is equivalent to

$$\mathcal{M}\left\{h_1 *_{\mathcal{M}} h_2\right\}(-s) = \mathcal{M}\left\{h_1\right\}(s)\, \mathcal{M}\left\{h_2\right\}(s). \tag{32}$$

Let us calculate the Mellin transforms that appear in (32). Using Formula (43) in [31], the Mellin transform of h_2 is given by:

$$\mathcal{M}\left\{h_2\right\}(s) = \frac{1}{\pi^{\frac{n-1}{2}}\, |x|^n\, 2^{n-1}} \frac{\Gamma(n-s)}{\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right) \Gamma\left(\frac{s}{2}\right)}. \tag{33}$$

Concerning the Mellin transform of h_1, we have by definition (see (16))

$$\mathcal{M}\left\{h_1\right\}(s) = \int_0^{+\infty} w^{s-1}\, E_{\beta_1,\beta_2}\left(-\tau w^2\right) dw. \tag{34}$$

Then, considering the change of variables $\tau w^2 = z$ in (34) we obtain

$$\mathcal{M}\left\{h_1\right\}(s) = \frac{1}{2\tau^{\frac{s}{2}}} \int_0^{+\infty} z^{\frac{s}{2}-1}\, E_{\beta_1,\beta_2}(-z) = \frac{1}{2\tau^{\frac{s}{2}}} \mathcal{M}\left\{E_{\beta_1,\beta_2}(-z)\right\}\left(\frac{s}{2}\right). \tag{35}$$

Taking into account (20), we have from (35)

$$\mathcal{M}\left\{h_1\right\}(s) = \frac{1}{2\tau^{\frac{s}{2}}} \frac{\Gamma\left(\frac{s}{2}\right) \Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\beta_2 - \frac{\beta_1 s}{2}\right)}. \tag{36}$$

Now, using the inverse Mellin transform defined in (17) applied to (32), together with (33) and (36), and the representation of the Fox H-function presented in (7), we finally get

$$\begin{aligned}
\mathcal{F}^{-1}\left\{E_{\beta_1,\beta_2}\left(-\tau\, |\kappa|^2\right)\right\}(x) &= \frac{1}{\pi^{\frac{n-1}{2}}\, (2\,|x|)^n} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(n-s)\, \Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right) \Gamma\left(\beta_2 - \frac{\beta_1 s}{2}\right)} \left(\frac{\sqrt{\tau}}{|x|}\right)^{-s} ds \qquad (37) \\
&= \frac{1}{\pi^{\frac{n-1}{2}}\, (2\,|x|)^n} H_{2,2}^{0,2}\left[\frac{\sqrt{\tau}}{|x|} \,\middle|\, \begin{array}{c} (1-n, 1), \left(0, \frac{1}{2}\right) \\ \left(\frac{1-n}{2}, \frac{1}{2}\right), \left(1-\beta_2, \frac{\beta_1}{2}\right) \end{array}\right],
\end{aligned}$$

which corresponds to our result.

■

By the previous lemma we can rewrite Theorem 3.2 in the case $d = 0$ using convolution integrals.

Theorem 3.5 *The solution of the generalized time-fractional diffusion equation with ψ-Hilfer derivative* (21) *with $d = 0$, subject to the conditions* (22) *is given, in terms of convolution integrals involving Fox H-functions, by*

$$u(x,t) = \int_{\mathbb{R}^n} f(z)\, G_1(x-z,t)\, dz + \int_{\mathbb{R}^n} \int_0^t q(z, \psi(t))\, G_2\left(x-z, \psi^{-1}\left(\psi(t)-\psi(w)\right)\right) \psi'(w)\, dw\, dz, \tag{38}$$

and the functions G_1 and G_2 are given by

$$G_1(x,t) = \frac{(\psi(t))^{\alpha+\mu(1-\alpha)-1}}{\pi^{\frac{n-1}{2}}(2|x|)^n} H^{0,2}_{2,2}\left[\frac{c_2(\psi(t))^{\frac{\alpha}{2}}}{\sqrt{c_1}|x|} \;\middle|\; \begin{matrix} (1-n,1),\left(0,\frac{1}{2}\right) \\ \left(\frac{1-n}{2},\frac{1}{2}\right),\left(1-\alpha-\mu(1-\alpha),\frac{\alpha}{2}\right) \end{matrix}\right],$$

$$G_2(x,t) = \frac{(\psi(t))^{\alpha-1}}{c_1\,\pi^{\frac{n-1}{2}}(2|x|)^n} H^{0,2}_{2,2}\left[\frac{c_2(\psi(t))^{\frac{\alpha}{2}}}{\sqrt{c_1}|x|} \;\middle|\; \begin{matrix} (1-n,1),\left(0,\frac{1}{2}\right) \\ \left(\frac{1-n}{2},\frac{1}{2}\right),\left(1-\alpha,\frac{\alpha}{2}\right) \end{matrix}\right].$$

Remark 3.6 *If we consider in* (21)-(22)

$$f(x) = \delta(x) = \prod_{i=1}^{n}\delta(x_i), \qquad q(x,t)=0, \qquad \mu = c_1 = c_2 = 1, \qquad d=0, \qquad \psi(t)=t, \qquad I=\mathbb{R}^+$$

our initial problem reduces to

$$\begin{cases} \left({}^C\partial^{\alpha}_{t,a^+} - c_2^2\,\Delta_x\right)u(x,t) = 0 \\ u(x,0) = \delta(x), \end{cases} \tag{39}$$

where ${}^C\partial^{\alpha}_{t,0^+}$ denotes the time-fractional derivative of order α in the Caputo sense. The solution $u(x,t)$ of (39) *corresponds to the first fundamental solution and from Theorem 3.5 it is given by*

$$\begin{aligned} u(x,t) &= \frac{1}{\pi^{\frac{n-1}{2}}(2|x|)^n} H^{0,2}_{2,2}\left[\frac{c_2 t^{\frac{\alpha}{2}}}{|x|} \;\middle|\; \begin{matrix} (1-n,1),\left(0,\frac{1}{2}\right) \\ \left(\frac{1-n}{2},\frac{1}{2}\right),\left(0,\frac{\alpha}{2}\right) \end{matrix}\right] \\ &= \frac{1}{\pi^{\frac{n-1}{2}}(2|x|)^n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(n-s)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\Gamma\left(1-\frac{\alpha s}{2}\right)}\left(\frac{c_2 t^{\frac{\alpha}{2}}}{|x|}\right)^{-s} ds. \end{aligned} \tag{40}$$

From the duplication formula for the Gamma function we have that

$$\frac{\Gamma(n-s)}{\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)} = \frac{2^{n-s-1}}{\sqrt{\pi}}\,\Gamma\left(\frac{n-s}{2}\right) \tag{41}$$

and, hence, expression (40) *becomes*

$$u(x,t) = \frac{1}{2\pi^{\frac{n}{2}}|x|^n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(1-\frac{\alpha s}{2}\right)}\left(\frac{2c_2 t^{\frac{\alpha}{2}}}{|x|}\right)^{-s} ds$$

which coincides with Formula (38) in [8], and shows consistency with the results obtained previously.

4 Fractional Moments

In this section, we calculate the fractional moments of order $\gamma > 0$ associated with the first fundamental solution G_1 given in Theorem 3.5:

$$G_1(x,t) = \frac{(\psi(t))^{\alpha+\mu(1-\alpha)-1}}{\pi^{\frac{n-1}{2}}(2|x|)^n} H^{0,2}_{2,2}\left[\frac{c_2(\psi(t))^{\frac{\alpha}{2}}}{\sqrt{c_1}|x|} \;\middle|\; \begin{matrix} (1-n,1),\left(0,\frac{1}{2}\right) \\ \left(\frac{1-n}{2},\frac{1}{2}\right),\left(1-\alpha-\mu(1-\alpha),\frac{\alpha}{2}\right) \end{matrix}\right]. \tag{42}$$

It is known that the Mellin transform (16) can be interpreted as the fractional moment of order $s-1$ of the function f. From the definition of the Mellin transform we have

$$\begin{aligned}\mathbf{M}_n^{\alpha,\mu;\psi;\gamma}(t) &= \int_0^{+\infty} r^\gamma\, G_1(r,t)\ dr \\ &= \int_0^{+\infty} r^{\gamma-n+1-1}\, r^n\, G_1(r,t)\ dr \\ &= \mathcal{M}\left\{r^n\, G_1(r,t)\right\}(\gamma-n+1).\end{aligned}$$

Taking into account (32) and the integral representation (37) we obtain:

$$\begin{aligned}\mathcal{M}\left\{r^n\, G_1(r,t)\right\}(\gamma-n+1) &= \frac{1}{2^n\pi^{\frac{n-1}{2}}}\left(\frac{c_2^2\,(\psi(t))^{\frac{\alpha}{2}}}{c_1}\right)^{\frac{s}{2}} \left.\frac{\Gamma\left(1+\frac{s}{2}\right)\Gamma(n+s)}{\Gamma\left(\frac{n+1}{2}+\frac{s}{2}\right)\Gamma\left(\alpha+\mu(1-\alpha)+\frac{\alpha s}{2}\right)}\right|_{s=\gamma-n+1} \\ &= \frac{1}{2^n\pi^{\frac{n-1}{2}}}\left(\frac{c_2^2\,(\psi(t))^{\frac{\alpha}{2}}}{c_1}\right)^{\frac{\gamma-n+1}{2}} \frac{\Gamma\left(\frac{3+\gamma-n}{2}\right)\Gamma(1+\gamma)}{\Gamma\left(\frac{\gamma+2}{2}\right)\Gamma\left(\alpha+\mu(1-\alpha)+\frac{\alpha(\gamma-n+1)}{2}\right)}.\end{aligned}$$

Therefore, the fractional moments are given by

$$\mathbf{M}_n^{\alpha,\mu;\psi;\gamma}(t) = \frac{1}{2^n\pi^{\frac{n-1}{2}}}\left(\frac{c_2^2\,(\psi(t))^{\frac{\alpha}{2}}}{c_1}\right)^{\frac{\gamma-n+1}{2}} \frac{\Gamma\left(\frac{3+\gamma-n}{2}\right)\Gamma(1+\gamma)}{\Gamma\left(\frac{\gamma+2}{2}\right)\Gamma\left(\alpha+\mu(1-\alpha)+\frac{\alpha(\gamma-n+1)}{2}\right)}. \tag{43}$$

When $\gamma=1,2$ expression (43) reduces to:

- *Mean Value* ($\gamma=1$)

$$\mathbf{M}_n^{\alpha,\mu;\psi;1}(t) = \frac{1}{2^{n-1}\pi^{\frac{n}{2}}}\left(\frac{c_2^2\,(\psi(t))^{\alpha}}{c_1}\right)^{1-\frac{n}{2}} \frac{\Gamma\left(2-\frac{n}{2}\right)}{\Gamma\left(\alpha+\mu(1-\alpha)+\frac{\alpha(2-n)}{2}\right)}.$$

- *Variance* ($\gamma=2$)

$$\mathbf{M}_n^{\alpha,\mu;\psi;2}(t) = \frac{1}{2^{n-1}\pi^{\frac{n-1}{2}}}\left(\frac{c_2^2\,(\psi(t))^{\alpha}}{c_1}\right)^{\frac{3-n}{2}} \frac{\Gamma\left(\frac{5-n}{2}\right)}{\Gamma\left(\alpha+\mu(1-\alpha)+\frac{\alpha(3-n)}{2}\right)}.$$

  Analysing this expression, we can infer that for large values of t, and whenever $\mathbf{M}_n^{\alpha,\mu;\psi;2}(t) < \mathbf{M}_n^{1,\mu;\psi;2}(t)$ for $n=1$ and $n=2$, the variance decays slowly when compared to the diffusion case, thus corresponding to a slow diffusion process. Furthermore, if $\mathbf{M}_n^{\alpha,\mu;\psi;2}(t) > \mathbf{M}_n^{1,\mu;\psi;2}(t)$ for $n=1$ and $n=2$ then we are dealing with a fast diffusion process. These two cases of diffusion processes appearing in the case of $0<\alpha<1$ are only possible due to the freedom of choice of the function ψ.

From (43) we have two special cases:

- When $\gamma=n-1$, with $n\in\mathbb{N}$, the moment is independent of the time variable, for all $\alpha\in]0,1]$ and $\mu\in[0,1]$, and is given by

$$\mathbf{M}_n^{\alpha,\mu;\psi;n-1}(t) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}.$$

- When $\gamma=n-2k-3$, with $n>2k+3$ and $k\in\mathbb{N}_0$, the moment becomes infinite, for all $\alpha\in]0,1[$ and $\mu\in[0,1]$. For example, when $k=0$ and $n=4$ the moment $\mathbf{M}_4^{\alpha,\mu;\psi;n-2k-3}$ is infinite for all $\alpha\in]0,1[$ and $\mu\in[0,1]$.

We would like to remark that in the one dimensional case ($n=1$) the moments were calculated only for positive values of x. The true moments of the fundamental solution should be calculated over the whole real line. However, in this case it is not possible to compute the fractional moments for every order $\gamma>0$ since the power function r^{γ} is not always well defined. Nevertheless, two cases are of special importante: the odd integer moments ($\gamma = 2k+1, k \in \mathbb{N}$) vanish, while the even integer moments are given by

$$\mathbf{M}_n^{\alpha,\mu;\psi;2k}(t) = \int_{-\infty}^{+\infty} r^{2k}\, G_1(r,t)\; dr \;= 2\int_0^{+\infty} r^{2k}\, G_1(r,t)\; dr$$
$$= \left(\frac{c_2^2\,(\psi(t))^{\alpha}}{c_1}\right)^k \frac{\Gamma(1+2k)}{\Gamma((1+k)\,\alpha + (1-\alpha)\,\mu)}.$$

Remark 4.1 *If we restrict the results presented in this section to the Caputo case, i.e., if we consider $\psi(t) = t$ and $\mu = 1$, all results reduce to the correspondent ones obtained in Section 6 of [8].*

5 Series representation of the first fundamental solution

In this section we deduce the series representation for the first fundamental solution associated with (21), which has a representation in terms of Fox H-function given by (42). Taking into account (37), expression (42) can be rewritten in terms of Mellin-Barnes integrals as follows

$$G_1(x,t) = \frac{(\psi(t))^{\alpha+\mu(1-\alpha)-1}}{\pi^{\frac{n-1}{2}}\,(2\,|x|)^n}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(n-s)\,\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\Gamma\left(\alpha+\mu(1-\alpha)-\frac{\alpha s}{2}\right)}\left(\frac{c_2\,(\psi(t))^{\frac{\alpha}{2}}}{\sqrt{c_1}\,|x|}\right)^{-s} ds.$$

Using (41) the previous expression becomes

$$G_1(x,t) = \frac{(\psi(t))^{\alpha+\mu(1-\alpha)-1}}{2\pi^{\frac{n}{2}}\,|x|^n}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\alpha+\mu(1-\alpha)-\frac{\alpha s}{2}\right)}\left(\frac{2c_2\,(\psi(t))^{\frac{\alpha}{2}}}{\sqrt{c_1}\,|x|}\right)^{-s} ds. \tag{44}$$

To explicitly evaluate the integral (44) we need to apply the Residue Theorem. Applying general convergence conditions of Mellin-Barnes integrals to (44) (see [17]), we conclude that for $0 < \alpha \leq 1$ and $0 \leq \mu \leq 1$ the integral is convergent and the contour of integration must be transformed to the loop $\mathcal{L}_{+\infty}$ starting and ending at infinity and encircling all the poles of the Gamma functions $\Gamma\left(1-\frac{s}{2}\right)$ and $\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)$. As the Gamma function $\Gamma(s)$ has simple poles at $s=-n$, with $n \in \mathbb{N}_0$, we have that $\Gamma\left(1-\frac{s}{2}\right)$ has poles at $s = 2k+2$, with $k \in \mathbb{N}_0$, and $\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)$ has poles at $s = 2k+n$, with $k \in \mathbb{N}_0$. From this it is evident that we have different types of poles according the parity of the dimension n.

5.1 The case of odd dimension

When n is odd, we have two non-coincident sequences of simple poles, $s = 2k+2$, $k \in \mathbb{N}_0$, and $s = 2k+n$, $k \in \mathbb{N}_0$. Applying the Residue Theorem for each Gamma function and taking into account (3), we get the following series representation for G_1

$$G_1(x,t) = \frac{(\psi(t))^{\alpha+\mu(1-\alpha)-1}}{\pi^{\frac{n}{2}}\;|x|^n}\left[\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\,\frac{\Gamma\left(-1-k+\frac{n}{2}\right)}{\Gamma(\alpha+\mu(1-\alpha)-\alpha(k+1))}\left(\frac{2c_2\,(\psi(t))^{\frac{\alpha}{2}}}{\sqrt{c_1}\,|x|}\right)^{-2k-2}\right.$$
$$\left.+\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\,\frac{\Gamma\left(1-k-\frac{n}{2}\right)}{\Gamma\left(\alpha+\mu(1-\alpha)-\alpha\left(k+\frac{n}{2}\right)\right)}\left(\frac{2c_2\,(\psi(t))^{\frac{\alpha}{2}}}{\sqrt{c_1}\,|x|}\right)^{-2k-n}\right].$$

Following [8], both series can be combined into a single series. To obtain this, we start by considering the change of variables $m = 2k+1$ and $m = 2k$ in the first and second series, respectively. Hence, we get

$$G_1(x,t) = \frac{(\psi(t))^{\alpha+\mu(1-\alpha)-1}}{\pi^{\frac{n}{2}}\;|x|^n}\left[\sum_{\substack{m=1\\ m\ \text{odd}}}^{+\infty}\frac{(-1)^{\frac{m-1}{2}}}{\left(\frac{m-1}{2}\right)!}\,\frac{\Gamma\left(-1-\frac{m-1}{2}+\frac{n}{2}\right)}{\Gamma\left(\alpha+\mu(1-\alpha)-\alpha\left(1+\frac{m-1}{2}\right)\right)}\left(\frac{\sqrt{c_1}\,|x|}{2c_2\,(\psi(t))^{\frac{\alpha}{2}}}\right)^{m+1}\right.$$
$$\left.+\sum_{\substack{m=0\\ m\ \text{even}}}^{+\infty}\frac{(-1)^{\frac{m}{2}}}{\left(\frac{m}{2}\right)!}\,\frac{\Gamma\left(1-\frac{m}{2}-\frac{n}{2}\right)}{\Gamma\left(\alpha+\mu(1-\alpha)-\frac{\alpha(m+n)}{2}\right)}\left(\frac{\sqrt{c_1}\,|x|}{2c_2\,(\psi(t))^{\frac{\alpha}{2}}}\right)^{m+n}\right].$$

To have the same exponent in both series we consider in the first series the change of the order of summation $m = p + n - 1$ and in the second series $m = p$, resulting in

$$G_1(x,t) = \frac{(\psi(t))^{\alpha+\mu(1-\alpha)-1}}{\pi^{\frac{n}{2}}\ |x|^n}\left[\sum_{\substack{p=2-n\\ p \text{ odd}}}^{-1} \frac{(-1)^{\frac{p+n-2}{2}}}{\Gamma\left(\frac{p+n}{2}\right)} \frac{\Gamma\left(-\frac{p}{2}\right)}{\Gamma\left(\alpha+\mu(1-\alpha)-\frac{\alpha(p+n)}{2}\right)}\left(\frac{\sqrt{c_1}\,|x|}{2c_2(\psi(t))^{\frac{\alpha}{2}}}\right)^{p+n}\right.$$

$$+\sum_{\substack{p=1\\ p \text{ odd}}}^{+\infty} \frac{(-1)^{\frac{p+n-2}{2}}}{\Gamma\left(\frac{p+n}{2}\right)} \frac{\Gamma\left(-\frac{p}{2}\right)}{\Gamma\left(\alpha+\mu(1-\alpha)-\frac{\alpha(p+n)}{2}\right)}\left(\frac{\sqrt{c_1}\,|x|}{2c_2(\psi(t))^{\frac{\alpha}{2}}}\right)^{p+n}$$

$$\left.+\sum_{\substack{p=0\\ p \text{ even}}}^{+\infty} \frac{(-1)^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}+1\right)} \frac{\Gamma\left(1-\frac{p}{2}-\frac{n}{2}\right)}{\Gamma\left(\alpha+\mu(1-\alpha)-\frac{\alpha(p+n)}{2}\right)}\left(\frac{\sqrt{c_1}\,|x|}{2c_2(\psi(t))^{\frac{\alpha}{2}}}\right)^{p+n}\right]. \tag{45}$$

Now, we analyse the coefficients of the odd and even series. For odd p, using properties of the Gamma function, we have

$$\Gamma\left(\frac{p+n}{2}\right) = \Gamma\left(\frac{p+1}{2}+\frac{n-1}{2}\right) = \left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} \Gamma\left(\frac{p}{2}+\frac{1}{2}\right) = \frac{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} 2^{1-p}\sqrt{\pi}\ \Gamma(p)}{\Gamma\left(\frac{p}{2}\right)}. \tag{46}$$

Using (46) and Euler's reflection formula we get

$$\frac{(-1)^{\frac{p+n-2}{2}}\ \Gamma\left(-\frac{p}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)} = \frac{(-1)^{\frac{p+n-2}{2}}\ \Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{p}{2}\right)}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} 2^{1-p}\sqrt{\pi}\ \Gamma(p)} = -\frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}\ 2^p}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}}\ p!}. \tag{47}$$

On the other hand, for p even, using Legendre's duplication formula an the following identities for the Gamma function

$$\Gamma(z-n) = \frac{(-1)^n\ \Gamma(z)}{(1-z)_n}, \quad n \in \mathbb{N}_0$$

and

$$\Gamma\left(\frac{1}{2}-z\right)\Gamma\left(\frac{1}{2}+z\right) = \frac{\pi}{\cos(\pi z)} \tag{48}$$

we obtain

$$\Gamma\left(1-\frac{p}{2}-\frac{n}{2}\right) = \Gamma\left(\frac{1}{2}-\frac{p}{2}-\frac{n-1}{2}\right) = \frac{(-1)^{\frac{n-1}{2}}\ \Gamma\left(\frac{1}{2}-\frac{p}{2}\right)}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}}}$$

$$= \frac{(-1)^{\frac{n-1}{2}}\ \pi}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} \cos\left(\frac{p\pi}{2}\right)\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)} = \frac{(-1)^{\frac{n-p-1}{2}}\sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right)}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}}\ 2^{1-p}\ \Gamma(p)}. \tag{49}$$

Hence, from (49) we get

$$\frac{(-1)^{\frac{p}{2}}\ \Gamma\left(1-\frac{p}{2}-\frac{n}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)} = \frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}\ 2^p}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}}\ p!}. \tag{50}$$

From (47) and (50), we conclude that the coefficients of the series in (45) are equal up to a minus sign in the odd series which can be included as $(-1)^p$ for odd and even p. Thus, adding the odd and the even series and considering the change $p = 2k + 2 - n$ in the finite sum, we obtain the simplified series representation of G_1 given by

$$G_1(x,t) = \frac{(\psi(t))^{\mu(1-\alpha)-1}}{4c_2^2\,\pi^{\frac{n}{2}}\ |x|^{n-2}} \sum_{k=0}^{\frac{n-3}{2}} \frac{\Gamma\left(-1-k+\frac{n}{2}\right)}{\Gamma(\alpha+\mu(1-\alpha)-\alpha(k+1))\ k!}\left(-\frac{c_1\,|x|^2}{4c_2^2(\psi(t))^{\alpha}}\right)^k$$

$$+\frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}\ (\psi(t))^{\left(1-\frac{n}{2}\right)\alpha+\mu(1-\alpha)-1}}{(4\pi\,c_2^2)^{\frac{n}{2}}} \sum_{p=0}^{+\infty} \frac{1}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}}\Gamma\left(\alpha+\mu(1-\alpha)-\frac{\alpha(p+n)}{2}\right)\ p!}\left(-\frac{\sqrt{c_1}\,|x|}{c_2(\psi(t))^{\frac{\alpha}{2}}}\right)^p. \tag{51}$$

5.2 The case of even dimension

When n is even, we have a finite sequence of simple poles coming from $\Gamma\left(1-\frac{s}{2}\right)$ at the points $s=2k+2$, for $k=0,1,\ldots,\frac{n}{2}-2$, and an infinite sequence of double poles coming from $\Gamma\left(\frac{n-s}{2}\right)$ at the points $s=2k+2$, for $k\geq\frac{n}{2}-1$. Applying the Residue Theorem we get the following series representation for G_1 :

$$\begin{aligned}
G_1(x,t) =& \frac{(\psi(t))^{\mu(1-\alpha)-1}}{4\,c_2^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}} \sum_{k=0}^{\frac{n}{2}-2} \frac{\Gamma\left(\frac{n}{2}-k-1\right)}{\Gamma\left(\alpha+\mu(1-\alpha)-\alpha(k+1)\right)\ k!} \left(-\frac{c_1\,|x|^2}{4c_2^2\,(\psi(t))^{\alpha}}\right)^k \\
&+ \frac{(-1)^{\frac{n}{2}+1}\,(\psi(t))^{\left(1-\frac{n}{2}\right)\alpha+\mu(1-\alpha)-1}}{(4\pi\,c_2^2)^{\frac{n}{2}}} \\
&\times \sum_{k=0}^{+\infty} \frac{\Psi\left(k+\frac{n}{2}\right)-\alpha\,\Psi\left(\alpha+\mu(1-\alpha)-\alpha\left(k+\frac{n}{2}\right)\right)+\Psi(k+1)+\ln\left(\frac{4c_2^2(\psi(t))^{\alpha}}{c_1|x|^2}\right)}{\Gamma\left(k+\frac{n}{2}\right)\ \Gamma\left(\alpha+\mu(1-\alpha)-\alpha\left(k+\frac{n}{2}\right)\right)\ k!} \left(\frac{c_1\,|x|^2}{4c_2^2\,(\psi(t))^{\alpha}}\right)^k,
\end{aligned} \tag{52}$$

where $\Psi(z)$ denotes the digamma function.

Remark 5.1 *When $\mu=1$ and $\alpha\left(k+\frac{n}{2}\right)\in\mathbb{N}$ we have an indetermination in the series coefficients of* (52) *due to the terms $\alpha\,\Psi\left(\alpha+\mu(1-\alpha)-\alpha\left(k+\frac{n}{2}\right)\right)$ and $\Gamma\left(\alpha+\mu(1-\alpha)-\alpha\left(k+\frac{n}{2}\right)\right)$. In these cases, it is possible to remove the indetermination by applying properties of the Gamma function and the relation $\Psi(1-z)=\pi\cot(\pi z)+\Psi(z)$ for the digamma function (see [1]). An application of this technique can be found in [8] for the particular case of $\psi(t)=t$, and it can be adapted in a straightforward way here.*

From the series representations obtained in the odd and even dimensions it is easy to see that the fundamental solution is finite at the point $x=(0,\ldots,0)$ only in the one-dimensional case, being infinite at this for all $n\geq 2$.

5.3 The one dimensional case and its probabilistic interpretation

The aim of this subsection is to prove that in the one-dimensional case the first fundamental solution associated with (21) corresponds to a probability density function for $0<\alpha\leq 1$ and $0\leq\mu<1$. The case $\mu=1$ was studied in [24] for $\psi(t)=t$, however, the arguments can be adapted in a straightforward way to our setting. Our results are based on [13] and extend their results to a more general Wright function. Considering $n=1$ in (51) and taking into account (4), we have

$$\begin{aligned}
G_1(x,t) &= \frac{(\psi(t))^{\frac{\alpha}{2}+\mu(1-\alpha)-1}}{2c_2} \sum_{p=0}^{+\infty} \frac{1}{\Gamma\left(\alpha+\mu(1-\alpha)-\frac{\alpha(p+1)}{2}\right)\ p!} \left(-\frac{\sqrt{c_1}\,|x|}{c_2\,(\psi(t))^{\frac{\alpha}{2}}}\right)^p \\
&= \frac{1}{2c_2}\,(\psi(t))^{\frac{\alpha}{2}+\mu(1-\alpha)-1}\,W_{-\frac{\alpha}{2},\frac{\alpha}{2}+\mu(1-\alpha)}\left(-\frac{\sqrt{c_1}\,|x|}{c_2\,(\psi(t))^{\frac{\alpha}{2}}}\right).
\end{aligned} \tag{53}$$

We restrict our calculations to the case where $x\in\mathbb{R}^+$. As the first parameter of the Wright function in (53) fulfills $-\frac{\alpha}{2}>-1$ then the function $G_1(x,t)$ is an entire function. Considering the new positive variable $r=|x|/(\psi(t))^{\frac{\alpha}{2}}$ and the positive constant $c=\sqrt{c_1}/c_2$, we can reduce our study to the Wright function with argument $-\frac{r}{c}$. Our goal now is to show that we have a completely monotone function in $\mathbb{R}^-$, with exponential decay when $\frac{r}{c}$ tends to infinity. We start computing some Laplace transform pairs. Using (5) we have

$$\mathcal{L}\left\{W_{-\frac{\alpha}{2},\frac{\alpha}{2}+\mu(1-\alpha)}\left(-\frac{r}{c}\right)\right\}(\mathbf{s}) = c\,E_{\frac{\alpha}{2},\alpha+\mu(1-\alpha)}(-c\,\mathbf{s}). \tag{54}$$

Moreover, using the definition of the Wright function (see (4)) it can be easily verify that

$$\mathcal{L}\left\{r^{\frac{\alpha}{2}+\mu(1-\alpha)-1}\,W_{-\frac{\alpha}{2},\frac{\alpha}{2}+\mu(1-\alpha)}\left(-c\,r^{-\frac{\alpha}{2}}\right)\right\}(\mathbf{s}) = \mathbf{s}^{-\frac{\alpha}{2}-\mu(1-\alpha)}\,e^{-c\,\mathbf{s}^{-\frac{\alpha}{2}}}. \tag{55}$$

Taking into account that $0<\alpha\leq 1$ and $0\leq\mu<1$ we have that $0<\frac{\alpha}{2}\leq\frac{1}{2}$ and $\alpha+\mu(1-\alpha)>\frac{\alpha}{2}$. Therefore, the Mittag-Leffler function that appears in (54) is completely monotone on the negative real axis (see [25]). This fact combined with the Bernstein Theorem allow us to conclude that the Wright function in (54) is a positive L_1 function. Moreover, from Theorem 2.3 we have the asymptotic behaviour

$$W_{-\frac{\alpha}{2},\frac{\alpha}{2}+\mu(1-\alpha)}\left(-cr\right) \sim A_0\, Y^{\frac{1-\alpha}{2}-\mu(1-\alpha)}\, e^{-Y}, \qquad \text{as } r \to +\infty, \tag{56}$$

where A_0 is defined by (6) and Y is given by

$$Y = (1-\alpha)\,\alpha^{\frac{\alpha}{2-\alpha}} \left(\frac{cr}{2}\right)^{\frac{2}{2-\alpha}}.$$

By (56) we can interpret the Wright function that appears in (53) as a generalization of the Gaussian and Airy functions. In addition, we have the following graphical representations of $W_{-\frac{\alpha}{2},\frac{\alpha}{2}+\mu(1-\alpha)}\left(-r\right)$ for $r \in [0,4]$, $c = 1$, and some particular values of α and μ

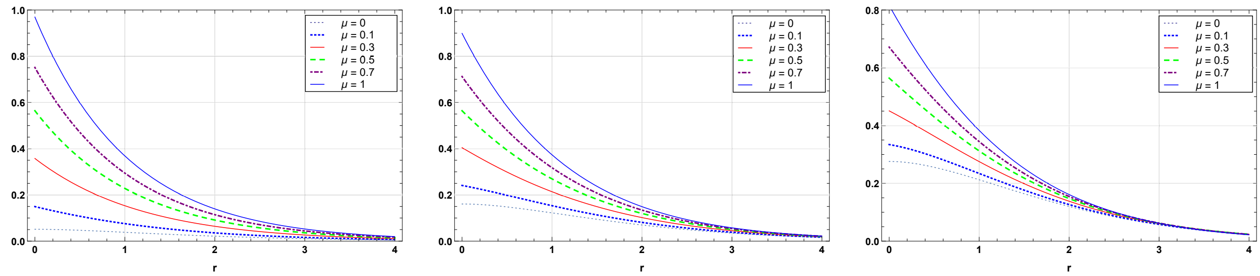

Figure 1: Plots of $W_{-\frac{\alpha}{2},\frac{\alpha}{2}+\mu(1-\alpha)}\left(-r\right)$ for $r \in [0,4]$, and $\alpha = 0.1, 0.3, 0.5$ (from left).

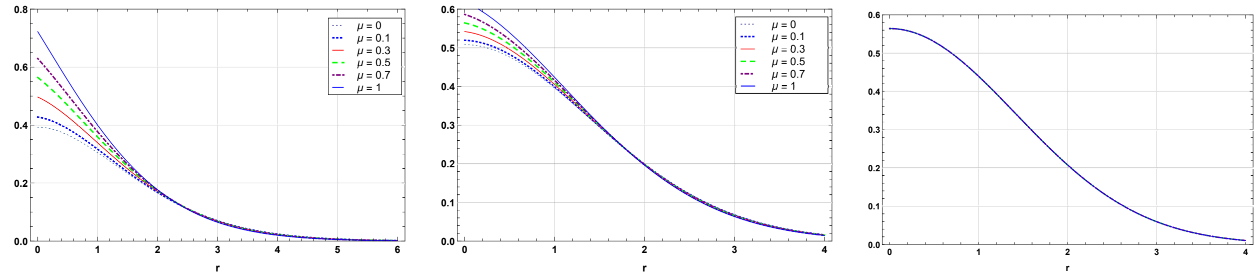

Figure 2: Plots of $W_{-\frac{\alpha}{2},\frac{\alpha}{2}+\mu(1-\alpha)}\left(-r\right)$ for $r \in [0,4]$, and $\alpha = 0.7, 0.9, 1$ (from left).

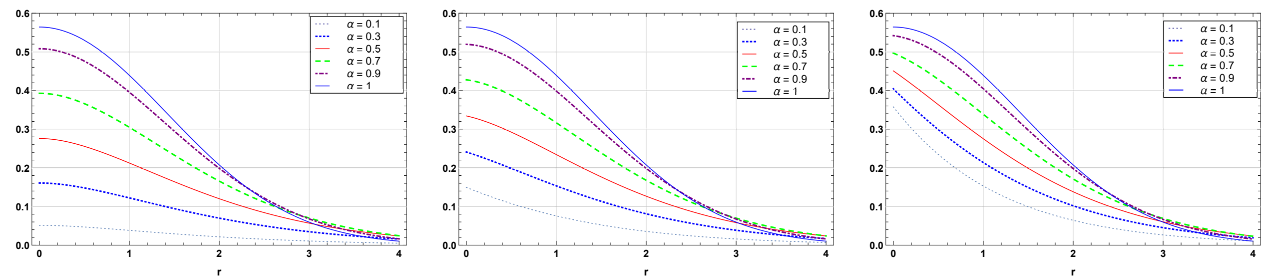

Figure 3: Plots of $W_{-\frac{\alpha}{2},\frac{\alpha}{2}+\mu(1-\alpha)}\left(-r\right)$ for $r \in [0,4]$, and $\mu = 0, 0.1, 0.3$ (from left).

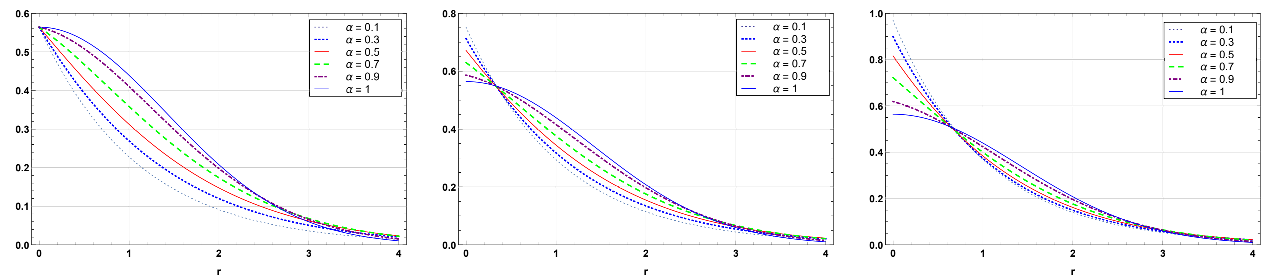

Figure 4: Plots of $W_{-\frac{\alpha}{2},\frac{\alpha}{2}+\mu(1-\alpha)}\left(-r\right)$ for $r \in [0,4]$, and $\mu = 0.5, 0.7, 1$ (from left).

The plots show that the Wright function in (53) is indeed positive and monotonically decreasing. Moreover, we observe a rapid decay, which is in accordance with (56). This conclusion combined with the L_1-integrability of $G_1(x,t)$, and expression (55), allow us to interpret $G_1(x,t)$ as a symmetric spatial probability density function evolving in time, with elongated exponential decay. Moreover, from (56) we have that

$$G_1(x,t) \sim \frac{A_0}{2c_2}(\psi(t))^{\frac{\alpha}{2}+\mu(1-\alpha)-1}\, Y^{\frac{1-\alpha}{2}-\mu(1-\alpha)}\, e^{-Y}, \quad \frac{|x|}{(\psi(t))^{\frac{\alpha}{2}}} \to +\infty,$$

where

$$Y = (1-\alpha)\,\alpha^{\frac{\alpha}{2-\alpha}} \left(\frac{\sqrt{c_1}\,|x|}{2c_2\,(\psi(t))^{\frac{\alpha}{2}}} \right)^{\frac{2}{2-\alpha}},$$

and A_0 is defined by the asymptotic expansion

$$\frac{\Gamma\left(1-\frac{\alpha}{2}-\mu(1-\alpha)+\frac{\alpha}{2}\psi(t)\right)}{2\pi\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2}\psi(t)}\left(1-\frac{\alpha}{2}\right)^{\left(1-\frac{\alpha}{2}\right)(1+\psi(t))}\Gamma(1+\psi(t))} = \sum_{m=0}^{M-1} \frac{(-1)^m A_m}{\Gamma\left(\left(1-\frac{\alpha}{2}\right)\psi(t)+\frac{\alpha}{2}+\mu(1-\alpha)+\frac{1}{2}+m\right)}$$
$$+\,O\left(\frac{1}{\Gamma\left(\left(1-\frac{\alpha}{2}\right)\psi(t)+\frac{\alpha}{2}+\mu(1-\alpha)+\frac{1}{2}+M\right)}\right)$$

valid for $\arg(\psi(t))$, $\arg\left(\frac{\alpha}{2}\psi(t)\right)$, and $\arg\left(1-\frac{\alpha}{2}+\mu(1-\alpha)+\frac{\alpha}{2}\psi(t)\right)$ all lying between $-\pi$ and π and $\psi(t)$ tending to infinity as $t \to +\infty$.

Remark 5.2 *When $\mu = 1$ (case of Caputo's fractional derivatives) the Wright function in* (53) *reduces to the so-called M-function (usually denoted by $M_\alpha(z)$ and studied in [23] in the context of fractional relaxation-oscillation and fractional diffusion-wave phenomena). The plots shown in Figure 4 for $\mu = 1$ agree with the plots that appear in Figures 4(a) and 4(b) in [23].*

6 Graphical representations

In this section, we present and discuss some plots of $G_1(x,t)$ (see (53)) for $n = 1$, $c_1 = c_2 = 1$, some values of the fractional parameters α and μ, and particular choices of the function ψ. The plots were generated using *Mathematica* software and the commands available in it.

6.1 ψ-Riemann-Liouville fractional derivatives

In the next subsections, we deal with time-fractional derivatives of Riemann-Liouville type, i.e., $\mu = 0$.

6.1.1 Riemann-Liouville fractional derivative

Here, we consider $\psi(t) = t$, $t \in \mathbb{R}^+$ in (21), which corresponds to the classical case of the Riemann-Liouville fractional derivative of order α (see [29, Sec. 5]). In the following Figures we present graphical representations of $G_1(x,t)$ for $x \in [-4,4]$, $t \in]0,4]$, and $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9, 1$.

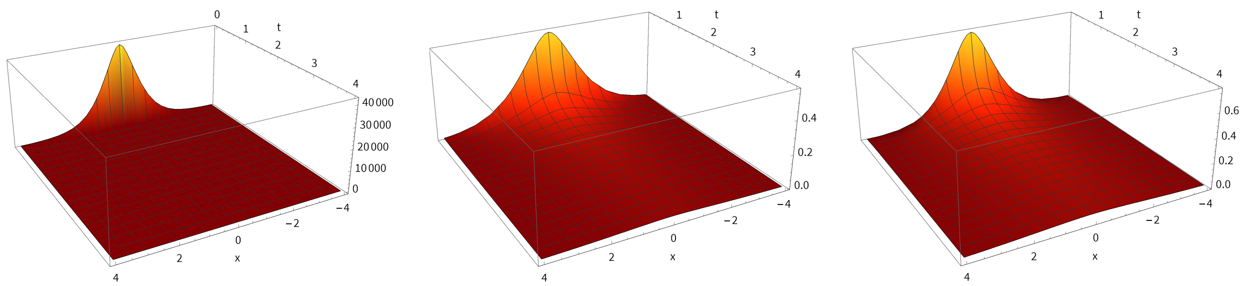

Figure 5: Plots of $G_1(x,t)$ for $\psi(t) = t$, $\mu = 0$, and $\alpha = 0.1, 0.3, 0.5$ (from left).

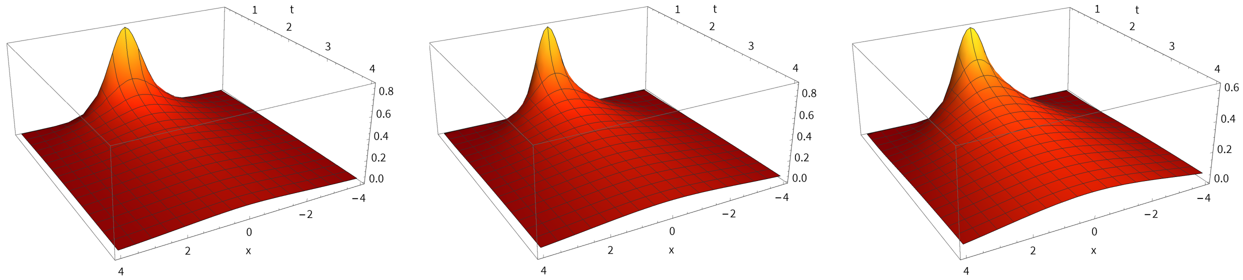

Figure 6: Plots of $G_1(x,t)$ for $\psi(t) = t$, $\mu = 0$, and $\alpha = 0.7, 0.9, 1$ (from left).

Analysing the plots, we see that the fundamental solution attains a maximum value at $x = 0$ and its first partial derivative with respect to x is continuous at this point. This is due to the vanishing of the second term in the series (53), which depends on $|x|$, and the differentiability of the remaining terms. Moreover, the fundamental solution is a positive function and exhibits an exponential decay, as was expected from the calculations presented in Section 5.3. The decay is more pronounced in space than in time. As α tends to one, the range of the plots reduces approximately from $[0, 40000]$ to $[0, 0.6]$.

6.1.2 Other fractional derivatives of Riemann-Liouville type

Here we present graphical representations of $G_1(x,t)$ for $x \in [-4,4]$, $\alpha = 0.1$, 0.3, 0.5, 0.7, 0.9, 1, and different choices of the function ψ:

- *Katugampola fractional derivative*: $\psi(t) = t^{\rho}$ with $\rho \in \mathbb{R}^+$ and $I = \mathbb{R}^+$.

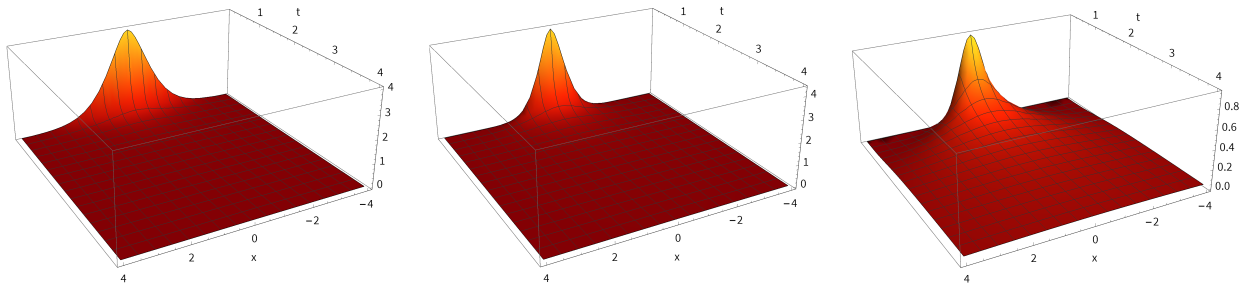

Figure 7: Plots of $G_1(x,t)$ for $\psi(t) = t^2$, $\mu = 0$, and $\alpha = 0.3, 0.5, 0.9$ (from left).

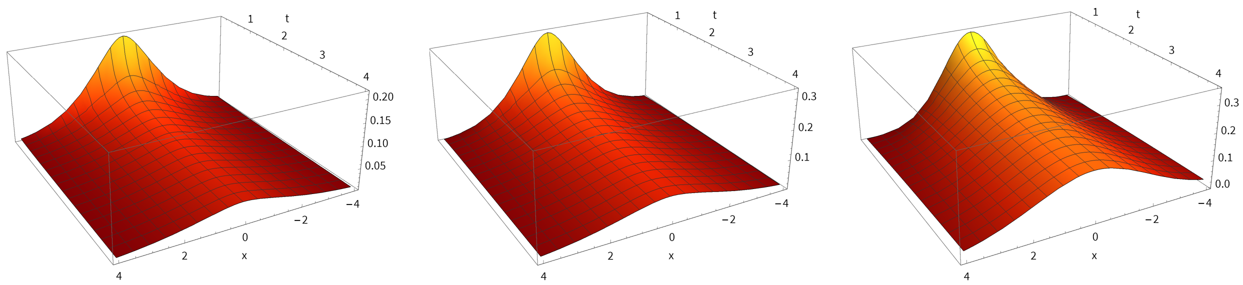

Figure 8: Plots of $G_1(x,t)$ for $\psi(t) = t^{\frac{1}{2}}$, $\mu = 0$, and $\alpha = 0.3, 0.5, 0.9$ (from left).

- *Hadamard fractional derivative*: $\psi(t) = \ln t$ and $I =]1, +\infty[$.

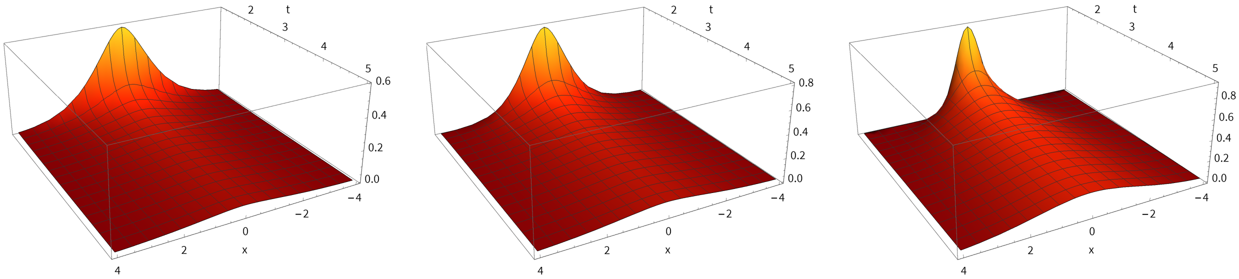

Figure 9: Plots of $G_1(x,t)$ for $\psi(t) = \ln(t)$, $\mu = 0$, and $\alpha = 0.3, 0.5, 0.9$ (from left).

- *Exponential type fractional derivative*: $\psi(t) = te^t$, and $I = \mathbb{R}^+$.

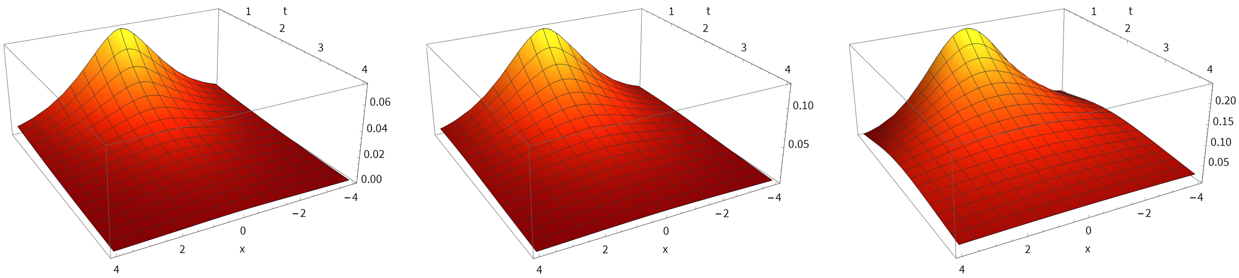

Figure 10: Plots of $G_1(x,t)$ for $\psi(t) = te^t$, $\mu = 0$, and $\alpha = 0.3, 0.5, 0.9$ (from left).

Analysing the plots, we see that the behaviour of the fundamental solutions is similar to that observed in Figures 5 and 6. However, we realise that different choices of the function ψ lead to different ranges of the plots and different decays of the fundamental solution as we move away from the origin.

6.2 ψ-Caputo fractional derivatives

In the next subsections, we deal with time-fractional derivatives of Caputo type, i.e., $\mu = 1$.

6.2.1 Caputo fractional derivative

Here, we consider $\psi(t) = t$, $t \in \mathbb{R}^+$ in (21), which corresponds to the classical case of the Caputo fractional derivative of order α (see [29, Sec. 5]). In the following Figures we present graphical representations of $G_1(x,t)$ for $x \in [-4,4]$, $t \in]0,4]$, and $\alpha = 0.1$, 0.3, 0.5, 0.7, 0.9, 1.

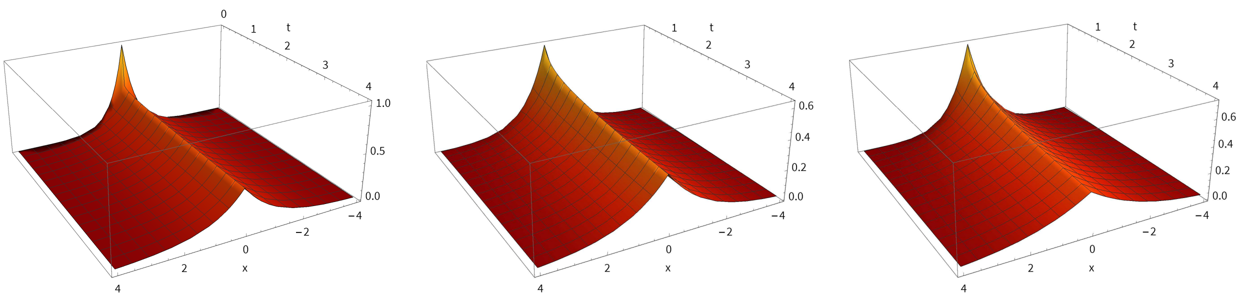

Figure 11: Plots of $G_1(x,t)$ for $\psi(t) = t$, $\mu = 1$, and $\alpha = 0.1, 0.3, 0.5$ (from left).

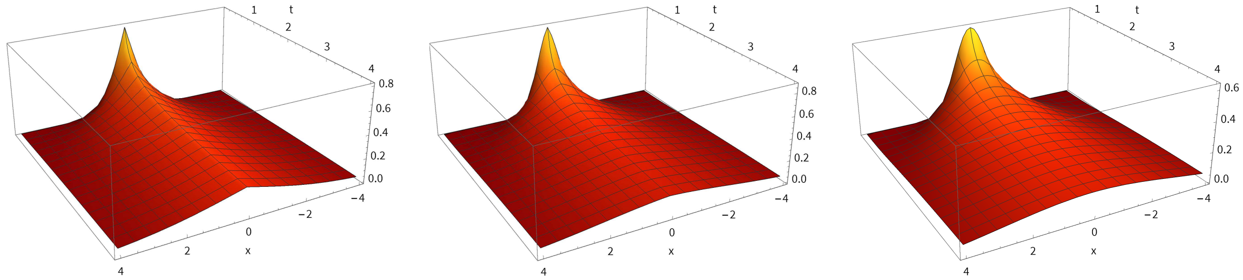

Figure 12: Plots of $G_1(x,t)$ for $\psi(t)=t$, $\mu=1$, and $\alpha=0.7, 0.9, 1$ (from left).

Analysing the plots, we see that the fundamental solution attains a maximum value at $x=0$ and its first partial derivative with respect to x is discontinuous at this point, unlike the Riemann-Liouville case. This is due to the non-vanishing of the second term in the series (53) that depends on $|x|$, which is the only non-differentiable term in the series. Moreover, the fundamental solution is a positive function and exhibits an exponential decay, as was expected from the calculations presented in Section 5.3. Here, too, the decay is more pronounced in space than in time. As α tends to one the range of the plots reduces approximately from $[0,1]$ to $[0,0.6]$. The Figures 11 and 12 are in accordance with those presented in Figure 2 of Section 7.1 of [8], for the slow-diffusion case.

6.2.2 Other fractional derivatives of Caputo type

Here we present graphical representations of $G_1(x,t)$ for $x\in[-4,4]$, $\alpha=0.1$, 0.3, 0.5, 0.7, 0.9, 1, and different choices of the function ψ:

- *Caputo-Katugampola fractional derivative*: $\psi(t)=t^{\rho}$ with $\rho\in\mathbb{R}^{+}$ and $I=\mathbb{R}^{+}$.

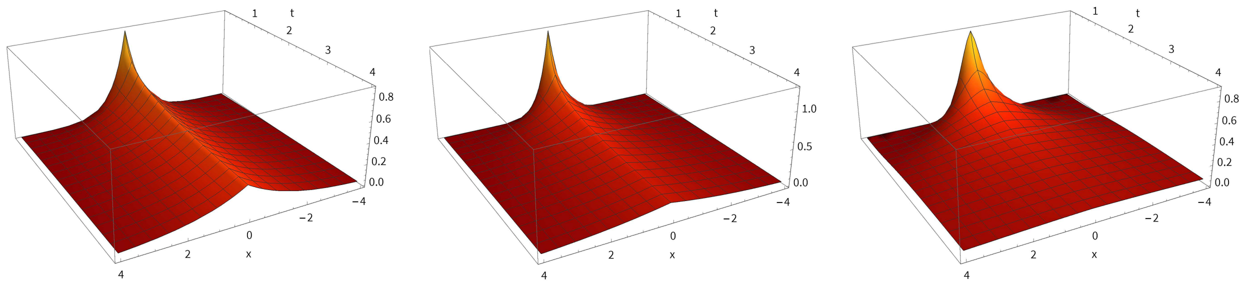

Figure 13: Plots of $G_1(x,t)$ for $\psi(t)=t^2$, $\mu=1$, and $\alpha=0.3, 0.5, 0.9$ (from left).

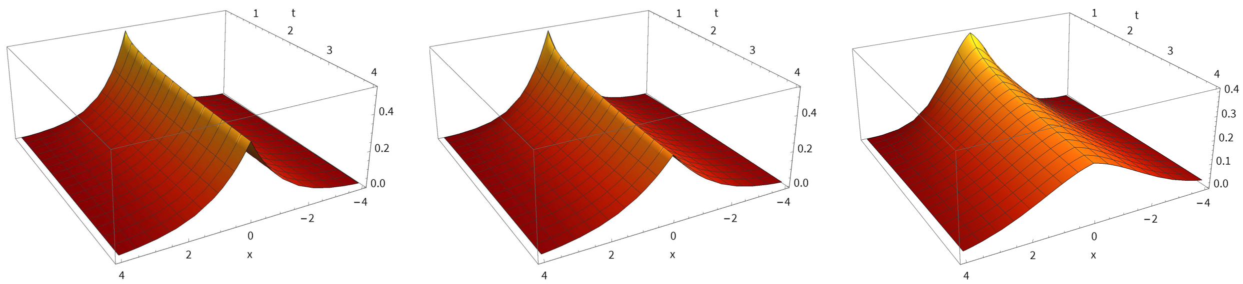

Figure 14: Plots of $G_1(x,t)$ for $\psi(t)=t^{\frac{1}{2}}$, $\mu=1$, and $\alpha=0.3, 0.5, 0.9$ (from left).

- *Caputo-Hadamard fractional derivative*: $\psi(t)=\ln(t)$ and $I=]1,+\infty[$.

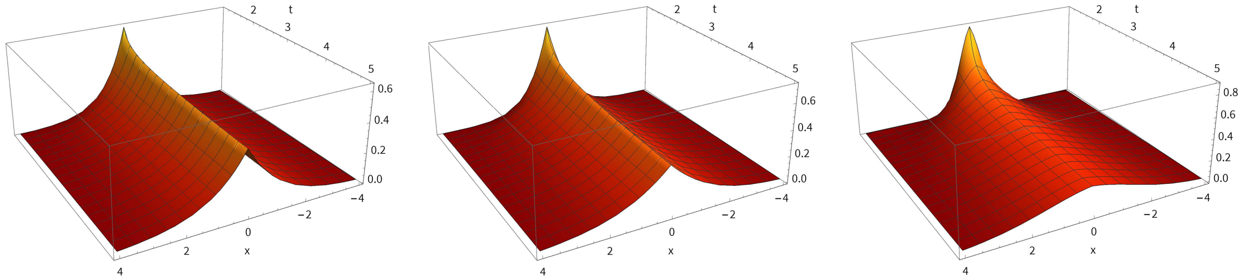

Figure 15: Plots of $G_1(x,t)$ for $\psi(t) = \ln t$, $\mu = 1$, and $\alpha = 0.3, 0.5, 0.9$ (from left).

- *Caputo-Exponential type fractional derivative*: $\psi(t) = te^t$, and $I = \mathbb{R}^+$.

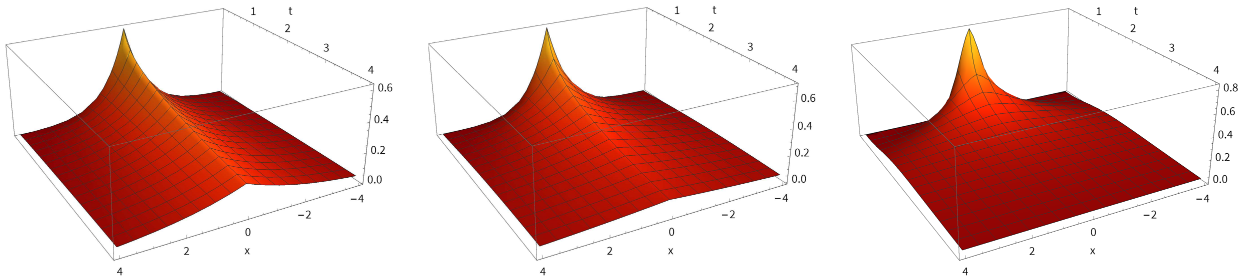

Figure 16: Plots of $G_1(x,t)$ for $\psi(t) = te^t$, $\mu = 1$, and $\alpha = 0.3, 0.5, 0.9$ (from left).

Analysing the plots, we see that the behaviour of the fundamental solutions is similar to that observed in Figures 11 and 12. As in the Riemann-Liouville case, different choices of the function ψ lead to different range of the plots and different decays of the fundamental solution as we move away from the origin.

6.3 Exponential type fractional derivative for $\mu = \frac{1}{2}$

In this subsection, we consider $\mu = \frac{1}{2}$, $\psi(t) = te^t$, and $I = \mathbb{R}^+$, which corresponds to the case where the time-fractional derivative in (21) is of order α, and type between Caputo and Riemann-Liouville (see [29, Sec. 5]). In the following Figures we present graphical representations of $G_1(x,t)$ for $x \in [-4,4]$, $t \in]0,4]$, and $\alpha = 0.3$, 0.5, 0.9.

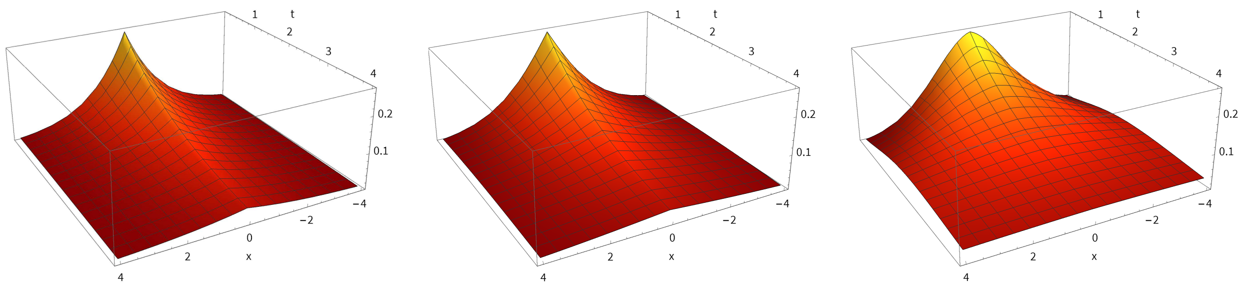

Figure 17: Plots of $G_1(x,t)$ $\psi(t) = te^t$, $\mu = \frac{1}{2}$, and $\alpha = 0.3, 0.5, 0.9$ (from left).

Analysing the plots, we see that the fundamental solution has an intermediate behaviour between the Riemann-Liouville and Caputo cases (see Figures 10 and 16). In fact, for small values of α the behaviour is more similar to the Caputo case, while when α tends to 1 the behaviour approaches the Riemann-Liouville case.

7 Conclusions

The diffusion equation containing fractional derivatives in time and/or in space are usually adopted to describe anomalous diffuse phenomena and, therefore, a detailed study of their solutions is required. In this work, we focused on the time-fractional diffusion equation in $\mathbb{R}^n \times \mathbb{R}^+$, where the time-fractional derivative is the ψ-Hilfer derivative of order $\alpha \in]0,1]$ and type $\mu \in [0,1]$. The ψ-Hilfer derivative has the freedom to encompass several fractional derivatives proposed in the literature for particular choices of the function ψ and the type of derivative μ. The derivative type allows a smooth interpolation between Caputo and Riemann-Liouville type fractional derivative operators. Our results give new contributions to the theory of the time-fractional diffusion equations.

The presented approach corresponds to a generalization of the techniques used in [8]. We were able to express the solution of the Cauchy problem associated with our equation in different integral representations. The study of the first fundamental solution was carried out, generalizing previous expressions and results in the literature. As a main result, we proved that in the one-dimensional case the first fundamental solution can be interpreted as a probability density function, for any admissible function ψ. The consistency of our results with previous works were demonstrated during the article considering particular choices of ψ and μ. We highlight that in order to implement the solution described in the paper it is necessary to develop suitable numerical methods. However, this is not in the scope of the paper and we leave it to future work.

Using similar techniques it is possible to study other fractional differential equations, for example, the time-telegraph equation with ψ-Hilfer derivatives. However, it is important to point out that the simultaneous presence of two time-fractional derivatives will imply more involved calculations and, therefore, more complicated results. For example, it is expected that the analogous of Theorem 3.2 and Lemma 3.4 is expected to involve the Prabhakar function, and the analogue of Theorem 3.5 to involve H-functions of two variables.

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