

A Decomposition Formula for the Weighted Commutator

Nelson Martins-Ferreira · Tim Van der Linden

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Abstract We decompose the weighted subobject commutator of M. Gran, G. Janelidze and A. Ursini as a join of a binary and a ternary commutator.

Keywords Semi-abelian category · Finitely cocomplete homological category · Weighted · Higher-order commutator · Co-smash product · Weighted commutator

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1 Introduction

In their article [6], M. Gran, G. Janelidze and A. Ursini introduce a *weighted normal commutator* which, depending on the chosen weight, captures classical commutators such as the Huq commutator [1, 3, 9] and the Smith commutator [1, 4, 14, 15]. It is constructed as

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N. Martins-Ferreira

Departamento de Matemática, Escola Superior de Tecnologia e Gestão, Centro para o Desenvolvimento Rápido e Sustentado do Produto, Instituto Politécnico de Leiria, Leiria, Portugal
e-mail: martins.ferreira@ipleiria.pt

N. Martins-Ferreira · T. Van der Linden

CMUC, University of Coimbra, 3001–454 Coimbra, Portugal

T. Van der Linden (✉)

Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, chemin du cyclotron 2 bte L7.01.02, B-1348 Louvain-la-Neuve, Belgium
e-mail: tim.vanderlinden@uclouvain.be

the normal closure of a so-called *weighted subobject commutator*. We show how this latter commutator may be decomposed as a join of a binary and a ternary commutator [7, 8] defined in terms of co-smash products [5]. We moreover explain that the corresponding concept of *weighted centrality* of arrows can be expressed in terms of the *admissibility* of certain diagrams in the first author’s sense [12].

2 The Weighted Subobject Commutator

In a finitely cocomplete homological category [1, 10], a **weighted cospan** is a triple of morphisms

$$\begin{array}{ccc}
 & W & \\
 & \downarrow w & \\
 X & \xrightarrow{x} D \xleftarrow{y} & Y
 \end{array} \tag{A}$$

in which (x, y) plays the role of cospan and w is the weight. Consider the pullback

$$\begin{array}{ccc}
 & W + Y & \\
 \pi_2 \nearrow & & \searrow \langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \rangle \\
 (W + X) \times_W (W + Y) & & W \\
 \pi_1 \searrow & & \nearrow \langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \rangle \\
 & W + X &
 \end{array}$$

and the induced outer diagram

$$\begin{array}{ccccc}
 W + X & \xrightarrow{\langle \begin{smallmatrix} 1_{W+X}, 1_W \\ 0 \end{smallmatrix} \rangle} & (W + X) \times_W (W + Y) & \xleftarrow{\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \rangle, 1_{W+Y} \rangle} & W + Y \\
 & \searrow \langle \begin{smallmatrix} w \\ x \end{smallmatrix} \rangle & \downarrow \varphi & \swarrow \langle \begin{smallmatrix} w \\ y \end{smallmatrix} \rangle & \\
 & & D & &
 \end{array}$$

In [6] the morphisms x and y are said to **commute over w** if and only if there exists a dotted arrow φ (called an **internal multiplication**) such that the above diagram is commutative.

As explained in [6], taking $W = 0$ captures commuting pairs in the Huq sense (x and y commute over 0 if and only if they Huq-commute), and $w = 1_D$ captures centralising equivalence relations in the Smith sense (the respective normalisations x and y of two equivalence relations R and S on D commute over 1_D if and only if R and S Smith-commute).

Now consider the canonical comparison morphism

$$\left\langle \begin{smallmatrix} 1_W & 1_W \\ 1_X & 0 \\ 0 & 1_Y \end{smallmatrix} \right\rangle : W + X + Y \rightarrow (W + X) \times_W (W + Y)$$

which, being a regular epimorphism [6] as the comparison between a sum and a product in the category of points over an object W in a regular Mal’sev category, induces a short exact sequence

$$0 \longrightarrow K \triangleright \longrightarrow W + X + Y \xrightarrow{\left\langle \begin{smallmatrix} 1_W & 1_W \\ 1_X & 0 \\ 0 & 1_Y \end{smallmatrix} \right\rangle} (W + X) \times_W (W + Y) \longrightarrow 0. \tag{B}$$

The (W, w) -**weighted subobject commutator** $\kappa : [(X, x), (Y, y)]_{(W, w)} \rightarrow D$ of x and y is the direct image of K along the induced arrow to D as in

$$\begin{array}{ccc}
 K & \xrightarrow{\quad} & W + X + Y \\
 \vdots & & \downarrow \left\langle \begin{matrix} w \\ x \\ y \end{matrix} \right\rangle \\
 [(X, x), (Y, y)]_{(W, w)} & \xrightarrow{\quad \kappa \quad} & D.
 \end{array}$$

It is clear from the exactness of the above sequence that x and y commute over w if and only if $[(X, x), (Y, y)]_{(W, w)}$ vanishes.

The normal closure of κ is called the (W, w) -**weighted normal commutator** of x and y and denoted by $N[(X, x), (Y, y)]_{(W, w)}$.

3 Admissibility

In order to analyse the weighted subobject commutator in terms of the binary and ternary commutators considered in [7, 8], we pass via an intermediate notion from [12]. An **admissibility diagram** is a diagram of shape

$$\begin{array}{ccccc}
 & & f & & \\
 & & \xrightarrow{\quad} & & \\
 A & & B & & C \\
 & \xleftarrow{r} & & \xleftarrow{s} & \\
 & & \beta & & \\
 & \searrow \alpha & \downarrow & \swarrow \gamma & \\
 & & D & &
 \end{array} \tag{C}$$

with $f \circ r = 1_B = g \circ s$ and $\alpha \circ r = \beta = \gamma \circ s$. Note that by taking the pullback of f with g , any admissibility diagram such as (C) may be extended to

$$\begin{array}{ccccc}
 & & C & & \\
 & \nearrow \pi_2 & & \searrow s & \\
 & & A \times_B C & & B \\
 & \nwarrow e_2 & & \nearrow g & \\
 & & A & & D \\
 & \nearrow e_1 & & \nwarrow f & \\
 & & A & & \\
 & \nwarrow \pi_1 & & \nearrow r & \\
 & & A & & \\
 & & & & \nearrow \alpha & \\
 & & & & &
 \end{array}$$

in which the pullback square is a double split epimorphism.

The triple (α, β, γ) is said to be **admissible with respect to** (f, r, g, s) if there is a (necessarily unique) morphism $\varphi : A \times_B C \rightarrow D$ such that $\varphi \circ e_1 = \alpha$ and $\varphi \circ e_2 = \gamma$.

4 Commuting Pairs in Terms of Admissibility

It is immediately clear from the definitions that the morphisms x and y commute over w if and only if the triple $(\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \rangle, w, \langle \begin{smallmatrix} w \\ y \end{smallmatrix} \rangle)$ is admissible with respect to the quadruple $(\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \rangle, \iota_W, \langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \rangle, \iota_W)$ as in the diagram

$$\begin{array}{ccccc}
 & & \langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \rangle & & \langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \rangle \\
 & & \xrightarrow{\quad} & & \xrightarrow{\quad} \\
 W + X & & W & & W + Y \\
 & \xleftarrow{\iota_W} & & \xleftarrow{\iota_W} & \\
 & & \downarrow w & & \\
 & \searrow \langle \begin{smallmatrix} w \\ x \end{smallmatrix} \rangle & & \swarrow \langle \begin{smallmatrix} w \\ y \end{smallmatrix} \rangle & \\
 & & D & &
 \end{array} \tag{D}$$

5 Admissibility in Terms of Commuting Pairs

Consider a diagram (C) and the induced weighted cospan

$$\begin{array}{ccc}
 & B & \\
 & \downarrow \beta & \\
 X = \text{Ker}(f) & \xrightarrow{\alpha \circ \text{ker}(f)} & D \xleftarrow{\gamma \circ \text{ker}(g)} \text{Ker}(g) = Y.
 \end{array}$$

We claim that the triple (α, β, γ) is admissible with respect to (f, r, g, s) if and only if $x = \alpha \circ \text{ker}(f)$ and $y = \gamma \circ \text{ker}(g)$ commute over $w = \beta: W = B \rightarrow D$. To see this, it suffices to compare Diagram (C) with the induced Diagram (D). In fact there is a regular epimorphism of admissibility diagrams from the latter to the former which keeps D fixed and makes

$$\begin{array}{ccccc}
 B + X & \xrightleftharpoons[\iota_B]{\langle \begin{smallmatrix} 1_B \\ 0 \end{smallmatrix} \rangle} & B & \xrightleftharpoons[\iota_B]{\langle \begin{smallmatrix} 1_B \\ 0 \end{smallmatrix} \rangle} & B + Y \\
 \downarrow \langle \text{ker}(f)^r \rangle & & \parallel & & \downarrow \langle \text{ker}(g)^s \rangle \\
 A & \xrightleftharpoons[r]{f} & B & \xrightleftharpoons[s]{g} & C
 \end{array}$$

commute. This already proves the “only if” in our claim. For the “if” suppose that x and y commute over β . For the induced arrow

$$\varphi: (B + X) \times_B (B + Y) \rightarrow D$$

to factor over the regular epimorphism

$$\langle \text{ker}(f)^r \rangle \times_B \langle \text{ker}(g)^s \rangle: (B + X) \times_B (B + Y) \rightarrow A \times_B C,$$

we only need that it vanishes on $\text{Ker}(\langle \text{ker}(f)^r \rangle) \times \text{Ker}(\langle \text{ker}(g)^s \rangle)$. This does indeed happen, because

$$\begin{aligned}
 \varphi \circ (\text{ker}(\langle \text{ker}(f)^r \rangle) \times \text{ker}(\langle \text{ker}(g)^s \rangle)) \circ \langle 1, 0 \rangle &= \varphi \circ \langle 1_{B+X}, \iota_B \circ \langle \begin{smallmatrix} 1_B \\ 0 \end{smallmatrix} \rangle \rangle \circ \text{ker}(\langle \text{ker}(f)^r \rangle) \\
 &= \langle \begin{smallmatrix} \beta \\ x \end{smallmatrix} \rangle \circ \text{ker}(\langle \text{ker}(f)^r \rangle) \\
 &= \alpha \circ \langle \text{ker}(f)^r \rangle \circ \text{ker}(\langle \text{ker}(f)^r \rangle)
 \end{aligned}$$

is trivial. Similarly, one can check that the arrow

$$\varphi \circ (\text{ker}(\langle \text{ker}(f)^r \rangle) \times \text{ker}(\langle \text{ker}(g)^s \rangle)) \circ \langle 0, 1 \rangle$$

is trivial.

6 Binary and Ternary Higgins Commutators

If $k: K \rightarrow X$ and $l: L \rightarrow X$ are subobjects of an object X in a finitely cocomplete homological category, then the (Higgins) commutator $[K, L] \leq X$ is the image of the induced morphism

$$K \diamond L \xrightarrow{\iota_{K,L}} K + L \xrightarrow{\langle \begin{smallmatrix} k \\ l \end{smallmatrix} \rangle} X,$$

where

$$K \diamond L = \text{Ker} \left(\left\langle \begin{matrix} 1_K & 0 \\ 0 & 1_L \end{matrix} \right\rangle : K + L \rightarrow K \times L \right).$$

As explained in [6], the Higgins commutator is another special case of the weighted subobject commutator recalled above. This commutator was first introduced in [7, 11]. Higher-order versions of it exist and are studied in [7, 8].

The object $K \diamond L$, as $K \diamond L \diamond M$ below, is an example of a **co-smash product** [5]. It is worth recalling from [11] that it may be computed as the intersection $K \flat L \wedge L \flat K$, where the object $K \flat L$ from [2] is the kernel in the split exact sequence

$$0 \longrightarrow K \flat L \rightrightarrows K + L \begin{matrix} \xrightarrow{\langle \begin{matrix} 1_K \\ 0 \end{matrix} \rangle} \\ \xleftarrow{\iota_K} \end{matrix} K \longrightarrow 0.$$

Furthermore, also the sequence

$$0 \longrightarrow K \diamond L \rightrightarrows K \flat L \begin{matrix} \xrightarrow{\iota_K} \\ \xleftarrow{\iota_L} \end{matrix} L \longrightarrow 0 \tag{E}$$

is split exact.

If $m : M \rightarrow X$ is another subobject of X , then the **ternary (Higgins) commutator** $[K, L, M] \leq X$ is defined as the image of the composite

$$K \diamond L \diamond M \xrightarrow{\iota_{K,L,M}} K + L + M \xrightarrow{\left\langle \begin{matrix} k \\ l \\ m \end{matrix} \right\rangle} X,$$

where $\iota_{K,L,M}$ is the kernel of the morphism

$$K + L + M \xrightarrow{\left\langle \begin{matrix} i_K & i_K & 0 \\ i_L & 0 & i_L \\ 0 & i_M & i_M \end{matrix} \right\rangle} (K + L) \times (K + M) \times (L + M).$$

It is well known that co-smash products are not associative, in general; furthermore, ternary co-smash products or commutators need not be decomposable into iterated binary ones: see [5, 7, 8].

Theorem 1 Consider a weighted cospan (A) such that x and y are normal monomorphisms (= kernels) in a finitely cocomplete homological category. Then x and y commute over w precisely when the commutators $[X, Y]$ and $[X, Y, \text{Im}(w)]$ vanish.

Proof First of all we show that x and y coincide with the images of $\langle \begin{matrix} w \\ x \end{matrix} \rangle \circ \text{ker}(\langle \begin{matrix} 1_W \\ 0 \end{matrix} \rangle)$ and $\langle \begin{matrix} w \\ y \end{matrix} \rangle \circ \text{ker}(\langle \begin{matrix} 1_W \\ 0 \end{matrix} \rangle)$, respectively, as in (D). To see this, we consider the diagram with short exact rows

$$\begin{array}{ccccccc} & & & X & & & \\ & & & \downarrow \iota_X & & & \\ 0 & \longrightarrow & W \flat X & \xrightarrow{\kappa_{B,X}} & W + X & \xrightarrow{\langle \begin{matrix} 1_W \\ 0 \end{matrix} \rangle} & W \longrightarrow 0 \\ & & \downarrow \xi & & \downarrow \langle \begin{matrix} w \\ x \end{matrix} \rangle & & \downarrow d \circ w \\ 0 & \longrightarrow & X & \xrightarrow{x} & D & \xrightarrow{d} & D_0 \longrightarrow 0 \end{array}$$

It is clear that $\langle \begin{matrix} 1_W \\ 0 \end{matrix} \rangle \circ \iota_X = 0$ induces the factorisation η_X^W of ι_X over the kernel $\kappa_{B,X}$ of $\langle \begin{matrix} 1_W \\ 0 \end{matrix} \rangle$. Similarly, since

$$d \circ \langle \begin{matrix} w \\ x \end{matrix} \rangle \circ \kappa_{B,X} = d \circ w \circ \langle \begin{matrix} 1_W \\ 0 \end{matrix} \rangle \circ \kappa_{B,X}$$

is trivial we obtain the dotted factorisation ξ . Now

$$x \circ \xi \circ \eta_X^W = \left\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \right\rangle \circ \kappa_{B,X} \circ \eta_X^W = \left\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \right\rangle \circ \iota_X = x,$$

so $\xi \circ \eta_X^W = 1_X$ because x is a monomorphism. In particular, ξ is a regular epimorphism. It follows that x is the image of $\left\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \right\rangle \circ \kappa_{B,X}$.

We know from the above discussion that x and y commute over w precisely when the triple $(\left\langle \begin{smallmatrix} w \\ x \end{smallmatrix} \right\rangle, w, \left\langle \begin{smallmatrix} w \\ y \end{smallmatrix} \right\rangle)$ is admissible with respect to $(\left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle, \iota_W, \left\langle \begin{smallmatrix} 1_W \\ 0 \end{smallmatrix} \right\rangle, \iota_W)$. Lemma 4.5 in [8] now tells us that this happens if and only if the commutators $[X, Y]$ and $[X, Y, \text{Im}(w)]$ vanish. □

Via Theorem 4.6 in [8] we now recover the known result that the *Smith is Huq* condition [13] holds if and only if, for any given cospan of normal monomorphisms (x, y) , the property of commuting over w is independent of the chosen weight w making (x, y, w) a weighted cospan.

We also see that the (W, w) -weighted normal commutator $N[(X, x), (Y, y)]_{(W,w)}$ of x and y is the normal closure of $[X, Y] \vee [X, Y, \text{Im}(w)]$ in D , since these two normal subobjects satisfy the same universal property. We shall, however, not insist further on this, because we can obtain the following refinement (Theorem 2).

Lemma 1 *If X, Y , and W are objects in a finitely cocomplete homological category, then there is a decomposition*

$$(X + Y) \diamond W \cong ((X \diamond Y \diamond W) \rtimes (X \diamond W)) \rtimes (Y \diamond W).$$

More precisely, there exists an object V and split short exact sequences

$$0 \longrightarrow V \triangleright \longrightarrow (X + Y) \diamond W \begin{smallmatrix} \longleftarrow \\ \longrightarrow \end{smallmatrix} Y \diamond W \longrightarrow 0$$

and

$$0 \longrightarrow X \diamond Y \diamond W \triangleright \longrightarrow V \begin{smallmatrix} \longleftarrow \\ \longrightarrow \end{smallmatrix} X \diamond W \longrightarrow 0.$$

Proof This Lemma 2.12 in [8], a result which was first obtained by M. Hartl and B. Loiseau. □

Theorem 2 *Given a weighted cospan (A) in a finitely cocomplete homological category, the (W, w) -weighted subobject commutator of monomorphisms x and y decomposes as*

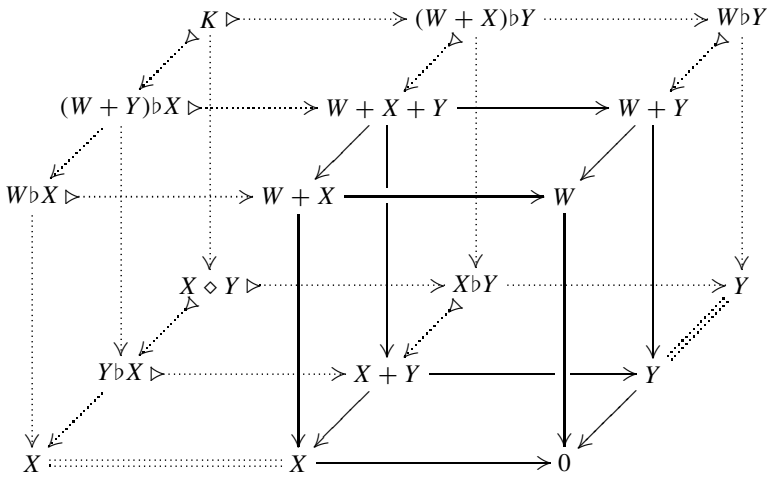
$$[(X, x), (Y, y)]_{(W,w)} = [X, Y] \vee [X, Y, \text{Im}(w)].$$

Proof We decompose the kernel K of the short exact sequence (B) into a join of the co-smash products $X \diamond Y$ and $X \diamond Y \diamond W$ considered as subobjects of K . The result then follows from Corollary 2.14 in [7]. Indeed, the image of the composite

$$X \diamond Y \diamond W \rightarrow W + X + Y \rightarrow D$$

is $[X, Y, \text{Im}(w)]$, which is a subobject of $[(X, x), (Y, y)]_{(W, w)}$. It is easily seen that also $[X, Y] \leq [(X, x), (Y, y)]_{(W, w)}$ and that these two inclusions are jointly regular epic.

Consider the cube of solid split epimorphisms



which, taking kernels horizontally, yields two 3×3 diagrams (or, equivalently, a 3×3 diagram of vertical split epimorphisms). Note that the bottom one has $X \diamond Y$, and the top one K , in its back left corner. It suffices to prove that, taking kernels vertically now, we obtain the split exact sequence

$$0 \longrightarrow X \diamond Y \diamond W \twoheadrightarrow K \xleftarrow{\cong} X \diamond Y \longrightarrow 0$$

in the back left corner of the induced $3 \times 3 \times 3$ diagram. Taking vertical kernels of the front and middle sections of the diagram above, we already obtain a morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & U & \longrightarrow & (X + Y) \blacktriangleright W & \xleftarrow{\cong} & Y \blacktriangleright W & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X \diamond W & \longrightarrow & X \blacktriangleright W & \xleftarrow{\cong} & W & \longrightarrow & 0 \end{array}$$

of short exact sequences. Using (E) we see that the sequence

$$0 \longrightarrow U \twoheadrightarrow (X + Y) \diamond W \xleftarrow{\cong} Y \diamond W \longrightarrow 0$$

is split exact. Noting that V in Lemma 1 is the object U , we see that the co-smash product $X \diamond Y \diamond W$ must coincide with the kernel of $U \rightarrow X \diamond W$, which we already know coincides with the needed kernel of $K \rightarrow X \diamond Y$. □

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