Time-fractional telegraph equation with ψ-Hilfer derivatives*

N. Vieira[‡], M. Ferreira[§,‡], and M.M. Rodrigues[‡]

[‡]CIDMA - Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.
Emails: nloureirovieira@gmail.com; mrodrigues@ua.pt; mferreira@ua.pt

[§]School of Technology and Management
Polytechnic of Leiria
P-2411-901, Leiria, Portugal.
E-mail: milton.ferreira@ipleiria.pt

July 31, 2022

Abstract

This paper deals with the investigation of the solution of the time-fractional telegraph equation in higher dimensions with ψ-Hilfer fractional derivatives. By application of the Fourier and ψ-Laplace transforms the solution is derived in closed form in terms of bivariate Mittag-Leffler functions in the Fourier domain and in terms of convolution integrals involving Fox H-functions of two-variables in the space-time domain. A double series representation of the first fundamental solution is deduced for the case of odd dimension. The results derived here are of general nature since our fractional derivatives allow to interpolate between Riemann-Liouville and Caputo fractional derivatives and the use of an arbitrary positive monotone increasing function ψ in the kernel allows to encompass most of the fractional derivatives in the literature. In the one dimensional case, we prove the conditions under which the first fundamental solution of our equation can be interpreted as a spatial probability density function evolving in time, generalizing the results of Orsingher and Beghin (2004). Some plots of the fundamental solutions for different fractional derivatives are presented and analysed, and particular cases are addressed to show the consistency of our results.

Keywords: Time-fractional telegraph equation; ψ-Hilfer fractional derivative; ψ-Laplace transform; Series and integral representations; Fractional moments; Probability density function.

MSC 2010: 35R11; 26A33; 35A08; 35A22; 35C15; 60G22.

1 Introduction

Nowadays, there are several definitions of fractional integrals and fractional derivatives in the literature that differ by the kernel used in its definition. To unify different integro-differential operators into general classes, the theory of fractional calculus evolved to a more general setting called generalised fractional calculus. Examples of such classes are the ψ-fractional calculus with respect to a given function ψ, the weighted ψ-fractional calculus, and fractional calculus with general analytic kernels. This variety of classes is justified by the need for operators with different structures to successfully model a large number of processes and phenomena that exist in the real world.

In [28,39] the authors proposed a fractional integral operator with respect to another function ψ, obtaining a general operator, in the sense that it is enough to choose a function ψ with certain properties to obtain the most

*The final version is published in *Chaos, Solitons & Fractals*, **162** (2022), Article ID: 112276 (26 pp). It as available via the website `https://doi.org/10.1016/j.chaos.2022.112276`

of the existing fractional integral operators. Attempting to unify several definitions of fractional derivatives into a single one, the concept of fractional derivative of a function with respect to another function was recently introduced. In 2017, Almeida [2] proposed a new fractional derivative called ψ-Caputo that generalizes a class of fractional derivatives in the Caputo (or Dzerbayshan-Caputo) sense. The same idea can be adapted to define the ψ-Riemmann-Liouville fractional derivative. In 2018, Sousa and Oliveira [46] unified both definitions using Hilfer's idea of interpolating between Riemann-Liouville and Caputo fractional derivatives by introducing a two-parameter family of fractional derivatives of order $\alpha > 0$ and type $\mu \in [0, 1]$ which depends on an arbitrary function ψ, and called it the ψ-Hilfer fractional derivative. Their approach allows obtaining as special cases some well-known fractional derivatives: Caputo, Riemann-Liouville, Hadamard, Katugampola, Chen, Jumarie, Prabhakar, Erdélyi-Kober, Weyl, among others (see [46, Sec. 5]).

The Hilfer and ψ-Hilfer fractional derivatives possess different degrees of arbitrariness that are important for applications. The type-parameter produces more stationary states, provides an extra degree of freedom on the initial condition, and increases the flexibility for the description of complex data. It was first used by Hilfer to describe the dynamics in glass formers over an extremely large-frequency window (see [23]). During the last years fractional differential equations with Hilfer and ψ-Hilfer derivatives were studied by several authors, see e.g. [12, 30, 41, 45, 48]. Moreover, the arbitrariness in the function ψ allows to construct different kernels for the fractional derivative with different properties of relaxation and memory processes, which are important to better describe the dynamics of a given problem and to model more complex physical phenomena. In [2], the author considered the ψ-Caputo derivative with different functions ψ to study the population growth within Malthus law and concluded that these generalised derivatives describe more efficiently the dynamics of the model due to the freedom of choice of the kernel. Recently, in [42], the author used the Hilfer fractional derivative to model alcohol concentration in human blood. The numerical results show that the arbitrariness of the fractional order and type parameters gives more flexibility in the characterization of the phenomena in different parts of the human body.

Motivated by our previous works on the study of fractional diffusion equations we investigate the time-fractional telegraph equation (TFTE) with ψ-Hilfer fractional derivatives. The classical telegraph equation was first derived by Lord Kelvin in the 19th century [47]. It is a hyperbolic partial differential equation of the form

$$c_2\,\partial^2_{tt} u\,(x,t) + \,c_1\,\partial_t u\,(x,t) - c_0^2\,\partial^2_{xx} u\,(x,t) + d\,u\,(x,t) = q\,(x,t)\,, \quad x \in \mathbb{R}, \quad t > 0.$$

This equation was proposed by Cattaneo in 1958 (see [10]) to overcome the problem of infinite propagation velocity in heat transmission. Over the years, this equation and its time-fractional versions appeared in the study of several phenomena such as transmission lines for all frequencies [27], random walks [4], solar particle transport [13], oceanic diffusion [34], wave propagation [50], damped small vibrations, anomalous diffusion and wave-like processes [6, 33, 35, 36], scalar part of the Maxwell equations.

The TFTE with time-fractional derivatives of orders $\alpha_1 \in]0, 1]$ and $\alpha_2 \in]1, 2]$ was studied from the analytical, numerical, and probabilistic points of view by several authors. In [9], Cascaval et al. discussed the well-posedness of some initial-boundary value problems for the TFTE as well as the asymptotic behaviour of the solutions. Beghin and Orsingher considered the particular case when $\alpha_2 = 2\alpha_1$ with α_1 a rational number, and proved that the solutions of the associated Cauchy problem can be represented as densities of processes obtained by means of the composition of the telegraph process with a process representing time (see [5]). In [36] the authors studied the neutral case of the TFTE and obtained an explicit Fourier representation of the fundamental solution (FS) and made a probabilistic interpretation of the FS in terms of stable probability density functions. Particular attention was given to the case $\alpha_1 = 1/2$ and $\alpha_2 = 1$ due to the connection of the telegraph process with Brownian motion. Some of these results were generalized by Camargo et al. in [8] for general α_1 and α_2 and studied later by Boyadjiev and Luchko in [6]. Chen et al. [11] discussed the solution of the TFTE with different types of boundary conditions by using the method of separating variables. In [41], the authors considered a generalised telegraph equation with time-fractional derivatives in the Hilfer and Hadamard senses and space-fractional derivatives in the sense of Riesz-Feller. In [32], Mamchuev considered the inhomogeneous TFTE with Caputo derivatives and obtained a general representation of the regular solution in a rectangular domain in terms of FS by the Green's function method. Górska et al. (see [22]) considered various types of generalized telegraph equations and determine the conditions under which solutions can be recognized as probability density distributions. We refer the interested reader to the recent survey paper [33] where its is presented a very complete review of the fractional telegraph process.

The works [15–18, 35, 37] are devoted to the study of the TFTE in the multidimensional case with n space variables, where the second derivative in space is replaced by the Euclidean Laplace operator. In [35] the authors solved the multi-dimensional TFTE with multi-term time-fractional derivatives and proved that its fundamental solution is the law of a stable isotropic multi-dimensional process time-changed. Ovidio et al. [37] constructed compositions of vector processes whose distributions are related to space-time fractional n-dimensional telegraph equations. In [16, 18] were employed Fourier, Laplace, and Mellin transform techniques to obtain the first and second FS. Moreover, the application of the Residue Theorem allowed obtaining double series representation for the FS of the TFTE in higher dimensions. Connections of the TFTE with fractional Clifford analysis and Sturm-Liouville theory were presented in [17] and [15].

In this paper, we present a unified approach for the complete TFTE in the multidimensional case using time-fractional ψ-Hilfer derivatives subject to some initial and boundary conditions. Our main results appear in Sections 3, 5, and 6. In Theorem 3.1 we obtain an integral representation of the solution of our problem by series of classical convolution and ψ-convolution integrals involving inverse Fourier integrals. For the particular case $d = 0$, these integrals are calculated explicitly and are shown to correspond to Fox H-functions of one real variable. Consequently, we obtain another integral representation of the solution that we describe in Theorem 3.3. Finally, our third main result answers the question under which conditions the FS of the TFTE with ψ-Hilfer derivatives can be understood as a probability density function in the one-dimensional case. We prove in Theorem 6.1 that this is possible only for ψ-Caputo fractional derivatives, and for every $0 < \alpha_1 < 1$ and $1 \leq \alpha_2 < 2$. This result generalises the correspondent one in [36] and gives a definite answer to this question. In Section 4, we obtain a double series representation of the FS for the odd space dimension and analyse its convergence using Horn's technique (see Appendix). Fractional moments of the FS are computed in Section 5 as functions of time and their asymptotic behaviours are deduced for short and long times. Regarding the second-order moment (also known as the mean-square displacement) we provide an interpretation of the behaviour of the FS in terms of sub- and superdiffusion for different choices of the function ψ and the fractional parameters. Throughout the paper, we present and analyse some graphical representations of the FS for different ψ-Hilfer derivatives. In the last section, particular cases are addressed to show the consistency of our results.

2 Preliminaries

We start this section by presenting some concepts related to fractional integrals and derivatives of a function f with respect to another function ψ (for more details see [46] and references therein).

Definition 2.1 *(cf. [46, Def. 4]) Let (a, b) be a finite or infinite interval on the real line $\mathbb{R}$, $\alpha > 0$, and ψ a positive and monotone increasing function on (a, b) having a continuous derivative ψ' in (a, b). The left fractional integral of a function f with respect to another function ψ on $[a, b]$ is defined by*

$$\left(I_{a^+}^{\alpha;\psi} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(w) \, (\psi(t) - \psi(w))^{\alpha-1} f(w) \, dw, \qquad t > a. \tag{1}$$

Next, we give the definition of the so-called ψ-Hilfer fractional derivative of a function f with respect to another function.

Definition 2.2 *(cf. [46, Def. 7]) Let $\alpha > 0$ and $m = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α. Let also $I = [a, b]$ be a finite or infinite interval on the real line and $f, \psi \in C^m[a, b]$ two functions such that ψ is a positive monotone increasing function and $\psi'(t) \neq 0$, for all $t \in I$. The ψ-Hilfer left fractional derivative ${}^H\mathbb{D}_{t,a^+}^{\alpha,\mu;\psi}$ of order α and type $\mu \in [0, 1]$ is defined by*

$$\left({}^H\mathbb{D}_{a^+}^{\alpha,\mu;\psi} f\right)(t) = I_{a^+}^{\mu(m-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^m I_{a^+}^{(1-\mu)(m-\alpha);\psi} f(t). \tag{2}$$

We observe that when $\mu = 0$ we recover the left Riemann-Liouville fractional derivative of a function with respect to ψ (see [46, Def. 5]) and when $\mu = 1$ we obtain the left Caputo fractional derivative of a function with respect to ψ (see [46, Def. 6]). In Section 5 of [46] is presented a list of several fractional integrals and fractional derivatives that can be obtained from (1) and (2), respectively, for different choices of ψ and μ. In the following table, we present the fractional derivatives that are considered in Subsections 4.1 and 6.2 to graphically represent the first fundamental solution of our problem in the one-dimensional case.

$\psi(\mathbf{t})$	$\mathbf{I}$	μ	Designation
t	$\mathbb{R}^+$	0	Riemann-Liouville
		1	Caputo
$t^\rho,\ \rho \in \mathbb{R}^+$	$\mathbb{R}^+$	0	Katugampola
		1	Caputo-Katugampola
$\ln t$	$]1, +\infty[$	0	Hadamard
		1	Caputo-Hadamard
$t\, e^t$	$\mathbb{R}^+$	0	Exponential type
		1	Caputo-Exponential type

Table 1: Some particular cases of ψ-Hilfer derivatives.

The previous definitions of fractional integrals and derivatives can be naturally extended to $\mathbb{R}^n$ considering partial fractional integrals and partial fractional derivatives (see Chapter 5 in [39]).

Now, we recall some special functions used in this article and some of their main properties. The Gamma function (see [1]) is defined by the following integral

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}\, e^{-t}\, dt, \qquad \mathrm{Re}(z) > 0,$$

and admits an analytic continuation to the whole complex plane as a meromorphic function with simple poles at the negative integers and zero. The Gamma function satisfies many identities. The following ones are used in this work:

$$\Gamma(z+n) = (z)_n\ \Gamma(z), \quad n \in \mathbb{N}_0, \tag{3}$$

$$\Gamma(z-n) = \frac{(-1)^n\ \Gamma(z)}{(1-z)_n}, \quad n \in \mathbb{N}_0, \tag{4}$$

$$\Gamma\left(z + \frac{1}{2}\right) = \frac{2^{1-2z}\sqrt{\pi}\ \Gamma(2z)}{\Gamma(z)}, \tag{5}$$

$$\Gamma(z)\ \Gamma(-z) = \frac{-\pi}{z\ \sin(\pi z)}, \tag{6}$$

$$\Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos(\pi z)\ \Gamma\left(\frac{1}{2} + z\right)}. \tag{7}$$

The residues of the poles of the Gamma function at $s = -k$, $k \in \mathbb{Z}_0^+$, are given by

$$\mathrm{res}_{s=-k}\Gamma(s) = \frac{(-1)^k}{k!}, \quad k \in \mathbb{Z}_0^+. \tag{8}$$

The three parameter Mittag-Leffler function $E_{\beta_1,\beta_2}^{\beta_3}(z)$ (see [20]), is defined in terms of power series by

$$E_{\beta_1,\beta_2}^{\beta_3}(z) = \sum_{k=0}^{\infty} \frac{(\beta_3)_k\, z^k}{k!\, \Gamma(\beta_1 k + \beta_2)}, \quad z \in \mathbb{C}, \quad \beta_1, \beta_2, \beta_3 \in \mathbb{R}, \quad \beta_1 > 0, \tag{9}$$

where $(\beta_3)_k$ is the Pochhammer symbol, and satisfies the following addition formula (see (5.1.12) in [20])

$$z\, E_{\beta_1,\beta_2}^{\beta_3}(z) = E_{\beta_1,\beta_2-\beta_1}^{\beta_3}(z) - E_{\beta_1,\beta_2-\beta_1}^{\beta_3-1}(z). \tag{10}$$

When $z = t \in \mathbb{R}^+$, the first terms in the power series (9) give the following asymptotic expansion for $t \to 0^+$:

$$t^{\beta_2-1}\, E_{\beta_1,\beta_2}^{\beta_3}\left(-at^{\beta_1}\right) \sim t^{\beta_2-1}\left(\frac{1}{\Gamma(\beta_2)} - \frac{a\beta_3\, t^{\beta_1}}{\Gamma(\beta_1+\beta_2)}\right) \tag{11}$$

with $\beta_2, a > 0$. When $t \to +\infty$, focusing just on the leading term of the expansion (9), one can infer that for $\beta_2 \neq \beta_1\beta_3$ (see formula (4.25) in [24])

$$t^{\beta_2-1}\, E_{\beta_1,\beta_2}^{\beta_3}\left(-at^{\beta_1}\right) \sim a^{-\beta_3}\, \frac{t^{\beta_2-\beta_1\beta_3-1}}{\Gamma(\beta_2-\beta_1\beta_3)}, \quad \text{as } t \to +\infty. \tag{12}$$

The multivariate Mittag-Leffler function $E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n)$ of n complex variables $z_1,\ldots,z_n \in \mathbb{C}$ with complex parameters $a_1,\ldots,a_n,b \in \mathbb{C}$ (with positive real parts) is defined by (see [31])

$$E_{(a_1,\ldots,a_n),b}(z_1,\ldots,z_n) = \sum_{k=0}^{+\infty} \sum_{\substack{l_1+\ldots+l_n=k \\ l_1,\ldots,l_n\geq 0}} \binom{k}{l_1,\ldots,l_n} \frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma\left(b+\sum_{i=1}^{n} a_i l_i\right)}, \tag{13}$$

where the multinomial coefficients are given by

$$\binom{k}{l_1,\ldots,l_n} := \frac{k!}{l_1! \times \ldots \times l_n!}.$$

When $n=2$ we obtain the bivariate Mittag-Leffler function which can be written as

$$E_{(a_1,a_2),b}(z_1,z_2) = \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} \frac{(l_1+l_2)!}{l_1!\, l_2!} \frac{z_1^{l_1}\, z_2^{l_2}}{\Gamma\left(b+a_1 l_1+a_2 l_2\right)}. \tag{14}$$

The Fox H-function $H_{p,q}^{m,n}(z)$ is defined by means of a Mellin-Barnes type integral

$$H_{p,q}^{m,n}\left[z \;\middle|\; \begin{matrix} (a_1,\alpha_1),\ldots,(a_p,\alpha_p) \\ \\ (b_1,\beta_1),\ldots,(b_q,\beta_q) \end{matrix} \right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma\left(b_j+\beta_j s\right)\ \prod_{i=1}^{n}\Gamma\left(1-a_i-\alpha_i s\right)}{\prod_{i=n+1}^{p}\Gamma\left(a_i+\alpha_i s\right)\ \prod_{j=m+1}^{q}\Gamma\left(1-b_j-\beta_j s\right)}\ z^{-s}\ ds, \tag{15}$$

where an empty product is interpreted as 1. Moreover, $z^{-s} = \exp\left[-s\left(\ln|z| + i\arg z\right)\right]$, where $\arg z$ is not necessarily the principal value, and the parameters $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}^+$, for $i=1,\ldots,p$ and $j=1,\ldots,q$, are restricted that none of the poles of the integrand coincide. The integration path $\mathcal{L}$ is the infinite contour, indented if necessary, separating all the left poles of the Gamma functions $\Gamma\left(b_j+\beta_j s\right)$, $j=1,\ldots,m$ from the right poles of the Gamma functions $\Gamma\left(1-a_i-\alpha_i s\right)$, $i=1,\ldots,n$, and has one of the following forms:

- $\mathcal{L}=\mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty+i\varphi_1$ and terminating at the point $-\infty+i\varphi_2$ with $-\infty<\varphi_1<\varphi_2<+\infty$;
- $\mathcal{L}=\mathcal{L}_{+\infty}$ is a right loop situated in a horizontal strip starting at the point $+\infty+i\varphi_1$ and terminating at the point $+\infty+i\varphi_2$ with $-\infty<\varphi_1<\varphi_2<+\infty$;
- $\mathcal{L}=\mathcal{L}_{i\gamma\infty}$ is a contour starting at the point $\gamma-i\infty$ and terminating at the point $\gamma+i\infty$, where $\gamma\in\mathbb{R}$.

The conditions for the analyticity and convergence of the Fox H-function, and the orientation of the contour[1] $\mathcal{L}$ are presented in Theorems 1.1 and 1.2 in [29].

The H-function (15) was extended to several complex variables. For the case of two complex variables it is defined via a double Mellin-Barnes type integral of the form (see [7])

$$\begin{aligned} &H_{p_1,q_1;\,p_2,q_2;\,p_3,q_3}^{0,n_1;\,m_2,n_2;\,m_3,n_3}\left[\begin{matrix} z_1 \\ \\ z_2 \end{matrix} \;\middle|\; \begin{matrix} \left(a_j;\alpha_j,A_j\right)_{1,p_1};\ \left(c_j,\gamma_j\right)_{1,p_2};\ \left(e_j,E_j\right)_{1,p_3} \\ \\ \left(b_j;\beta_j,B_j\right)_{1,q_1};\ \left(d_j,\delta_j\right)_{1,q_2};\ \left(f_j,F_j\right)_{1,q_3} \end{matrix} \right] \\ &\qquad = \frac{1}{(2\pi i)^2}\int_{\mathcal{L}_2}\int_{\mathcal{L}_1} \phi(s,w)\ \phi_1(s)\ \phi_2(w)\ z_1^s\, z_2^w\, ds\, dw, \end{aligned} \tag{16}$$

where

$$\phi(s,w) = \frac{\prod_{i=1}^{n_1}\Gamma\left(1-a_i+\alpha_i s+A_i w\right)}{\prod_{i=n_1+1}^{p_1}\Gamma\left(a_i-\alpha_i s-A_i w\right)\ \prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j s+B_j w\right)},$$

$$\phi_1(s) = \frac{\prod_{j=1}^{m_2}\Gamma\left(d_j-\delta_j s\right)\ \prod_{i=1}^{n_2}\Gamma\left(1-c_i+\gamma_i s\right)}{\prod_{j=m_2+1}^{q_2}\Gamma\left(1-d_j+\delta_j s\right)\ \prod_{i=n_2+1}^{p_2}\Gamma\left(c_i-\gamma_i s\right)},$$

$$\phi_2(w) = \frac{\prod_{j=1}^{m_3}\Gamma\left(f_j-F_j w\right)\ \prod_{i=1}^{n_3}\Gamma\left(1-e_i+E_i w\right)}{\prod_{j=m_3+1}^{q_3}\Gamma\left(1-f_j+F_j w\right)\ \prod_{i=n_3+1}^{p_3}\Gamma\left(e_i-E_i w\right)}.$$

[1]To see some examples of the contour $\mathcal{L}$ in particular cases, the interested reader can consult the webpage `https://homepage.tudelft.nl/11r49/teaching/specfunc/hyper/barnes.html`

Here, $x, y \in \mathbb{C}$, $m_i, n_i, p_i, q_i \in \mathbb{Z}$ such that $0 \leq m_i \leq q_i$, $0 \leq n_i \leq p_i$ $(i = 1, 2, 3)$; $a_i, b_j, c_i, d_j, e_i, f_j \in \mathbb{C}$, $\alpha_i, A_i, \beta_j, B_j, \gamma_i, \delta_j, E_i, F_j \in \mathbb{R}^+$, an empty product is interpreted as 1, and the sequence of parameters (a_i), (b_j), (c_i), (d_j), (e_i), and (f_j) are restricted that none of the poles of the integrand coincide. The contour $\mathcal{L}_1$ in the complex s-plane, and the contour $\mathcal{L}_2$ in the complex w-plane, are of Mellin-Barnes type with indentations, if necessary, to ensure that they separate one set of poles from the other. The conditions for the analyticity and convergence of this special function, its general properties, and the orientation of the contours $\mathcal{L}_1$ and $\mathcal{L}_2$ are studied in [7, 25].

In this work, some integral transforms are used, namely, the ψ-Laplace, the Fourier, and the Mellin transforms. The ψ-Laplace transform of a real valued function f with respect to ψ is defined by (see [45, Def. 13])

$$\mathcal{L}_\psi \{f(t)\}(\mathbf{s}) = \widetilde{f}_\psi(\mathbf{s}) = \int_0^{+\infty} e^{-\mathbf{s}\,\psi(t)}\, \psi'(t)\, f(t)\, dt, \quad \operatorname{Re}(\mathbf{s}) \in \mathbb{C}$$

where ψ is a positive monotone increasing function in $\mathbb{R}_0^+$ such that $\psi(0^+) = 0$. The ψ-Laplace transform may be written as the following operator composition involving the classical Laplace transform (cf. [45, Thm. 4]) $\mathcal{L}_\psi = \mathcal{L} \circ Q_{\psi^{-1}}$, where $\left(Q_{\psi^{-1}} f\right)(t) = f\left(\psi^{-1}(t)\right)$ is the composition operator of f with ψ^{-1}. As a consequence, if f is a function whose classical Laplace transform is $\widetilde{f}(\mathbf{s})$, then the ψ-Laplace transform of $f(\psi(t))$ is also $\widetilde{f}(\mathbf{s})$ (see [45, Cor. 2])

$$\mathcal{L}\{f(t)\}(\mathbf{s}) = \widetilde{f}(\mathbf{s}) \quad \Rightarrow \quad \mathcal{L}_\psi\{f(\psi(t))\}(\mathbf{s}) = \widetilde{f}(\mathbf{s}).$$

Concerning the inverse ψ-Laplace transform, it can be written as the following operator composition

$$\mathcal{L}_\psi^{-1} = Q_\psi \circ \mathcal{L}^{-1}, \quad \text{where } (Q_\psi f)(t) = f(\psi(t)), \tag{17}$$

i.e.,

$$\mathcal{L}_\psi^{-1}\left\{\widetilde{f}_\psi(\mathbf{s})\right\}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\mathbf{s}\,\psi(t)}\, \widetilde{f}_\psi(\mathbf{s})\, d\mathbf{s},$$

where $\operatorname{Re}(\mathbf{s}) = c$. We note that the definition of the ψ-Laplace can be adapted to any interval $[a, +\infty[\subseteq \mathbb{R}_0^+$, with ψ satisfying $\psi(a^+) = 0$. This is important in our work so that the ψ-Hilfer derivative covers the largest number of fractional derivatives. When the ψ-Laplace transform is applied to the ψ-Hilfer derivative, we get (see [45, Thm. 6])

$$\mathcal{L}_\psi\left\{{}^H\mathbb{D}_{a^+}^{\alpha,\mu;\psi} f(t)\right\}(\mathbf{s}) = \mathbf{s}^\alpha\, \widetilde{f}_\psi(\mathbf{s}) - \sum_{j=0}^{m-1} \mathbf{s}^{m-\mu(m-\alpha)-1-j}\left(I_{a^+}^{(1-\mu)(m-\alpha)-j,\psi} f\right)(a^+), \tag{18}$$

where $m = [\alpha] + 1$ and the initial-value terms $\left(I_{a^+}^{(1-\mu)(m-\alpha)-j,\psi} f\right)(a^+)$ are evaluated at the limit $t \to a^+$. The ψ-Laplace convolution of two functions is defined by (see [45, Def. 15])

$$(f *_\psi g)(t) = \int_0^t f\left(\psi^{-1}(\psi(t) - \psi(w))\right)\, \psi'(w)\, g(w)\, dw, \qquad t \in \mathbb{R}^+, \tag{19}$$

and the correspondent Convolution Theorem is (see [45, Thm. 8])

$$\mathcal{L}_\psi\{(f *_\psi g)(t)\}(\mathbf{s}) = \mathcal{L}_\psi\{f\}(\mathbf{s})\, \mathcal{L}_\psi\{g\}(\mathbf{s}). \tag{20}$$

The ψ-Laplace transform of the power function is given by (see (2.1.1.1) in [38])

$$\mathcal{L}_\psi\left\{\psi(t)^{\nu-1}\right\}(\mathbf{s}) = \frac{\Gamma(\nu)}{\mathbf{s}^\nu}, \qquad \nu > 0. \tag{21}$$

For the exponential function we have (see 2.2.2.5 in [38])

$$\mathcal{L}_\psi\left\{\psi(t)^{-\frac{3}{2}}\, e^{-\frac{\nu}{\psi(t)}}\right\}(\mathbf{s}) = \sqrt{\frac{\pi}{\nu}}\, e^{-2\sqrt{\nu \mathbf{s}}}, \qquad \nu > 0. \tag{22}$$

From expression (5.1.33) in [20], the ψ-Laplace transform of the three parameter Mittag-Leffler function is given by

$$\mathcal{L}_{\psi}\left\{\psi\left(t\right)^{\beta_2-1}\ E_{\beta_1,\beta_2}^{\beta_3}\left(\lambda\psi\left(t\right)^{\beta_1}\right)\right\}(\mathbf{s})\ =\frac{\mathbf{s}^{\beta_1\beta_3-\beta_2}}{\left(\mathbf{s}^{\beta_1}-\lambda\right)^{\beta_3}}. \tag{23}$$

Moreover, from relation (17.6) in [26] we have that

$$\mathcal{L}_{\psi}\left\{\psi\left(t\right)^{\alpha-\gamma}\sum_{p=0}^{+\infty}\left(-a\psi\left(t\right)^{\alpha-\beta}\right)^{p}E_{\alpha,\alpha+(\alpha-\beta)p-\gamma+1}^{p+1}\left(-b\psi\left(t\right)^{\alpha}\right)\right\}(\mathbf{s})=\frac{\mathbf{s}^{\gamma-1}}{\mathbf{s}^{\alpha}+a\mathbf{s}^{\beta}+b}, \tag{24}$$

where $\mathrm{Re}\left(\alpha\right),\mathrm{Re}\left(\beta\right),\mathrm{Re}\left(\gamma\right)\in\mathbb{R}^{+}$, $\left|\frac{a\mathbf{s}^{\beta}}{\mathbf{s}^{\alpha}+\beta}\right|<1$, and provided that the series in (24) is convergent.

The second integral transform of foremost importance that we will use is the n-dimensional Fourier transform. For a real-valued integrable function f on $\mathbb{R}^n$, it is defined by (see [28])

$$\mathcal{F}\left\{f\left(x\right)\right\}(\kappa)\ =\widehat{f}\left(\kappa\right)\ =\int_{\mathbb{R}^n}e^{i\kappa\cdot x}\,f\left(x\right)\,dx,\qquad x,\kappa\in\mathbb{R}^n,$$

while the corresponding inverse Fourier transform is formally given by

$$f\left(x\right)\ =\mathcal{F}^{-1}\left\{\widehat{f}\left(\kappa\right)\right\}(x)\ =\frac{1}{\left(2\pi\right)^{n}}\int_{\mathbb{R}^n}e^{-i\kappa\cdot x}\,\widehat{f}\left(\kappa\right)\,d\kappa. \tag{25}$$

Associated with the Fourier transform we have the well-known Convolution Theorem:

$$\mathcal{F}\left\{\left(f*_x g\right)(x)\right\}(\kappa)\ =\mathcal{F}\left\{f\right\}(\kappa)\ \mathcal{F}\left\{g\right\}(\kappa), \tag{26}$$

where the convolution $*_x$ is given by

$$\left(f*_x g\right)(x)\ =\int_{\mathbb{R}^n}f\left(x-z\right)\,g\left(z\right)\,dz. \tag{27}$$

The n-dimensional Laplace operator $\Delta_x=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ has Fourier symbol $-\left|\kappa\right|^2$, i.e.:

$$\mathcal{F}\left\{\Delta_x f\left(x\right)\right\}(\kappa)\ =-\left|\kappa\right|^2\mathcal{F}\left\{f\left(x\right)\right\}(\kappa). \tag{28}$$

For the Cauchy distribution in $\mathbb{R}$ we have the following Fourier relation:

$$\mathcal{F}\left\{\frac{a_1}{a_2}\,e^{-a_2|x|}\right\}(\kappa)\ =\frac{2a_1}{a_2^2+\kappa^2},\quad x,\,\kappa\in\mathbb{R},\quad a_1,\,a_2>0. \tag{29}$$

Another important integral transform that we use in this work is the Mellin transform. For f locally integrable on $]0,+\infty[$ it is defined by (see [28])

$$\mathcal{M}\left\{f\left(w\right)\right\}(s)\ =f^{*}\left(s\right)\ =\int_0^{+\infty}w^{s-1}\,f\left(w\right)\,dw,\qquad s\in\mathbb{C}, \tag{30}$$

and the inverse Mellin transform is formally given by

$$f\left(w\right)\ =\mathcal{M}^{-1}\left\{f^{*}\left(s\right)\right\}(w)\ =\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}w^{-s}\,f\left(s\right)\,ds,\qquad w>0,\quad c=\mathrm{Re}\left(s\right). \tag{31}$$

The condition for the existence of (30) is that $-p<c<q$ (called the fundamental strip), where p, q, are the order of f at the origin and $+\infty$, respectively. The integration in (31) is performed along the imaginary axis and the result does not depend on the choice of c inside the fundamental strip. More information about this transform and its properties can be found, for example, in [28]. The Mellin convolution between two functions is defined by

$$\left(f*_{\mathcal{M}}g\right)(x)\ =\int_0^{+\infty}f\left(\frac{x}{u}\right)\,g\left(u\right)\,\frac{du}{u}, \tag{32}$$

and satisfies the Mellin Convolution Theorem (see formula (1.4.40) in [28])

$$\mathcal{M}\left\{f*_{\mathcal{M}}g\right\}(s)=\mathcal{M}\left\{f\right\}(s)\ \mathcal{M}\left\{g\right\}(s).$$

The following relation holds (see (1.4.30) in [28])

$$\mathcal{M}\left\{f\left(\frac{1}{x}\right)\right\}(s) = \mathcal{M}\{f\}(-s). \tag{33}$$

The Mellin-transform of the three-parameter Mittag-Leffler function is given by (see Formula (4.9.3) in [20])

$$\mathcal{M}\left\{E_{\beta_1,\beta_2}^{\beta_3}(-z)\right\}(s) = \frac{\Gamma(s)\,\Gamma(\beta_3 - s)}{\Gamma(\beta_3)\,\Gamma(\beta_2 - \beta_1 s)}. \tag{34}$$

Throughout the paper, we assume that all the involved functions are ψ-Laplace, Fourier, and Mellin transformable.

3 Generalized time-fractional telegraph equation with ψ-Hilfer derivatives

In this work, we consider the following time-fractional telegraph equation

$$c_2\, {}^H\partial_{t,a^+}^{\alpha_2,\mu_2;\psi} u(x,t) + c_1\, {}^H\partial_{t,a^+}^{\alpha_1,\mu_1;\psi} u(x,t) - c_0^2\,\Delta_x u(x,t) + d^2\, u(x,t) = q(x,t), \tag{35}$$

subject to the following initial and boundary conditions

$$\lim_{|x|\mapsto+\infty} u(x,t) = 0 \qquad \left(I_{t,a^+}^{(1-\mu_1)(1-\alpha_1);\psi} u\right)(x,a^+) = f(x) \tag{36}$$

$$\left(I_{t,a^+}^{(1-\mu_2)(2-\alpha_2);\psi} u\right)(x,a^+) = g_1(x) \qquad \frac{\partial}{\partial t}\left[\left(I_{t,a^+}^{(1-\mu_2)(2-\alpha_2);\psi} u\right)\right](x,a^+) = g_2(x), \tag{37}$$

where the second condition in (36) and the conditions (37) are evaluated at the limit $t \to a^+$. Moreover, $c_1 \geq 0$, $c_2 > 0$, $d \in \mathbb{R}$, $(x,\, t) \in \mathbb{R}^n \times I$, with $I = [a,b]$ being a finite or infinite interval on $\mathbb{R}^+$, Δ_x is the classical Laplace operator in $\mathbb{R}^n$, the partial time-fractional derivatives of orders $\alpha_1 \in]0,1]$ and $\alpha_2 \in]1,2]$, and types $\mu_1, \mu_2 \in [0,1]$, respectively, are the ψ-Hilfer derivatives given by (2), ψ is a function under the conditions of Definition 2.2, q belongs to $L_1(\mathbb{R}^n \times I)$, and $f, g_1, g_2 \in L_1(\mathbb{R}^n)$. We look for solutions $u(x,t)$ of our problem in the space $C^2(\mathbb{R}^n) \times C^2(a,b)$ with possible exception at $x = 0$.

To obtain the analytical solution of (35)-(37) we start by applying to (35) the Fourier transform to the space variable x and the ψ-Laplace transform to the variable t, and then we solve the equation in the Fourier-Laplace domain. After that, we invert the ψ-Laplace transform and then invert the Fourier transform of the result. For the inversion of the ψ-Laplace transform we take into account the operational rules presented in [45], while the inversion of the Fourier transform is performed via the Mellin transform.

3.1 Solution in the Fourier-Laplace domain

Let us start by applying to (35) the ψ-Laplace transform with respect to the time variable $t \in I$ and the n-dimensional Fourier transform with respect to the space variable $x \in \mathbb{R}^n$. Taking into account relations (18) and (28), and the initial conditions in (36) and (37), then (35) transforms in the Fourier-Laplace domain to the equation

$$\begin{aligned} &c_2\,\mathbf{s}^{\alpha_2}\,\widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s}) - c_2\,\widehat{g}_1(\kappa)\,\mathbf{s}^{1-\mu_2(2-\alpha_2)} - c_2\,\widehat{g}_2(\kappa)\,\mathbf{s}^{-\mu_2(2-\alpha_2)} \\ &+ c_1\,\mathbf{s}^{\alpha_1}\,\widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s}) - c_1\,\widehat{f}(\kappa)\,\mathbf{s}^{-\mu_1(1-\alpha_1)} + c_0^2\,|\kappa|^2\,\widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s}) + d^2\,\widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s}) = \widehat{\widetilde{q}}_\psi(\kappa,\mathbf{s}). \end{aligned} \tag{38}$$

Solving the above equation in order to $\widehat{\widetilde{u}}_\psi$, we obtain:

$$\begin{aligned} \widehat{\widetilde{u}}_\psi(\kappa,\mathbf{s}) = {}& \frac{c_1}{c_2}\widehat{f}(\kappa)\,\frac{\mathbf{s}^{-\mu_1(1-\alpha_1)}}{\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\mathbf{s}^{\alpha_1} + \frac{c_0^2}{c_2}\,|\kappa|^2 + \frac{d^2}{c_2}} + \widehat{g}_1(\kappa)\,\frac{\mathbf{s}^{1-\mu_2(2-\alpha_2)}}{\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\mathbf{s}^{\alpha_1} + \frac{c_0^2}{c_2}\,|\kappa|^2 + \frac{d^2}{c_2}} \\ &+ \widehat{g}_2(\kappa)\,\frac{\mathbf{s}^{-\mu_2(2-\alpha_2)}}{\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\mathbf{s}^{\alpha_1} + \frac{c_0^2}{c_2}\,|\kappa|^2 + \frac{d^2}{c_2}} + \frac{1}{c_2}\widehat{\widetilde{q}}_\psi(\kappa,\mathbf{s})\,\frac{1}{\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\mathbf{s}^{\alpha_1} + \frac{c_0^2}{c_2}\,|\kappa|^2 + \frac{d^2}{c_2}}, \end{aligned} \tag{39}$$

where $\widehat{f}$, $\widehat{g}_1$, and $\widehat{g}_2$ are the Fourier transforms of the functions f, g_1, and g_3, respectively. Expression (39) corresponds to the solution in the Fourier-Laplace domain of our problem (35)-(37).

3.2 Solution in the space-time domain

Now, we show how to obtain our solution in the space-time domain. Applying the inverse ψ-Laplace transform to (39) and taking into account (24) and (20), we obtain

$$
\begin{aligned}
&\widehat{u}(\kappa,t)\\
&=\frac{c_1}{c_2}\,\widehat{f}(\kappa)\,\psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)}\\
&\qquad\times\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^{p}E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_1(1-\alpha_1)}\left(-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\\
&\quad+\widehat{g}_1(\kappa)\,\psi(t)^{\alpha_2-2+\mu_2(2-\alpha_2)}\\
&\qquad\times\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^{p}E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p-1+\mu_2(2-\alpha_2)}\left(-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\\
&\quad+\widehat{g}_2(\kappa)\,\psi(t)^{\alpha_2-1+\mu_2(2-\alpha_2)}\\
&\qquad\times\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^{p}E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_2(2-\alpha_2)}\left(-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\\
&\quad+\frac{1}{c_2}\,\widehat{q}(\kappa,\psi(t))*_{\psi}\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^{p}\psi(t)^{\alpha_2-1}\,E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p}\left(-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right),
\end{aligned}
\tag{40}
$$

where the ψ-convolution is given by (19). From the definition of the bivariate Mittag-Leffler function (see (14)) we can rewrite (40) as

$$
\begin{aligned}
&\widehat{u}(\kappa,t)\\
&=\frac{c_1}{c_2}\,\widehat{f}(\kappa)\,\psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)}\,E_{(\alpha_2-\alpha_1,\alpha_2),\alpha_2+\mu_1(1-\alpha_1)}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1},-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\\
&\quad+\widehat{g}_1(\kappa)\,\psi(t)^{\alpha_2-2+\mu_2(2-\alpha_2)}\,E_{(\alpha_2-\alpha_1,\alpha_2),\alpha_2-1+\mu_2(2-\alpha_2)}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1},-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\\
&\quad+\widehat{g}_2(\kappa)\,\psi(t)^{\alpha_2-1+\mu_2(2-\alpha_2)}\,E_{(\alpha_2-\alpha_1,\alpha_2),\alpha_2+\mu_2(2-\alpha_2)}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1},-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\\
&\quad+\frac{1}{c_2}\,\widehat{q}(\kappa,\psi(t))*_{\psi}\left[\psi(t)^{\alpha_2-1}\,E_{(\alpha_2-\alpha_1,\alpha_2),\alpha_2}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1},-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\right].
\end{aligned}
\tag{41}
$$

Therefore, in the Fourier domain the solution involves series of three-parameter Mittag-Leffler functions of one variable or just bivariate Mittag-Leffler functions. It follows from the following asymptotic expansion of $E^{\beta_3}_{\beta_1,\beta_2}(-x)$ in the negative semi-axes (see Theorem 5.4 in [20])

$$
E^{\beta_3}_{\beta_1,\beta_2}(-x)\sim\frac{x^{-\beta_3}}{\Gamma(\beta_3)}\sum_{k=0}^{+\infty}\frac{(-1)^k\,\Gamma(k+x)}{\Gamma(\beta_2-\beta_1(k+\beta_3))}\,x^{-k},\quad t\to+\infty,
$$

where $0<\beta_1<2$, that the three-parameter Mittag-Leffler functions in (41) belong to the space $L_1(\mathbb{R}^n)$. Hence, applying the inverse Fourier transform and taking into account (25) and (26), we obtain from (40)

$$
\begin{aligned}
&u(x,t)\\
&=\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)}\\
&\quad\times\sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^{p}f(x)*_x\mathcal{F}^{-1}\left\{E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_1(1-\alpha_1)}\left(-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\right\}(x,t)
\end{aligned}
$$

$$
\begin{aligned}
&+\psi(t)^{\alpha_2-2+\mu_2(2-\alpha_2)} \\
&\times \sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p g_1(x) *_x \mathcal{F}^{-1}\left\{E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p-1+\mu_2(2-\alpha_2)}\left(-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\right\}(x,t) \\
&+\psi(t)^{\alpha_2-1+\mu_2(2-\alpha_2)} \\
&\times \sum_{p=0}^{+\infty}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p g_2(x) *_x \mathcal{F}^{-1}\left\{E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_2(2-\alpha_2)}\left(-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\right\}(x,t) \\
&+\frac{1}{c_2}q(x,\psi(t)) *_x *_\psi \sum_{p=0}^{+\infty}\left\{\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p\right. \\
&\qquad\qquad \left.\times\left[\psi(t)^{\alpha_2-1}\;\mathcal{F}^{-1}\left\{E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p}\left(-\frac{1}{c_2}\left(c_0^2\,|\kappa|^2+d^2\right)\psi(t)^{\alpha_2}\right)\right\}(x,t)\right]\right\}. \quad (42)
\end{aligned}
$$

Considering the following formula presented in [39] for the inverse Fourier transform of radial functions:

$$
\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot\kappa}\,\varphi(|\kappa|)\,d\kappa=\frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}\varphi(w)\,w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|w)\,dw, \tag{43}
$$

which is valid for any function $\varphi\in L_1(\mathbb{R}^n)$ (see Lemma 25.1 in [39]), we obtain the following result.

Theorem 3.1 *The solution of the generalized time-fractional telegraph equation with ψ-Hilfer derivatives* (35) *subject to the conditions* (36) *and* (37) *is given, in terms of convolution integrals, by*

$$
\begin{aligned}
&u(x,t) \\
&=\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)}\sum_{p=0}^{+\infty}\left\{\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p\right. \\
&\quad\left.\times f(x) *_x\left[\frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_1(1-\alpha_1)}\left(-\frac{1}{c_2}\left(c_0^2\,w^2+d^2\right)\psi(t)^{\alpha_2}\right)w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|w)\,dw\right]\right\} \\
&\quad+\psi(t)^{\alpha_2-2+\mu_2(2-\alpha_2)}\sum_{p=0}^{+\infty}\left\{\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p\right. \\
&\quad\left.\times g_1(x) *_x\left[\frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p-1+\mu_2(2-\alpha_2)}\left(-\frac{1}{c_2}\left(c_0^2\,w^2+d^2\right)\psi(t)^{\alpha_2}\right)w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|w)\,dw\right]\right\} \\
&\quad+\psi(t)^{\alpha_2-1+\mu_2(2-\alpha_2)}\sum_{p=0}^{+\infty}\left\{\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p\right. \\
&\quad\left.\times g_2(x) *_x\left[\frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_2(2-\alpha_2)}\left(-\frac{1}{c_2}\left(c_0^2\,w^2+d^2\right)\psi(t)^{\alpha_2}\right)w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|w)\,dw\right]\right\} \\
&+\frac{1}{c_2}\sum_{p=0}^{+\infty}\left\{\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p\right. \\
&\left.\times q(x,\psi(t)) *_x *_\psi\left[\psi(t)^{\alpha_2-1}\,\frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty}E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p}\left(-\frac{1}{c_2}\left(c_0^2\,w^2+d^2\right)\psi(t)^{\alpha_2}\right)w^{\frac{n}{2}}\,J_{\frac{n}{2}-1}(|x|w)\,dw\right]\right\},
\end{aligned} \tag{44}
$$

where the convolutions $*_x$ and $*_\psi$ are defined by (27) *and* (19)*, respectively.*

The consideration of the parameter d in (42) leads to cumbersome formulas for the inverse Fourier transform. To avoid this, we will consider from now on the case $d=0$. The following lemma shows a fundamental formula for the explicit Fourier inversion of the terms in (42).

Lemma 3.2 *Let $\beta_1, \beta_2 \in \mathbb{C}$ such that $\mathrm{Re}(\beta_1) > 0$, $\tau \in \mathbb{R}^+$, and $\kappa \in \mathbb{R}^n$. The following multidimensional Fourier-type relation is valid*

$$\mathcal{F}^{-1}\left\{E^{\beta_3}_{\beta_1,\beta_2}\left(-\tau\,|\kappa|^2\right)\right\}(x) \;=\; \frac{1}{\pi^{\frac{n}{2}}\,|x|^n\,\Gamma(\beta_3)}\, H^{0,2}_{2,1}\left[\begin{array}{c|c} \dfrac{4\tau}{|x|^2} & \begin{array}{c}(1-n,1)\,,\,(1-\beta_3,1)\\ \\ (1-\beta_2,\beta_1)\end{array}\end{array}\right], \tag{45}$$

where H is the Fox H-function defined in (15).

Proof: We start the proof by noting that we want to apply the inverse Fourier transform to a radial function in the variable κ. Then by (42) we have

$$\begin{aligned}\mathcal{F}^{-1}\left\{E^{\beta_3}_{\beta_1,\beta_2}\left(-\tau\,|\kappa|^2\right)\right\}(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot\kappa}\, E^{\beta_3}_{\beta_1,\beta_2}\left(-\tau\,|\kappa|^2\right)\, d\kappa\\ &= \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty} E^{\beta_3}_{\beta_1,\beta_2}\left(-\tau w^2\right)\, w^{\frac{n}{2}}\, J_{\frac{n}{2}-1}(|x|w)\, dw. \end{aligned}\tag{46}$$

To explicitly compute the integral on the right-hand side of (46) we will use the Mellin transform. First, we rewrite the integral as a Mellin convolution. In fact, considering

$$h_1(w) \;=E^{\beta_3}_{\beta_1,\beta_2}\left(-\tau w^2\right) \qquad \text{and} \qquad h_2(w) \;= \frac{1}{(2\pi)^{\frac{n}{2}}\,|x| w^{\frac{n}{2}-1}}\, J_{\frac{n}{2}-1}\left(\frac{1}{w}\right), \tag{47}$$

and the definition of the Mellin convolution in (32), we have

$$\begin{aligned}\mathcal{M}\left\{h_1 *_{\mathcal{M}} h_2\right\}\left(\frac{1}{|x|}\right) &= \int_0^{+\infty} h_1(w)\; h_2\left(\frac{1}{|x|w}\right)\,\frac{dw}{w}\\ &= \int_0^{+\infty} E^{\beta_3}_{\beta_1,\beta_2}\left(-\tau w^2\right)\,\frac{w^{\frac{n}{2}+1}\,|x|^{\frac{n}{2}+1}}{(2\pi)^{\frac{n}{2}}\,|x|^n}\, J_{\frac{n}{2}-1}(|x|w)\,\frac{dw}{w}\\ &= \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}}\int_0^{+\infty} E^{\beta_3}_{\beta_1,\beta_2}\left(-\tau w^2\right)\, w^{\frac{n}{2}}\, J_{\frac{n}{2}-1}(|x|w)\, dw.\end{aligned}$$

From (33) it follows that

$$\mathcal{M}\left\{h_1 *_{\mathcal{M}} h_2\right\}(s) \;=\mathcal{M}\left\{h_1\right\}(-s)\;\mathcal{M}\left\{h_2\right\}(-s)\,,$$

which is equivalent to

$$\mathcal{M}\left\{h_1 *_{\mathcal{M}} h_2\right\}(-s) \;=\mathcal{M}\left\{h_1\right\}(s)\;\mathcal{M}\left\{h_2\right\}(s)\,. \tag{48}$$

Let us calculate the Mellin transforms that appear in (48). The Mellin transform of h_2 was already calculated in [49] (see Formula (43)):

$$\mathcal{M}\left\{h_2\right\}(s) \;= \frac{1}{\pi^{\frac{n-1}{2}}\,|x|^n\,2^{n-1}}\,\frac{\Gamma(n-s)}{\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\,\Gamma\left(\frac{s}{2}\right)}. \tag{49}$$

Concerning the Mellin transform of h_1, we take into account the definition of the Mellin transform (see (30))

$$\mathcal{M}\left\{h_1\right\}(s) \;= \int_0^{+\infty} w^{s-1}\, E^{\beta_3}_{\beta_1,\beta_2}\left(-\tau w^2\right)\, dw. \tag{50}$$

Making the change of variables $\tau w^2 = z$ in (50) we obtain

$$\mathcal{M}\left\{h_1\right\}(s) \;= \frac{1}{2\tau^{\frac{s}{2}}}\int_0^{+\infty} z^{\frac{s}{2}-1}\, E^{\beta_3}_{\beta_1,\beta_2}(-z) \;= \frac{1}{2\tau^{\frac{s}{2}}}\,\mathcal{M}\left\{E^{\beta_3}_{\beta_1,\beta_2}(-z)\right\}\left(\frac{s}{2}\right). \tag{51}$$

Taking into account (34), we have from (51)

$$\mathcal{M}\left\{h_1\right\}(s) \;= \frac{1}{2\tau^{\frac{s}{2}}}\,\frac{\Gamma\left(\frac{s}{2}\right)\,\Gamma\left(\beta_3-\frac{s}{2}\right)}{\Gamma(\beta_3)\,\Gamma\left(\beta_2-\frac{\beta_1 s}{2}\right)}. \tag{52}$$

Now, using the inverse Mellin transform defined in (31) applied to (48), together with (49) and (52), we obtain

$$\mathcal{F}^{-1}\left\{E_{\beta_1,\beta_2}^{\beta_3}\left(-\tau\left|\kappa\right|^2\right)\right\}(x)=\frac{1}{\pi^{\frac{n-1}{2}}\left(2\left|x\right|\right)^n\Gamma\left(\beta_3\right)}\frac{1}{2\pi i}\int_{\mathcal{L}_1}\frac{\Gamma\left(n-s\right)\Gamma\left(\beta_3-\frac{s}{2}\right)}{\Gamma\left(\frac{n+1}{2}-\frac{s}{2}\right)\Gamma\left(\beta_2-\frac{\beta_1 s}{2}\right)}\left(\frac{\sqrt{\tau}}{\left|x\right|}\right)^{-s}ds. \tag{53}$$

Making the change of variables $z=-\frac{s}{2}$ in (53) and using the representation of the Fox H-function presented in (15), we finally get

$$\begin{aligned}\mathcal{F}^{-1}\left\{E_{\beta_1,\beta_2}^{\beta_3}\left(-\tau\left|\kappa\right|^2\right)\right\}(x)&=\frac{1}{\pi^{\frac{n}{2}}\left|x\right|^n\Gamma\left(\beta_3\right)}\frac{1}{2\pi i}\int_{\mathcal{L}_1}\frac{\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(\beta_3+s\right)}{\Gamma\left(\beta_2+\frac{\beta_1 s}{2}\right)}\left(\frac{4\tau}{\left|x\right|^2}\right)^s ds\\&=\frac{1}{\pi^{\frac{n}{2}}\left|x\right|^n\Gamma\left(\beta_3\right)}H_{2,1}^{0,2}\left[\frac{4\tau}{\left|x\right|^2}\left|\begin{array}{c}\left(1-\frac{n}{2},1\right),\left(1-\beta_3,1\right)\\ \\ \left(1-\beta_2,\beta_1\right)\end{array}\right.\right],\end{aligned}$$

which corresponds to our result.

■

By the previous lemma we can rewrite Theorem 3.1 in the case $d=0$ using convolution integrals that involve four functions G_1, G_2, G_3, and G_4, that we will describe in the next theorem.

Theorem 3.3 *The solution of the generalized time-fractional telegraph equation with ψ-Hilfer derivative* (35) *with $d=0$, subject to the conditions* (36)-(37) *is given, in terms of convolution integrals involving Fox H-functions, by*

$$\begin{aligned}u\left(x,t\right)=&\int_{\mathbb{R}^n}f\left(z\right)G_1\left(x-z,t\right)dz+\int_{\mathbb{R}^n}g_1\left(z\right)G_2\left(x-z,t\right)dz\\&+\int_{\mathbb{R}^n}g_2\left(z\right)G_3\left(x-z,t\right)dz+\int_{\mathbb{R}^n}\int_0^t q\left(z,\psi\left(t\right)\right)G_4\left(x-z,\psi^{-1}\left(\psi\left(t\right)-\psi\left(w\right)\right)\right)\psi'\left(w\right)dw\,dz,\end{aligned} \tag{54}$$

where G_1, G_2, G_3, and G_4 are given by

$$\begin{aligned}G_1\left(x,t\right)=&\frac{c_1\psi\left(t\right)^{\alpha_2-1+\mu_1\left(1-\alpha_1\right)}}{c_2\,\pi^{\frac{n}{2}}\left|x\right|^n}\\&\times\sum_{p=0}^{+\infty}\frac{1}{p!}\left(-\frac{c_1}{c_2}\psi\left(t\right)^{\alpha_2-\alpha_1}\right)^p H_{2,1}^{0,2}\left[\frac{4c_0^2\,\psi\left(t\right)^{\alpha_2}}{c_2\left|x\right|^2}\left|\begin{array}{c}\left(1-\frac{n}{2},1\right),\left(-p,1\right)\\ \\ \left(1-\alpha_2-\left(\alpha_2-\alpha_1\right)p-\mu_1\left(1-\alpha_1\right),\alpha_2\right)\end{array}\right.\right],\end{aligned} \tag{55}$$

$$\begin{aligned}G_2\left(x,t\right)=&\frac{\psi\left(t\right)^{\alpha_2-2+\mu_2\left(2-\alpha_2\right)}}{\pi^{\frac{n}{2}}\left|x\right|^n}\\&\times\sum_{p=0}^{+\infty}\frac{1}{p!}\left(-\frac{c_1}{c_2}\psi\left(t\right)^{\alpha_2-\alpha_1}\right)^p H_{2,1}^{0,2}\left[\frac{4c_0^2\,\psi\left(t\right)^{\alpha_2}}{c_2\left|x\right|^2}\left|\begin{array}{c}\left(1-\frac{n}{2},1\right),\left(-p,1\right)\\ \\ \left(2-\alpha_2-\left(\alpha_2-\alpha_1\right)p-\mu_2\left(2-\alpha_2\right),\alpha_2\right)\end{array}\right.\right],\end{aligned} \tag{56}$$

$$\begin{aligned}G_3\left(x,t\right)=&\frac{\psi\left(t\right)^{\alpha_2-1+\mu_2\left(2-\alpha_2\right)}}{\pi^{\frac{n}{2}}\left|x\right|^n}\\&\times\sum_{p=0}^{+\infty}\frac{1}{p!}\left(-\frac{c_1}{c_2}\psi\left(t\right)^{\alpha_2-\alpha_1}\right)^p H_{2,1}^{0,2}\left[\frac{4c_0^2\,\psi\left(t\right)^{\alpha_2}}{c_2\left|x\right|^2}\left|\begin{array}{c}\left(1-\frac{n}{2},1\right),\left(-p,1\right)\\ \\ \left(1-\alpha_2-\left(\alpha_2-\alpha_1\right)p-\mu_2\left(2-\alpha_2\right),\alpha_2\right)\end{array}\right.\right],\end{aligned} \tag{57}$$

and

$$G_4(x,t)=\frac{\psi(t)^{\alpha_2-1}}{c_2\,\pi^{\frac{n}{2}}\,|x|^n}$$

$$\times\sum_{p=0}^{+\infty}\frac{1}{p!}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p H_{2,1}^{0,2}\left[\frac{4c_0^2\,\psi(t)^{\alpha_2}}{c_2\,|x|^2}\;\middle|\;\begin{array}{c}\left(1-\frac{n}{2},1\right),\,(-p,1)\\ \\ (1-\alpha_2-(\alpha_2-\alpha_1)\,p,\alpha_2)\end{array}\right]. \tag{58}$$

Formulas (54)-(58) give a very general expression for the solution of our problem, which turns its physical interpretation complicated. However, for special choices of the dimension, the parameters, and the functions involved, the solution may simplify.

Corollary 3.4 *For $c_1=0$ in* (54)-(58) *we obtain the solution of the generalized time-fractional wave equation with ψ-Hilfer derivative, which amounts to consider $G_1(x,t)=0$ in* (55) *and the first term $p=0$ in the series* (56)-(58).

Corollary 3.5 *If we consider in the previous theorem*

$$c_0=c_2=1,\qquad g_2(x)=q(x,t)=0,\qquad f(x)=g_1(x)=\delta(x),\qquad \mu_1=1,\qquad \alpha_1=0,\qquad c_1=-\lambda$$

with $\lambda\in\mathbb{R}$, the solution given by (54) *corresponds to the eigenfunctions of the generalized time-fractional wave equation with ψ-Hilfer derivatives in $\mathbb{R}^n\times\mathbb{R}^+$, with initial and boundary conditions* (36)-(37).

The functions $G_i,\ i=1,2,3,4$ that appear in Theorem 3.3 can be rewritten in terms of Fox H-functions of two variables, thus avoiding the appearance of a series of Fox H-functions of one variable. We show how to proceed to G_1, being the reasoning the same for the other functions. Applying the Residue Theorem to the series in p and Lemma 3.2 to the first term of (44), and taking into account (15) and (16), we get the following representation of G_1 in terms of double Mellin-Barnes integrals and Fox H-function of two variables

$$\begin{aligned}G_1(x,t)\;&=\frac{c_1\psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)}}{c_2\,\pi^{\frac{n}{2}}\,|x|^n}\\ &\times\frac{1}{(2\pi i)^2}\int_{\mathcal{L}_1}\int_{\mathcal{L}_2}\frac{\Gamma(1+w+s)\,\Gamma(-w)\,\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(\alpha_2+\mu_1(1-\alpha_1)+(\alpha_2-\alpha_1)\,w+\alpha_2 s)}\left(\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^w\left(\frac{4c_0^2\,\psi(t)^{\alpha_2}}{c_2\,|x|^2}\right)^s dw\,ds\\ &=\frac{c_1\psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)}}{c_2\,\pi^{\frac{n}{2}}\,|x|^n}\,H_{1,1;1,0;0,1}^{0,1;0,1;1,0}\left[\begin{array}{c}\frac{4c_0^2\,\psi(t)^{\alpha_2}}{c_2\,|x|^2}\\ \\ \frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\end{array}\;\middle|\;\begin{array}{c}(0;1,1);\,\left(1-\frac{n}{2},1\right);\,--\\ \\ (1-\alpha_2-\mu_1(1-\alpha_1);\alpha_2,\alpha_2-\alpha_1);\,--;\,(0,1)\end{array}\right].\end{aligned} \tag{59}$$

The other functions are obtained by the same reasoning applied to the remaining terms of (44):

$$\begin{aligned}&G_2(x,t)\\ &=\frac{\psi(t)^{\alpha_2-2+\mu_2(2-\alpha_2)}}{\pi^{\frac{n}{2}}\,|x|^n}\,H_{1,1;1,0;0,1}^{0,1;0,1;1,0}\left[\begin{array}{c}\frac{4c_0^2\,\psi(t)^{\alpha_2}}{c_2\,|x|^2}\\ \\ \frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\end{array}\;\middle|\;\begin{array}{c}(0;1,1);\,\left(1-\frac{n}{2},1\right);\,--\\ \\ (2-\alpha_2-\mu_2(2-\alpha_2);\alpha_2,\alpha_2-\alpha_1);\,--;\,(0,1)\end{array}\right],\end{aligned} \tag{60}$$

$$\begin{aligned}&G_3(x,t)\\ &=\frac{\psi(t)^{\alpha_2-1+\mu_2(2-\alpha_2)}}{\pi^{\frac{n}{2}}\,|x|^n}\,H_{1,1;1,0;0,1}^{0,1;0,1;1,0}\left[\begin{array}{c}\frac{4c_0^2\,\psi(t)^{\alpha_2}}{c_2\,|x|^2}\\ \\ \frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\end{array}\;\middle|\;\begin{array}{c}(0;1,1);\,\left(1-\frac{n}{2},1\right);\,--\\ \\ (1-\alpha_2-\mu_2(2-\alpha_2);\alpha_2,\alpha_2-\alpha_1);\,--;\,(0,1)\end{array}\right],\end{aligned} \tag{61}$$

and

$$G_4(x,t) = \frac{\psi(t)^{\alpha_2-1}}{c_2\,\pi^{\frac{n}{2}}\,|x|^n}\, H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c|c} \dfrac{4c_0^2\,\psi(t)^{\alpha_2}}{c_2\,|x|^2} & (0;1,1)\,;\,\left(1-\dfrac{n}{2},1\right);\,-- \\ \dfrac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1} & (1-\alpha_2;\alpha_2,\alpha_2-\alpha_1)\,;\,--;\,(0,1) \end{array}\right]. \tag{62}$$

Applying the conditions (3.5) of Theorem 3.1 in [25], we can guarantee that the double Mellin-Barnes integral associated to the Fox H-functions of two variables in (59)-(62) are convergent for all $x\in\mathbb{R}^n$ $c_0,c_1,c_2\in\mathbb{R}^+$, and all admissible functions ψ.

4 Double series representation for the first fundamental solution

In this section we deduce a double series representation for the first fundamental solution $\mathbf{G_1}$ of our problem. This is obtained considering $d=q(x,t)=0$ in (35) and

$$f(x) = g_1(x) = \delta(x) = \prod_{j=1}^{n}\delta(x_j) \qquad \text{and} \qquad g_2(x) = 0 \tag{63}$$

in (36)-(37). Then $\mathbf{G_1}$ is given by $\mathbf{G_1}(x,t) = G_1(x,t)+G_2(x,t)$, where G_1 and G_2 are given by (55) and (56), respectively. Due to the similarities between the expressions of G_1 and G_2, we present only the deduction of the double series representation for G_1, being the reasoning the same for G_2. Taking into account the representation of the Fox H-functions in terms of Mellin-Barnes integrals given in (15), we have

$$\begin{aligned} G_1(x,t) &= \frac{c_1\psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)}}{c_2\,\pi^{\frac{n}{2}}\,|x|^n} \\ &\times\sum_{p=0}^{+\infty}\frac{1}{p!}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p \underbrace{\frac{1}{2\pi i}\int_{\mathcal{L}_1}\frac{\Gamma(1+p-s)\,\Gamma\left(\frac{n}{2}-s\right)}{\Gamma(\alpha_2+\mu_1(1-\alpha_1)+(\alpha_2-\alpha_1)p-\alpha_2 s)}\left(\frac{4c_0^2\,\psi(t)^{\alpha_2}}{c_2\,|x|^2}\right)^{-s}ds}_{\mathbf{I}}, \end{aligned} \tag{64}$$

where $\mathcal{L}_1=\mathcal{L}_{+\infty}$ since $\Delta=\alpha_2-2<0$ (cf. Theorem 1.1 in [29]). To find a series representation for the contour integral $\mathbf{I}$ we need to take into account the poles of the Gamma functions in the numerator. When n is odd we deal only with sequences of simple poles, and when n is even we deal with sequences of simple and/or double poles. Assuming that n is odd, we have two non-coincident infinite sequences of simple poles at $s=q+p+1$ for $q\in\mathbb{N}_0$, and at $s=k+\frac{n}{2}$, for $k\in\mathbb{N}_0$, coming from the gamma functions $\Gamma(1+p-s)$ and $\Gamma\left(\frac{n}{2}-s\right)$, respectively. Therefore, applying the Residue Theorem and taking into account (8) we obtain the following series representation:

$$\begin{aligned} \mathbf{I} &= \sum_{q=0}^{+\infty}\frac{(-1)^q}{q!}\,\frac{\Gamma\left(\frac{n}{2}-1-p-q\right)}{\Gamma(\mu_1(1-\alpha_1)-\alpha_1 p-\alpha_2 q)}\left(\frac{c_2\,|x|^2}{4c_0^2\,\psi(t)^{\alpha_2}}\right)^{p+q+1} \\ &+\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\,\frac{\Gamma\left(1-\frac{n}{2}+p-k\right)}{\Gamma\left(\alpha_2\left(1-\frac{n}{2}\right)+\mu_1(1-\alpha_1)+(\alpha_2-\alpha_1)p-\alpha_2 k\right)}\left(\frac{c_2\,|x|^2}{4c_0^2\,\psi(t)^{\alpha_2}}\right)^{k+\frac{n}{2}}. \end{aligned} \tag{65}$$

The previous series can be combined into a single series. To obtain this, we start by considering the change of variables $m=2q+1$ and $m=2k$ in the first and second series in brackets, respectively. Hence, we get

$$\begin{aligned} \mathbf{I} &= \sum_{\substack{m=1\\ m\text{ odd}}}^{+\infty}\frac{(-1)^{\frac{m-1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}\,\frac{\Gamma\left(\frac{n-1}{2}-p-\frac{m}{2}\right)}{\Gamma\left(\frac{\alpha_2}{2}+\mu_1(1-\alpha_1)-\alpha_1 p-\frac{\alpha_2 m}{2}\right)}\left(\frac{c_2\,|x|^2}{4c_0^2\,\psi(t)^{\alpha_2}}\right)^{p+\frac{m}{2}+\frac{1}{2}} \\ &+\sum_{\substack{m=0\\ m\text{ even}}}^{+\infty}\frac{(-1)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}+1\right)}\,\frac{\Gamma\left(1-\frac{n}{2}+p-\frac{m}{2}\right)}{\Gamma\left(\alpha_2\left(1-\frac{n}{2}\right)+\mu_1(1-\alpha_1)+(\alpha_2-\alpha_1)p-\frac{\alpha_2 m}{2}\right)}\left(\frac{c_2\,|x|^2}{4c_0^2\,\psi(t)^{\alpha_2}}\right)^{\frac{m}{2}+\frac{n}{2}}. \end{aligned} \tag{66}$$

To have the same exponent in both series we consider the changes $m = r + n - 1 - 2p$ and $m = r$ in the first and the second series, respectively, resulting in

$$\begin{aligned}\mathbf{I} = &\sum_{\substack{r=2p-n+2\\ r \text{ odd}}}^{-1} \frac{(-1)^{\frac{r+n}{2}-p-1}}{\Gamma\left(\frac{r+n}{2}-p\right)} \frac{\Gamma\left(-\frac{r}{2}\right)}{\Gamma\left(\alpha_2\left(1-\frac{n}{2}\right)+\mu_1\left(1-\alpha_1\right)+\left(\alpha_2-\alpha_1\right)p-\frac{\alpha_2 r}{2}\right)} \left(\frac{c_2\,|x|^2}{4c_0^2\,\psi\left(t\right)^{\alpha_2}}\right)^{\frac{r+n}{2}}\\
&+\sum_{\substack{r=1\\ r \text{ odd}}}^{+\infty} \frac{(-1)^{\frac{r+n}{2}-p-1}}{\Gamma\left(\frac{r+n}{2}-p\right)} \frac{\Gamma\left(-\frac{r}{2}\right)}{\Gamma\left(\alpha_2\left(1-\frac{n}{2}\right)+\mu_1\left(1-\alpha_1\right)+\left(\alpha_2-\alpha_1\right)p-\frac{\alpha_2 r}{2}\right)} \left(\frac{c_2\,|x|^2}{4c_0^2\,\psi\left(t\right)^{\alpha_2}}\right)^{\frac{r+n}{2}}\\
&+\sum_{\substack{r=0\\ r \text{ even}}}^{+\infty} \frac{(-1)^{\frac{r}{2}}}{\Gamma\left(\frac{r}{2}+1\right)} \frac{\Gamma\left(1+p-\frac{r+n}{2}\right)}{\Gamma\left(\alpha_2\left(1-\frac{n}{2}\right)+\mu_1\left(1-\alpha_1\right)+\left(\alpha_2-\alpha_1\right)p-\frac{\alpha_2 r}{2}\right)} \left(\frac{c_2\,|x|^2}{4c_0^2\,\psi\left(t\right)^{\alpha_2}}\right)^{\frac{r+n}{2}}.\end{aligned} \tag{67}$$

Now, we analyse the coefficients of the odd and even series. For odd r, using (4) and after straightforward calculations we obtain

$$\frac{(-1)^{\frac{r+n}{2}-p-1}\ \Gamma\left(-\frac{r}{2}\right)}{\Gamma\left(\frac{r+n}{2}-p\right)\ 2^{r+n}} = \frac{(-1)^{\frac{r+n-2}{2}-p}\ \Gamma\left(-\frac{r}{2}\right)\ \left(1-\frac{r+n}{2}\right)_p}{\Gamma\left(\frac{r+n}{2}\right)\ 2^{r+n}}. \tag{68}$$

Considering the following equality proved in [19, Sec. 3.2.1] for r odd

$$\frac{(-1)^{\frac{r+n-2}{2}}\ \Gamma\left(-\frac{r}{2}\right)}{\Gamma\left(\frac{r+n}{2}\right)} = -\frac{(-1)^{\frac{n-1}{2}}\ \sqrt{\pi}\ 2^r}{\left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}\ r!}$$

we obtain that

$$\frac{(-1)^{\frac{r+n}{2}-p-1}\ \Gamma\left(-\frac{r}{2}\right)}{\Gamma\left(\frac{r+n}{2}-p\right)\ 2^{r+n}} = -\frac{(-1)^{\frac{n-1}{2}}\ \sqrt{\pi}\ \left(1-\frac{r+n}{2}\right)_p}{2^n\ r!\ \left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}}. \tag{69}$$

On the other hand, for p even, using (3) and after straightforward calculations we get

$$\frac{(-1)^{\frac{r}{2}}\ \Gamma\left(1+p-\frac{r+n}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)\ 2^{r+n}} = \frac{(-1)^{\frac{r}{2}}\ \Gamma\left(1-\frac{r+n}{2}\right)\left(1-\frac{r+n}{2}\right)_p}{\Gamma\left(\frac{r}{2}+1\right)\ 2^{r+n}}. \tag{70}$$

By the equality proved in [19, Sec. 3.2.1] for r even

$$\frac{(-1)^{\frac{r}{2}}\ \Gamma\left(1-\frac{n}{2}-\frac{r}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)} = \frac{(-1)^{\frac{n-1}{2}}\ \sqrt{\pi}\ 2^r}{\left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}\ r!},$$

we obtain that

$$\frac{(-1)^{\frac{r}{2}}\ \Gamma\left(1+p-\frac{r+n}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)\ 2^{r+n}} = \frac{(-1)^{\frac{n-1}{2}}\ \sqrt{\pi}\ \left(1-\frac{r+n}{2}\right)_p}{2^n\ r!\ \left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}}. \tag{71}$$

Hence, from (69) and (71) we see that the coefficients of the series are equal up to a minus sign in the odd series, which can be included as $(-1)^r$ for odd and even r. Thus, adding the even and odd series and considering the change $r = 2q + 2 - n + 2p$ in the finite sum we get

$$\begin{aligned}\mathbf{I} = &\frac{c_2\,|x|^2}{4c_0^2\,\psi\left(t\right)^{\alpha_2}} \sum_{q=0}^{\frac{n-3-2p}{2}} \frac{\Gamma\left(\frac{n}{2}-1-p-q\right)\left(-\frac{c_1|x|^2}{4c_0^2}\psi\left(t\right)^{-\alpha_1}\right)^p}{\Gamma\left(\mu_1\left(1-\alpha_1\right)-\alpha_1 p-\alpha_2 q\right)\ p!\,q!} \left(-\frac{c_2\,|x|^2}{4c_0^2\,\psi\left(t\right)^{\alpha_2}}\right)^q\\
&+(-1)^{\frac{n-1}{2}}\ \sqrt{\pi}\ \left(\frac{c_2\,|x|^2}{4c_0^2\,\psi\left(t\right)^{\alpha_2}}\right)^{\frac{n}{2}}\\
&\times\sum_{r=0}^{+\infty} \frac{\left(1-\frac{r+n}{2}\right)_p\ \left(-\frac{c_1}{c_2}\,\psi\left(t\right)^{\alpha_2-\alpha_1}\right)^p}{\Gamma\left(\alpha_2\left(1-\frac{n}{2}\right)+\mu_1\left(1-\alpha_1\right)+\left(\alpha_2-\alpha_1\right)p-\frac{\alpha_2 r}{2}\right)\left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}\ p!\,r!} \left(-\frac{\sqrt{c_2}\,|x|}{c_0\psi\left(t\right)^{\frac{\alpha_2}{2}}}\right)^r.\end{aligned} \tag{72}$$

Finally, from (64) and (72), and proceeding in a similar way to G_2, we arrive to the following simplified double series representations of G_1 and G_2 for the case of n odd:

$$G_1(x,t) = \frac{c_1 \psi(t)^{\mu_1(1-\alpha_1)-1}}{4c_0^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}} \sum_{p=0}^{+\infty} \sum_{q=0}^{\frac{n-3-2p}{2}} \frac{\Gamma\left(\frac{n}{2}-1-p-q\right)\left(-\frac{c_1|x|^2}{4c_0^2}\psi(t)^{-\alpha_1}\right)^p}{\Gamma\left(\mu_1(1-\alpha_1)-\alpha_1 p-\alpha_2 q\right)\,p!\,q!}\left(-\frac{c_2\,|x|^2}{4c_0^2\,\psi(t)^{\alpha_2}}\right)^q$$

$$+\frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}c_1\,c_2^{\frac{n}{2}-1}\psi(t)^{\alpha_2\left(1-\frac{n}{2}\right)+\mu_1(1-\alpha_1)-1}}{\left(4c_0^2\,\pi\right)^{\frac{n}{2}}}$$

$$\times\sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty}\frac{\left(1-\frac{r+n}{2}\right)_p\left(-\frac{c_1}{c_2}\psi(t)^{\alpha_2-\alpha_1}\right)^p}{\Gamma\left(\alpha_2\left(1-\frac{n}{2}\right)+\mu_1(1-\alpha_1)+(\alpha_2-\alpha_1)p-\frac{\alpha_2 r}{2}\right)\left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}\,p!\,r!}\left(-\frac{\sqrt{c_2}\,|x|}{c_0\psi(t)^{\frac{\alpha_2}{2}}}\right)^r, \quad (73)$$

and

$$G_2(x,t) = \frac{c_2\,\psi(t)^{\mu_2(2-\alpha_2)-2}}{4c_0^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}} \sum_{p=0}^{+\infty} \sum_{q=0}^{\frac{n-3-2p}{2}} \frac{\Gamma\left(\frac{n}{2}-1-p-q\right)\left(-\frac{c_1|x|^2}{4c_0^2}\psi(t)^{-\alpha_1}\right)^p}{\Gamma\left(\mu_2(2-\alpha_2)-1-\alpha_1 p-\alpha_2 q\right)\,p!\,q!}\left(-\frac{c_2\,|x|^2}{4c_0^2\,\psi(t)^{\alpha_2}}\right)^q$$

$$+\frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}c_2^{\frac{n}{2}}\,\psi(t)^{\alpha_2\left(1-\frac{n}{2}\right)+\mu_2(2-\alpha_2)-2}}{\left(4c_0^2\,\pi\right)^{\frac{n}{2}}}$$

$$\times\sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty}\frac{\left(1-\frac{r+n}{2}\right)_p\left(-\frac{c_1}{c_2}\psi(t)^{\alpha_2-\alpha_1}\right)^p}{\Gamma\left(\alpha_2\left(1-\frac{n}{2}\right)-1+\mu_2(2-\alpha_2)+(\alpha_2-\alpha_1)p-\frac{\alpha_2 r}{2}\right)\left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}\,p!\,r!}\left(-\frac{\sqrt{c_2}\,|x|}{c_0\psi(t)^{\frac{\alpha_2}{2}}}\right)^r. \quad (74)$$

The absolute convergence of the series in (73) and (74) is guaranteed by applying the ratio test provided by Horn's technique. In fact, applying the Lemma A.1 that we proved in the Appendix results that the series in (73) and (74) are absolutely convergent for all $(x,t)\in\mathbb{R}^n\times I$ and $n\in\mathbb{N}$. Furthermore, taking into account the formula (128) we can interpret the double series in (73) and (74) as generalised Lauricella series.

An important special case can be deduced from (73) and (74). Considering $c_1=0$ we obtain the series representation of the first fundamental solution of the time-fractional wave equation with ψ-Hilfer derivative for n odd, which is given by

$$\mathbf{G_1}(x,t) = \frac{c_2\,\psi(t)^{\mu_2(2-\alpha_2)-2}}{4c_0^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}} \sum_{q=0}^{\frac{n-3-2p}{2}} \frac{\Gamma\left(\frac{n}{2}-1-q\right)}{\Gamma\left(\mu_2(2-\alpha_2)-1-\alpha_2 q\right)\,q!}\left(-\frac{c_2\,|x|^2}{4c_0^2\,\psi(t)^{\alpha_2}}\right)^q$$

$$+\frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}c_2^{\frac{n}{2}}\,\psi(t)^{\alpha_2\left(1-\frac{n}{2}\right)+\mu_2(2-\alpha_2)-2}}{\left(4c_0^2\,\pi\right)^{\frac{n}{2}}}\sum_{r=0}^{+\infty}\frac{\left(-\frac{\sqrt{c_2}|x|}{c_0\psi(t)^{\frac{\alpha_2}{2}}}\right)^r}{\Gamma\left(\alpha_2\left(1-\frac{n}{2}\right)-1+\mu_2(2-\alpha_2)-\frac{\alpha_2 r}{2}\right)\left(\frac{r+1}{2}\right)_{\frac{n-1}{2}}\,r!}.$$

From the series representations obtained previously it is easy to see that the fundamental solution is finite at the point $x=(0,\ldots,0)$ only in the one-dimensional case, being infinite at $x=(0,\ldots,0)$ for all $n\geq 2$, with n odd. The case of even dimension can be treated in a similar way as it was done in our paper [18]. To not overload this section, we omit the presentation of the calculations for this case. Finally, we would like to remark that there is a second fundamental solution $\mathbf{G_2}$ assuming $d=q(x,t)=0$ and $f(x)=g_1(x)=0$ in (35), and $g_2(x)=\delta(x)$ in (36) and (37). Then $\mathbf{G_2}$ is given by $\mathbf{G_2}(x,t)=G_3(x,t)$, where G_3 corresponds to (57). Similarly as it was done for $\mathbf{G_1}$ it is possible to obtain a double series representation of $\mathbf{G_2}$.

Remark 4.1 *In [3] it was presented a series representation of the first fundamental solution in the one-dimensional case when the time-fractional derivatives are in the Riemann-Liouville sense. Here, we obtained a general expression for the first fundamental solution that encompasses several types of fractional derivatives and arbitrary dimension.*

4.1 Graphical representation of the first fundamental solution in the one dimensional case

In this section we present and discuss some plots of the first fundamental solution $\mathbf{G_1}$ in the one dimensional case, i.e., $n=1$, for the following types of time-fractional derivatives: Caputo-type ($\mu_1=\mu_2=1$), Riemann-

Liouville type ($\mu_1 = \mu_2 = 0$), and intermediate types μ_1 and μ_2. For the three types we consider some particular values of the fractional parameters α_1 and α_2, and particular choices of the function ψ. Considering $n = 1$ in (73) and (74) the finite sums vanish and, consequently, we get the following double series representation of $\mathbf{G_1}$:

$$\begin{aligned}\mathbf{G_1}(x,t) &= \frac{c_1\,\psi(t)^{\frac{\alpha_2}{2}+\mu_1(1-\alpha_1)-1}}{2c_0\sqrt{c_2}} \sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p \left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p}{\Gamma\left(\frac{\alpha_2}{2}+\mu_1(1-\alpha_1)+(\alpha_2-\alpha_1)\,p-\frac{\alpha_2 r}{2}\right)\,p!\,r!}\left(-\frac{\sqrt{c_2}\,|x|}{c_0\psi(t)^{\frac{\alpha_2}{2}}}\right)^r \\ &+ \frac{\sqrt{c_2}\,\psi(t)^{\frac{\alpha_2}{2}+\mu_2(2-\alpha_2)-2}}{2c_0} \sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p \left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)^p}{\Gamma\left(\frac{\alpha_2}{2}-1+\mu_2(2-\alpha_2)+(\alpha_2-\alpha_1)\,p-\frac{\alpha_2 r}{2}\right)\,p!\,r!}\left(-\frac{\sqrt{c_2}\,|x|}{c_0\psi(t)^{\frac{\alpha_2}{2}}}\right)^r. \qquad (75)\end{aligned}$$

In the following subsections we consider $c_0 = c_1 = c_2 = 1$ in (75).

4.1.1 Riemann-Liouville type operators

Here we present the graphical representation of $\mathbf{G_1}$ when the time-fractional derivatives are of Riemann-Liouville type ($\mu_1 = \mu_2 = 0$) of orders α_1 and α_2.

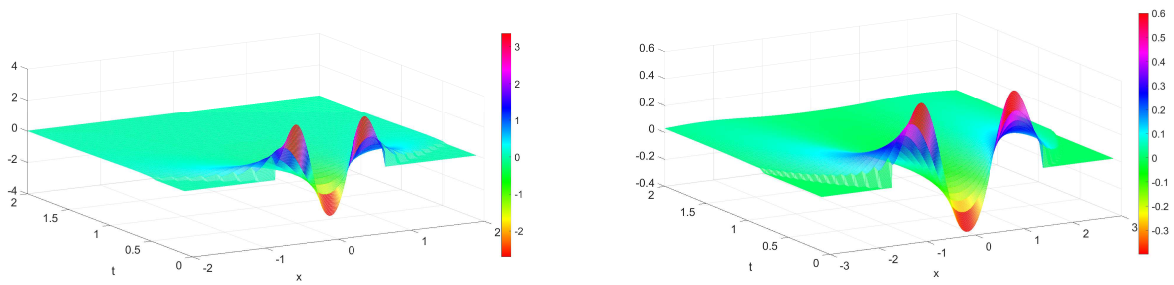

Figure 1: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = t$ with $t \in \mathbb{R}^+$ (Riemann-Liouville fractional derivatives), and $\psi(t) = t^2$ with $t \in \mathbb{R}^+$ (Katugampola fractional derivatives). The order of the derivatives are $\alpha_1 = 0.25$ and $\alpha_2 = 1.25$.

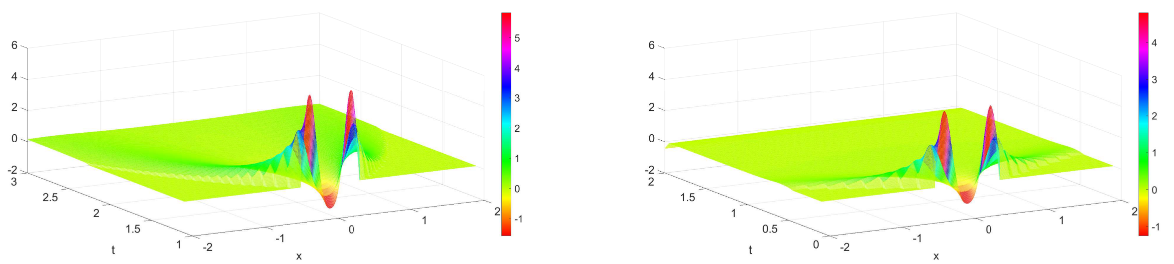

Figure 2: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = \ln t$ with $t \in]1,+\infty[$ (Hadamard fractional derivatives), and $\psi(t) = te^t$ with $t \in \mathbb{R}^+$. The order of the derivatives are $\alpha_1 = 0.50$ and $\alpha_2 = 1.50$.

From the previous plots, we conclude that the $\mathbf{G_1}$ is an even, positive and negative function. The fundamental solution has two symmetric maximum points that move from the origin as α_2 and α_1 increase, and a local negative minimum when $x = 0$. The plots possess similar behaviour for different choices of ψ, however the range of the plots and the horizontal shrink is different for each function ψ, due to the nonlinearity of the time varying.

4.1.2 Caputo type operators

In this section we present the graphical representation of $\mathbf{G_1}$ when the time-fractional derivatives are of Caputo type ($\mu_1 = \mu_2 = 1$) of orders α_1 and α_2.

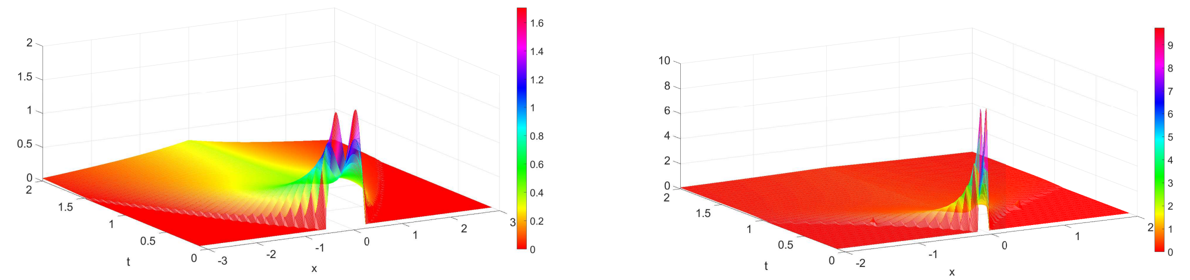

Figure 3: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = t$ with $t \in \mathbb{R}^+$ (Caputo fractional derivatives), and $\psi(t) = t^2$ with $t \in \mathbb{R}^+$ (Caputo-Katugampola fractional derivatives). The order of the derivatives are $\alpha_1 = 0.50$ and $\alpha_2 = 1.50$.

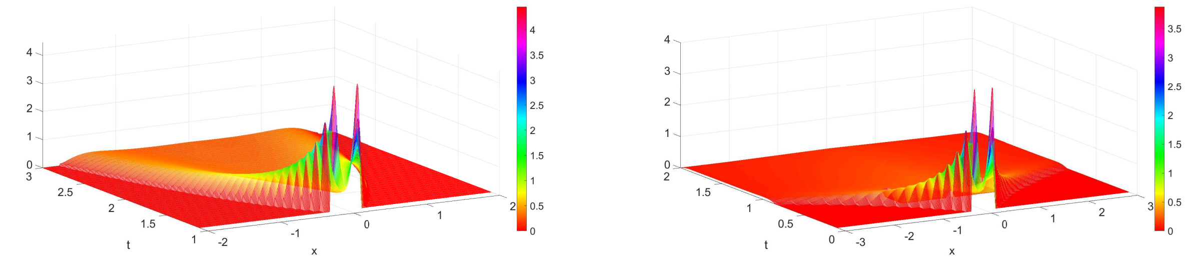

Figure 4: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = \ln t$ and $t \in]1,+\infty[$(Caputo-Hadamard fractional derivatives), and $\psi(t) = te^t$ and $t \in \mathbb{R}^+$. The order of the derivatives are $\alpha_1 = 0.75$ and $\alpha_2 = 1.75$.

We can observe now that the fundamental solution is an even positive function. The different choices of the function ψ imply different vertical ranges and horizontal shrink. A detailed analysis of the Caputo case will be performed in Section 6.2.1.

4.1.3 Derivatives of intermediate types

In this section we present the graphical representation of $\mathbf{G_1}$ for different types of derivatives, i.e., for different values of $\mu_1, \mu_2 \in]0,1[$ in (75).

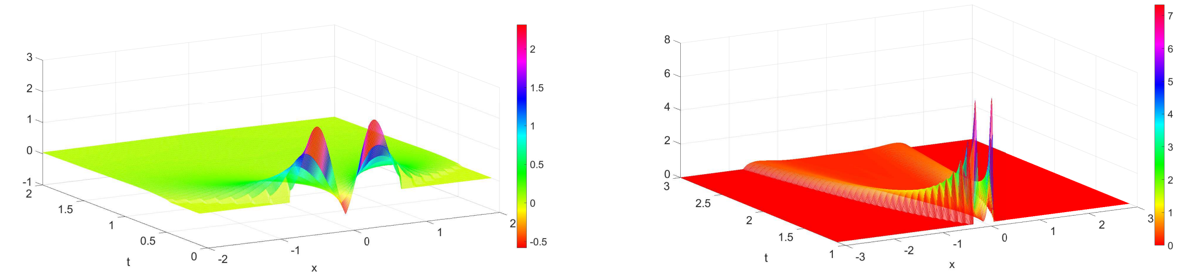

Figure 5: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = t$ with $t \in \mathbb{R}^+$, $\mu_1 = 0.20$, $\mu_2 = 0.30$, $\alpha_1 = 0.25$, and $\alpha_2 = 1.25$, and $\psi(t) = \ln t$ with $t \in]1,+\infty[$, $\mu_1 = 0.60$, $\mu_2 = 0.40$, $\alpha_1 = 0.50$, and $\alpha_2 = 1.50$.

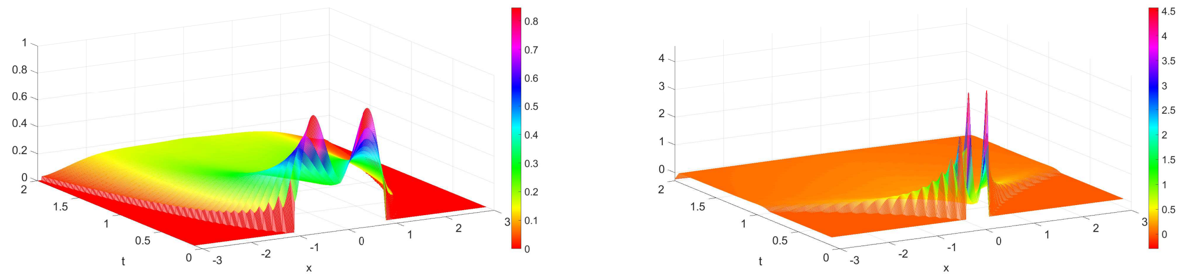

Figure 6: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = t^{\frac{1}{2}}$ with $t \in \mathbb{R}^+$, $\mu_1 = 0.40$, $\mu_2 = 0.60$, $\alpha_1 = 0.50$, and $\alpha_2 = 1.50$, and $\psi(t) = te^t$ with $t \in \mathbb{R}^+$, $\mu_1 = 0.60$, $\mu_2 = 0.80$, $\alpha_1 = 0.75$, and $\alpha_2 = 1.75$.

The previous plots correspond to an interpolation between the Riemann-Liouville and Caputo cases. In fact, for small values of μ_1 and μ_2, we see that the fundamental solution is negative near the origin and has a shape closer to that observed for the Riemann-Liouville case. As μ_1 and μ_2 increase the fundamental solution becomes a non-negative function and the shape is more similar to that observed in the Caputo case.

5 Moments of the first fundamental solution

In this section we deduce the expression for some moments of the first fundamental solution $\mathbf{G_1}$ introduced in Section 4. We separate the analysis of the case when $n = 1$ from the case $n \geq 2$.

5.1 One dimensional case

For $n = 1$, the moments $\mathbf{M}^\gamma$ of order $\gamma > 0$ of $\mathbf{G_1}$ are given by

$$\mathbf{M}^\gamma(t) = \int_{\mathbb{R}} x^\gamma \, \mathbf{G_1}(x,t) \, dx.$$

The previous integral cannot be calculated directly. Therefore, we will compute the moments using Fourier-Laplace techniques. Considering $n = 1$ in (39), we have that

$$\mathcal{L}_\psi \left\{ \widehat{\mathbf{G_1}}(\kappa,t) \right\}(\kappa,\mathbf{s}) = \frac{\mathbf{s}^{1-\mu_2(2-\alpha_2)} + \frac{c_1}{c_2}\,\mathbf{s}^{-\mu_1(1-\alpha_1)}}{\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1} + \frac{c_0^2}{c_2}\,\kappa^2}. \tag{76}$$

Inverting the Fourier transform in (76) using (29) leads to

$$\mathcal{L}_\psi \left\{ \mathbf{G_1}(x,t) \right\}(x,\mathbf{s}) = \frac{\sqrt{c_2}}{2c_0} \frac{\mathbf{s}^{1-\mu_2(2-\alpha_2)} + \frac{c_1}{c_2}\,\mathbf{s}^{-\mu_1(1-\alpha_1)}}{\left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}\right)^{\frac{1}{2}}} \exp\left(-\frac{\sqrt{c_2}\,|x|}{c_0} \left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\, s^{\alpha_1} \right)^{\frac{1}{2}} \right). \tag{77}$$

The ψ-Laplace transform of the moments of order $\gamma > 0$ of $\mathbf{G_1}$ are given, via (77), by

$$\mathcal{L}_\psi \left\{ \mathbf{M}^\gamma(t) \right\}(\mathbf{s}) = \frac{\sqrt{c_2}}{c_0} \frac{\mathbf{s}^{1-\mu_2(2-\alpha_2)} + \frac{c_1}{c_2}\,\mathbf{s}^{-\mu_1(1-\alpha_1)}}{\left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}\right)^{\frac{1}{2}}} \int_0^{+\infty} x^\gamma \exp\left(-\frac{\sqrt{c_2}\,|x|}{c_0} \left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\, s^{\alpha_1} \right)^{\frac{1}{2}} \right) dx.$$

Taking into account the formula $\int_0^{+\infty} x^{b-1}\, e^{-ax}\, dx = \Gamma(b)\, a^{-b}$, we have that

$$\mathcal{L}_\psi \left\{ \mathbf{M}^\gamma(t) \right\}(\mathbf{s}) = \frac{c_0^\gamma\, \Gamma(\gamma+1)}{c_2^{\frac{\gamma}{2}}} \frac{\mathbf{s}^{1-\mu_2(2-\alpha_2)} + \frac{c_1}{c_2}\,\mathbf{s}^{-\mu_1(1-\alpha_1)}}{\left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}\right)^{\frac{\gamma}{2}+1}}. \tag{78}$$

Inverting the ψ-Laplace transform, we obtain

$$\mathbf{M}^\gamma(t) = \frac{c_0^\gamma\, \Gamma(\gamma+1)}{c_2^{\frac{\gamma}{2}}} \left[\mathcal{L}_\psi^{-1} \left\{ \frac{\mathbf{s}^{1-\mu_2(2-\alpha_2)}}{\left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}\right)^{\frac{\gamma}{2}+1}} \right\}(t) + \mathcal{L}_\psi^{-1} \left\{ \frac{\mathbf{s}^{-\mu_1(1-\alpha_1)}}{\left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}\right)^{\frac{\gamma}{2}+1}} \right\}(t) \right].$$

Taking into account (23) we have

$$\mathcal{L}_{\psi}^{-1}\left\{\frac{\mathbf{s}^{1-\mu_2(2-\alpha_2)}}{\left(\mathbf{s}^{\alpha_2}+\frac{c_1}{c_2}\mathbf{s}^{\alpha_1}\right)^{\frac{\gamma}{2}+1}}\right\}(t)=\mathcal{L}_{\psi}^{-1}\left\{\frac{\mathbf{s}^{(\alpha_2-\alpha_1)\left(\frac{\gamma}{2}+1\right)-\alpha_2\left(\frac{\gamma}{2}+1\right)-\mu_2(2-\alpha_2)+1}}{\left(\mathbf{s}^{\alpha_2-\alpha_1}-\left(-\frac{c_1}{c_2}\right)\right)^{\frac{\gamma}{2}+1}}\right\}(t)$$

$$=\psi(t)^{\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_2(2-\alpha_2)-2}\;E^{\frac{\gamma}{2}+1}_{\alpha_2-\alpha_1,\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_2(2-\alpha_2)-2}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)\quad(79)$$

and

$$\mathcal{L}_{\psi}^{-1}\left\{\frac{\frac{c_1}{c_2}\,\mathbf{s}^{-\mu_1(1-\alpha_1)}}{\left(\mathbf{s}^{\alpha_2}+\frac{c_1}{c_2}\mathbf{s}^{\alpha_1}\right)^{\frac{\gamma}{2}+1}}\right\}(t)=\mathcal{L}_{\psi}^{-1}\left\{\frac{\frac{c_1}{c_2}\mathbf{s}^{(\alpha_2-\alpha_1)\left(\frac{\gamma}{2}+1\right)-\alpha_2\left(\frac{\gamma}{2}+1\right)-\mu_1(1-\alpha_1)}}{\left(\mathbf{s}^{\alpha_2-\alpha_1}-\left(-\frac{c_1}{c_2}\right)\right)^{\frac{\gamma}{2}+1}}\right\}(t)$$

$$=\frac{c_1}{c_2}\psi(t)^{\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_1(1-\alpha_1)-1}\;E^{\frac{\gamma}{2}+1}_{\alpha_2-\alpha_1,\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_1(1-\alpha_1)}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right).\quad(80)$$

Hence, we conclude that the moments of order γ of $\mathbf{G_1}$ are given in terms of three-parameter Mittag-Leffler functions by

$$\mathbf{M}^{\gamma}(t)=\frac{c_0^{\gamma}\,\Gamma(\gamma+1)}{c_2^{\frac{\gamma}{2}}}\Bigg[\psi(t)^{\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_2(2-\alpha_2)-2}\;E^{\frac{\gamma}{2}+1}_{\alpha_2-\alpha_1,\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_2(2-\alpha_2)-1}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)$$
$$+\frac{c_1}{c_2}\,\psi(t)^{\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_1(1-\alpha_1)-1}\;E^{\frac{\gamma}{2}+1}_{\alpha_2-\alpha_1,\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_1(1-\alpha_1)}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)\Bigg].\quad(81)$$

From (11) we have the following asymptotic behaviour near the starting point $t=a$

$$\mathbf{M}^{\gamma}(t)\sim\frac{c_0^{\gamma}\,\Gamma(\gamma+1)}{c_2^{\frac{\gamma}{2}}}\,\psi(t)^{\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_2(2-\alpha_2)-2},\quad t\to a^+,$$

and from (12) we have the following asymptotic behaviour at the infinity

$$\mathbf{M}^{\gamma}(t)\sim\frac{c_0^{\gamma}\,\Gamma(\gamma+1)}{c_2^{\frac{\gamma}{2}}}\,\psi(t)^{\alpha_1\left(\frac{\gamma}{2}+1\right)+\mu_1(1-\alpha_1)-1},\quad t\to+\infty.$$

In particular when $\gamma=1$ (mean) we have

$$\mathbf{M}^{1}(t)\sim\frac{c_0}{\sqrt{c_2}}\,\psi(t)^{\frac{3\alpha_2}{2}+\mu_2(2-\alpha_2)-2},\quad t\to a^+,\qquad \mathbf{M}^{1}(t)\sim\frac{c_0}{\sqrt{c_2}}\,\psi(t)^{\frac{3\alpha_1}{2}+\mu_1(1-\alpha_1)-1},\quad t\to+\infty,$$

and for $\gamma=2$ (variance)

$$\mathbf{M}^{2}(t)\sim\frac{2c_0}{c_2}\,\psi(t)^{2\alpha_2+\mu_2(2-\alpha_2)-2},\quad t\to a^+,\qquad \mathbf{M}^{2}(t)\sim\frac{2c_0}{c_2}\,\psi(t)^{2\alpha_1+\mu_1(1-\alpha_1)-1},\quad t\to+\infty.\quad(82)$$

The variance, also called the mean square displacement of a particle in a diffusion process, allows to recognize the type of diffusion we are dealing with, by comparison with the variance in the normal diffusion process. From (82), and comparing diffusion processes with the same ψ-function, we get the following inequalities in the short and in the long time

$$\psi(t)^{2\alpha_2+\mu_2(2-\alpha_2)-2}>\psi(t)^2,\quad t\to a^+\qquad\text{and}\qquad\psi(t)^{2\alpha_1+\mu_1(1-\alpha_1)-1}<\psi(t),\quad t\to+\infty,\quad(83)$$

for all $\alpha_1\in]0,1]$, $\alpha_2\in]1,2]$, $\mu_i\in[0,1]$, $i=1,2$, and where the right-hand sides of (83) correspond to the limit cases $\alpha_2=2$ and $\alpha_1=1$, respectively. Hence, we conclude that in this case, the process is superdiffusive in the short time and subdiffusive in the long time, for the same function ψ. When comparing with the normal diffusion process $\psi(t)=t$ in the right-hand side of inequalities (83), both types of anomalous diffusion can appear in the short and long times, depending on the choice of the function ψ in the left-hand side of inequalities (83), as we can see in Figures 7 and 8.

In the following plots we show the asymptotic behaviour of the variance for small times in the cases $\psi(t) = t^2$, $\psi(t) = t^{0.5}$, and $\psi(t) = \ln t$, with $\alpha_2 = 1.5$ and different values of μ_2. For the comparison we represent also the normal diffusion that corresponds to the case $\psi(t) = t$ and $\alpha_2 = 2$.

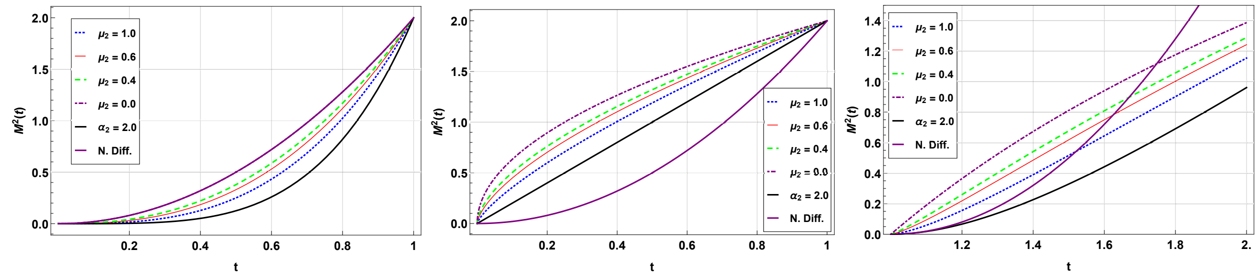

Figure 7: Plots of $\mathbf{M}^2(t)$ for short time with $\alpha_2 = 1.5$, $\mu_2 = 0.0, 0.4, 0.6, 1.0$, and $\psi(t) = t^2$, $\psi(t) = t^{0.5}$, and $\psi(t) = \ln t$ (from left).

From the previous plots we conclude that in the short time the process is subdiffusive for $\psi(t) = t^2$ and superdiffusive for $\psi(t) = t^{0.5}$, when comparing with the normal diffusion process. For the case $\psi(t) = \ln t$ we can observe that initially we have superdiffusion and the transition for subdiffusion occurs at different instants in the interval. Now, we present the plots for the asymptotic behaviour of the variance near the infinity for $\psi(t) = t^2$, $\psi(t) = t^{0.5}$, and $\psi(t) = \ln t$, with $\alpha_1 = 0.8$ and different values of μ_1.

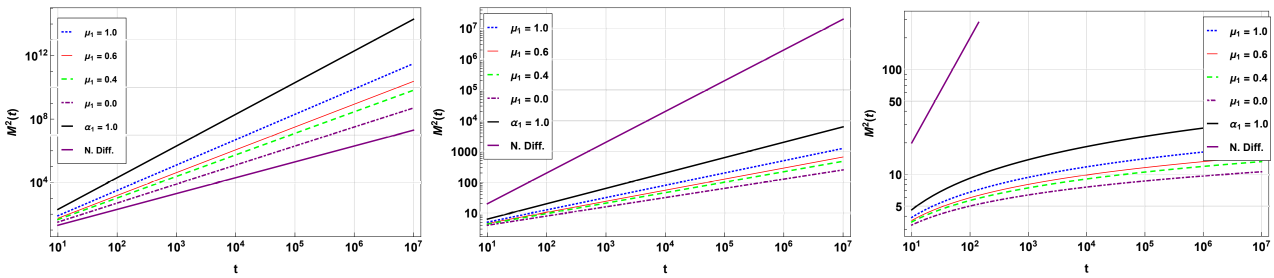

Figure 8: Plots of $\mathbf{M}^2(t)$ for long time with $\alpha_1 = 0.5$, $\mu_1 = 0.0, 0.4, 0.6, 1.0$, and $\psi(t) = t^2$, $\psi(t) = t^{0.5}$, and $\psi(t) = \ln t$ (from left).

From the analysis of the plots in Figure 8 we conclude that in the long time the process is superdiffusive for $\psi(t) = t^2$ and subdiffusive for $\psi(t) = t^{0.5}$ and $\psi(t) = \ln t$, when comparing with the normal diffusion process. We point out that for different values of the parameters $\alpha_1, \alpha_2, \mu_1, \mu_2$ the type of diffusion process may change.

5.2 The case of higher dimension

Here we consider the case when $n \geq 2$. Because of the structure of (76), the use of formula (43) to invert the Fourier transform of (76) leads to complicated expressions. Hence, we will not proceed as it was done for the one dimensional case, but we consider an approach consisting in the reinterpretation of the moments in terms of the Laplace transform (see e.g. [36]). First we compute the integer derivatives of order $\gamma \in \mathbb{N}$ of (76) with respect to r at $r = 0$, yielding

$$D_r^\gamma \left[\widehat{\widetilde{\mathbf{G}_1}}_{\psi} \right](0,s) = \begin{cases} 0, & \gamma \text{ is odd} \\ \dfrac{(-1)^{\frac{\gamma}{2}}\, \gamma!\, c_0^\gamma}{c_2^{\frac{\gamma}{2}}} \, \dfrac{\mathbf{s}^{1-\mu_2(2-\alpha_2)} + \frac{c_1}{c_2}\, \mathbf{s}^{-\mu_1(1-\alpha_1)}}{\left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\, \mathbf{s}^{\alpha_1} \right)^{\frac{\gamma}{2}+1}}, & \gamma \text{ is even} \end{cases}. \tag{84}$$

On the other hand, taking into account the definition of the Fourier transform for radial functions in $\mathbb{R}^n$ (see [39]) and due to the convergence of the improper integrals, we have

$$D_r^\gamma\left[\widehat{\widehat{\mathbf{G_1}}_\psi}\right](0,s) = D_r^\gamma\left[\int_0^{+\infty} e^{-s\,\psi(t)}\ \underbrace{\frac{(2\pi)^{\frac{n}{2}}}{r^{\frac{n}{2}-1}}\int_0^{+\infty}\mathbf{G_1}(w,t)\ J_{\frac{n}{2}-1}(wr)\ w^{\frac{n}{2}}\ dw}_{\widehat{\mathbf{G_1}}(r,t)}\ \psi'(t)\ dt\right](0,s)$$

$$= \int_0^{+\infty} e^{-s\,\psi(t)} \int_0^{+\infty} D_r^\gamma\left[\frac{(2\pi)^{\frac{n}{2}}}{r^{\frac{n}{2}-1}}\ J_{\frac{n}{2}-1}(wr)\ w^{\frac{n}{2}}\right]_{r=0}\mathbf{G_1}(w,t)\ dw\,\psi'(t)\ dt. \tag{85}$$

Since

$$D_r^\gamma\left[\frac{(2\pi)^{\frac{n}{2}}}{r^{\frac{n}{2}-1}}\ J_{\frac{n}{2}-1}(wr)\ w^{\frac{n}{2}}\right]_{r=0} = \begin{cases} 0, & \gamma \text{ is odd} \\ \dfrac{(-1)^{\frac{\gamma}{2}}\ (\gamma-1)!!\,\pi^{\frac{n}{2}}\ w^{n+\gamma-1}}{2^{\frac{\gamma}{2}-1}\ \Gamma\left(\frac{\gamma+n}{2}\right)}, & \gamma \text{ is even} \end{cases},$$

then when γ is even (85) becomes

$$D_r^\gamma\left[\widehat{\widehat{\mathbf{G_1}}_\psi}\right](0,s) = \frac{(-1)^{\frac{\gamma}{2}}\ (\gamma-1)!!\,\pi^{\frac{n}{2}}}{2^{\frac{\gamma}{2}-1}\ \Gamma\left(\frac{\gamma+n}{2}\right)} \int_0^{+\infty} e^{-s\,\psi(t)} \int_0^{+\infty} w^{n+\gamma-1}\,\mathbf{G_1}(w,t)\ dw\,\psi'(t)\ dt$$

$$= \frac{(-1)^{\frac{\gamma}{2}}\ (\gamma-1)!!\,\pi^{\frac{n}{2}}}{2^{\frac{\gamma}{2}-1}\ \Gamma\left(\frac{\gamma+n}{2}\right)}\,\mathcal{L}_\psi\left\{\mathbf{M}^{n+\gamma-1}\right\}(\mathbf{s}), \tag{86}$$

where $\mathbf{M}^{n+\gamma-1}$ is the moment of order $n+\gamma-1$ of $\mathbf{G_1}$. Moreover, from (86) and (84) we get

$$\mathcal{L}_\psi\left\{\mathbf{M}^{n+\gamma-1}\right\}(\mathbf{s}) = \frac{\gamma!!\,2^{\frac{\gamma}{2}-1}\,c_0^\gamma\ \Gamma\left(\frac{\gamma+n}{2}\right)}{c_2^{\frac{\gamma}{2}}\ \pi^{\frac{n}{2}}}\ \frac{\mathbf{s}^{1-\mu_2(2-\alpha_2)} + \frac{c_1}{c_2}\,\mathbf{s}^{-\mu_1(1-\alpha_1)}}{\left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}\right)^{\frac{\gamma}{2}+1}}.$$

Inverting the ψ-Laplace transform, we have

$$\mathbf{M}^{n+\gamma-1}(t) = \frac{\gamma!!\,2^{\frac{\gamma}{2}-1}\,c_0^\gamma\ \Gamma\left(\frac{\gamma+n}{2}\right)}{c_2^{\frac{\gamma}{2}}\ \pi^{\frac{n}{2}}}\left[\mathcal{L}_\psi^{-1}\left\{\frac{\mathbf{s}^{1-\mu_2(2-\alpha_2)}}{\left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}\right)^{\frac{\gamma}{2}+1}}\right\}(t) + \mathcal{L}_\psi^{-1}\left\{\frac{\mathbf{s}^{-\mu_1(1-\alpha_1)}}{\left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}\right)^{\frac{\gamma}{2}+1}}\right\}(t)\right]. \tag{87}$$

Taking into account (79) and (80), we conclude that the moments of order $n+\gamma-1$ of $\mathbf{G_1}$ are given by

$$\mathbf{M}^{n+\gamma-1}(t) = \frac{\gamma!!\,2^{\frac{\gamma}{2}-1}\,c_0^\gamma\ \Gamma\left(\frac{\gamma+n}{2}\right)}{c_2^{\frac{\gamma}{2}}\ \pi^{\frac{n}{2}}}$$
$$\times\left[\psi(t)^{\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_2(2-\alpha_2)-2}\ E^{\frac{\gamma}{2}+1}_{\alpha_2-\alpha_1,\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_2(2-\alpha_2)-1}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)\right.$$
$$\left.+\ \frac{c_1}{c_2}\,\psi(t)^{\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_1(1-\alpha_1)-1}\ E^{\frac{\gamma}{2}+1}_{\alpha_2-\alpha_1,\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_1(1-\alpha_1)}\left(-\frac{c_1}{c_2}\,\psi(t)^{\alpha_2-\alpha_1}\right)\right], \tag{88}$$

where γ is a nonnegative even integer. From (11) we have the following asymptotic behaviour of (88)

$$\mathbf{M}^{n+\gamma-1}(t) \sim \frac{\gamma!!\,2^{\frac{\gamma}{2}-1}\,c_0^\gamma\ \Gamma\left(\frac{\gamma+n}{2}\right)}{c_2^{\frac{\gamma}{2}}\ \pi^{\frac{n}{2}}\,\Gamma\left(\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_2(2-\alpha_2)-1\right)}\,\psi(t)^{\alpha_2\left(\frac{\gamma}{2}+1\right)+\mu_2(2-\alpha_2)-2}, \quad \text{as } t\to a^+,$$

and from (12) we conclude that

$$\mathbf{M}^{n+\gamma-1}(t) \sim \frac{\gamma!!\,2^{\frac{\gamma}{2}-1}\,c_0^\gamma\ \Gamma\left(\frac{\gamma+n}{2}\right)}{c_2^{\frac{\gamma}{2}}\ \pi^{\frac{n}{2}}\,\Gamma\left(\alpha_1\left(\frac{\gamma}{2}+1\right)+\mu_1(1-\alpha_1)\right)}\,\psi(t)^{\alpha_1\left(\frac{\gamma}{2}+1\right)+\mu_1(1-\alpha_1)-1}, \quad \text{as } t\to+\infty.$$

We remark that this approach does not give all the moments for any dimension. For example, assuming $n=2$ in (88) we can only compute the moments of odd order, because γ is even from (84). To obtain the moments of even order a different approach must be developed.

6 The one dimensional case and its probabilistic interpretation

In this section we prove that the first fundamental solution $\mathbf{G_1}$ introduced in Section 5 with $n = 1$ can be interpreted as a true probability density function, for some particular choices of the parameters μ_1 and μ_2. These conditions show us to what extent it is possible to generalize the results presented in [36].

Theorem 6.1 *For $n = 1$, the first fundamental solution $\mathbf{G_1}$ of the generalized time-fractional telegraph equation* (35) *corresponds to a true probability density function for all $0 < \alpha_1 < 1$ and $1 \leq \alpha_2 < 2$ if and only if $\mu_1 = \mu_2 = 1$, i.e., when we deal with ψ-Caputo fractional derivatives.*

Proof: The first fundamental solution $\mathbf{G_1}$ can only be considered as a probability density function if and only if $\widehat{\mathbf{G_1}}(0,t) = 1$ and $\mathbf{G_1}(x,t)$ is nonnegative for all $(x,t) \in \mathbb{R} \times I$, with $I = [a,b]$ being a finite or an infinite interval on $\mathbb{R}^+$ (see assumptions and conditions assumed in (35)-(37)). Let us start with the first property. Taking into account that $E^{\beta_3}_{\beta_1,\beta_2}(0) = \frac{1}{\Gamma(\beta)}$, we have from (39) that

$$
\begin{aligned}
1 &= \widehat{\mathbf{G_1}}(0,t) \\
&= \sum_{p=0}^{+\infty} \left(-\frac{c_1}{c_2} \psi(t)^{\alpha_2-\alpha_1} \right)^p \left[\frac{c_1}{c_2} \psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)} \; E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p+\mu_1(1-\alpha_1)}(0) \right. \\
&\qquad\qquad \left. + \psi(t)^{\alpha_2-2+\mu_2(2-\alpha_2)} \; E^{p+1}_{\alpha_2,\alpha_2+(\alpha_2-\alpha_1)p-1+\mu_2(2-\alpha_2)}(0) \right] \\
&= \sum_{p=0}^{+\infty} \left(-\frac{c_1}{c_2} \psi(t)^{\alpha_2-\alpha_1} \right)^p \left[\frac{c_1}{c_2} \psi(t)^{\alpha_2-1+\mu_1(1-\alpha_1)} \frac{1}{\Gamma\left((\alpha_2-\alpha_1)p+\alpha_2+\mu_1(1-\alpha_1)\right)} \right. \\
&\qquad\qquad \left. + \psi(t)^{\alpha_2-2+\mu_2(2-\alpha_2)} \frac{1}{\Gamma\left((\alpha_2-\alpha_1)p+\alpha_2-1+\mu_2(2-\alpha_2)\right)} \right] \\
&= -\psi(t)^{-(1-\mu_1)(1-\alpha_1)} \sum_{p=0}^{+\infty} \left(-\frac{c_1}{c_2} \psi(t)^{\alpha_2-\alpha_1} \right)^{p+1} \frac{1}{\Gamma\left((\alpha_2-\alpha_1)p+\alpha_2+\mu_1(1-\alpha_1)\right)} \\
&\quad + \psi(t)^{-(1-\mu_2)(2-\alpha_2)} \sum_{p=0}^{+\infty} \left(-\frac{c_1}{c_2} \psi(t)^{\alpha_2-\alpha_1} \right)^{p} \frac{1}{\Gamma\left((\alpha_2-\alpha_1)p+\alpha_2-1+\mu_2(2-\alpha_2)\right)} \\
&= -\psi(t)^{-(1-\mu_1)(1-\alpha_1)} \sum_{q=1}^{+\infty} \left(-\frac{c_1}{c_2} \psi(t)^{\alpha_2-\alpha_1} \right)^{q} \frac{1}{\Gamma\left((\alpha_2-\alpha_1)q+\mu_1(1-\alpha_1)+\alpha_1\right)} \\
&\quad + \psi(t)^{-(1-\mu_2)(2-\alpha_2)} \frac{1}{\Gamma\left(\alpha_2-1+\mu_2(2-\alpha_2)\right)} \\
&\quad + \psi(t)^{-(1-\mu_2)(2-\alpha_2)} \sum_{p=1}^{+\infty} \left(-\frac{c_1}{c_2} \psi(t)^{\alpha_2-\alpha_1} \right)^{p} \frac{1}{\Gamma\left((\alpha_2-\alpha_1)p+\alpha_2-1+\mu_2(2-\alpha_2)\right)}.
\end{aligned}
\tag{89}
$$

For all $0 < \alpha_1 < 1$ and $1 < \alpha_2 < 2$, the two series in (89) cancel if and only if $\mu_1 = \mu_2 = 1$. In these conditions we immediately get that $\widehat{\mathbf{G_1}}(0,t) = 1$, for all $t \in I$. From now on, we assume that $\mu_1 = \mu_2 = 1$. Let's now see in which conditions $\mathbf{G_1}(x,t)$ is a nonnegative function for all $x \in \mathbb{R}$ and $t \in I$. Following the ideas presented in [36] we analyse the ψ-Laplace transform of $\widehat{\mathbf{G_1}}(\kappa,t)$ with respect to the time-variable t. Considering $\mu_1 = \mu_2 = 1$ in (77) we have

$$
\mathcal{L}_\psi \left\{ \mathbf{G_1}(x,t) \right\}(x,\mathbf{s}) = \frac{\sqrt{c_2}}{2c_0\mathbf{s}} \left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2} \mathbf{s}^{\alpha_1} \right)^{\frac{1}{2}} \exp\left(-\frac{\sqrt{c_2}\,|x|}{c_0} \left(\mathbf{s}^{\alpha_2} + \frac{c_1}{c_2} s^{\alpha_1} \right)^{\frac{1}{2}} \right). \tag{90}
$$

The ψ-Laplace transform (90) can be further inverted in a simple way when $c_1 = 0$ (which corresponds to the case of the generalized time-fractional wave equation with ψ-Caputo derivative). For a random variable $X \sim S_\alpha(\tau, 1, 0)$, also called a stable distribution with index α, scale parameter τ, and without skewness and shiftness (see [40]), the expected value of $e^{-\gamma X}$ is given by

$$E\left[e^{-\mathbf{s}X}\right] = \exp\left(-\frac{\tau^{\alpha}}{\cos\left(\frac{\alpha\pi}{2}\right)}\,\mathbf{s}^{\alpha}\right), \quad 0<\alpha\leq 2, \quad \alpha\neq 1, \quad \tau, \mathbf{s}\in\mathbb{R}^{+}, \tag{91}$$

which is the same as the Laplace transform of X (see [40, Prop. 1.2.12]). Hence for $c_1 = 0$, the expression (90) can be written as

$$\mathcal{L}_{\psi}\left\{\mathbf{G_1}\left(x,t\right)\right\}\left(x,\mathbf{s}\right) = \frac{\sqrt{c_2}}{2c_0}\,\mathbf{s}^{\frac{\alpha_2}{2}-1}\,E\left[e^{-\mathbf{s}X}\right], \tag{92}$$

where

$$X \sim S_{\frac{\alpha_2}{2}}\left(\frac{\sqrt{c_2}\,|x|}{c_0}\cos\left(\frac{\alpha_2\pi}{4}\right), 1, 0\right), \quad 1<\alpha_2\leq 2. \tag{93}$$

Denoting by $p\left(|x|,t\right)$ the probability density function of X we can invert the ψ-Laplace transform in (92), via the Convolution Theorem present in (20), as follows

$$\mathbf{G_1}\left(x,t\right) = \frac{\sqrt{c_2}}{2c_0}\,\frac{\psi\left(t\right)^{\frac{\alpha_2}{2}}}{\Gamma\left(1-\frac{\alpha_2}{2}\right)} *_{\psi} p\left(|x|,\psi\left(t\right)\right), \tag{94}$$

where, by (17), holds

$$\mathcal{L}_{\psi}^{-1}\left\{\exp\left(-\frac{\sqrt{c_2}\,|x|}{c_0}\,\mathbf{s}^{\frac{\alpha_2}{2}}\right)\right\}\left(t\right) = p\left(|x|,\psi\left(t\right)\right), \tag{95}$$

and by (21)

$$\mathcal{L}_{\psi}^{-1}\left\{\mathbf{s}^{\frac{\alpha_2}{2}-1}\right\}\left(t\right) = \frac{\psi\left(t\right)^{\frac{\alpha_2}{2}}}{\Gamma\left(1-\frac{\alpha_2}{2}\right)}. \tag{96}$$

Hence, from (95) and (96), and the definition of the ψ-convolution presented in (19), we conclude that $\mathbf{G_1}\left(x,t\right)$ is a nonnegative function for all $1<\alpha_2<2$. For the limit case of $\alpha_2 = 1$ we have, by (22) that

$$\mathcal{L}_{\psi}^{-1}\left\{\exp\left(-\frac{\sqrt{c_2}\,|x|}{c_0}\,\mathbf{s}^{\frac{1}{2}}\right)\right\}\left(t\right) = \frac{\sqrt{c_2}\,|x|}{2c_0\sqrt{\pi}}\psi\left(t\right)^{-\frac{3}{2}}\exp\left(-\frac{c_2\,|x|}{4c_0^2\,\psi\left(t\right)}\right). \tag{97}$$

Moreover, since

$$\mathcal{L}_{\psi}^{-1}\left\{\mathbf{s}^{-\frac{1}{2}}\right\}\left(t\right) = \frac{1}{\sqrt{\pi\,\psi\left(t\right)}} \tag{98}$$

we get

$$\mathbf{G_1}\left(x,t\right) = \frac{\sqrt{c_2}}{2c_0}\,\frac{1}{\sqrt{\pi\,\psi\left(t\right)}} *_{\psi} \frac{\sqrt{c_2}\,|x|}{2c_0\sqrt{\pi}}\psi\left(t\right)^{-\frac{3}{2}}\exp\left(-\frac{c_2\,|x|}{4c_0^2\,\psi\left(t\right)}\right). \tag{99}$$

Since ψ is a nonnegative function we can also guarantee that $\mathbf{G_1}\left(x,t\right)$ is a nonnegative function for $\alpha_2 = 1$. Let us now study the case where $c_1 \neq 0$. Taking into account the following identity

$$\frac{2}{a^2+x^2} = \int_0^{+\infty}\frac{1}{w^2}\exp\left(-\frac{a^2}{2w}\right)\exp\left(-\frac{x^2}{2w}\right)dw, \tag{100}$$

we have that (76), with $\mu_1 = \mu_2 = 1$, can be written as

$$\begin{aligned}\mathcal{L}_{\psi}\left\{\widehat{\mathbf{G_1}}\left(\kappa,t\right)\right\}\left(\kappa,\mathbf{s}\right) &= \frac{c_2}{2c_0^2}\left(\mathbf{s}^{\alpha_2-1}+\frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1-1}\right)\frac{2}{\frac{c_2}{c_0^2}\left(\mathbf{s}^{\alpha_2}+\frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1}\right)+\kappa^2}\\ &= \frac{c_2}{2c_0^2}\left(\mathbf{s}^{\alpha_2-1}+\frac{c_1}{c_2}\,\mathbf{s}^{\alpha_1-1}\right)\int_0^{+\infty}\frac{1}{w^2}\exp\left(-\frac{\kappa^2}{2w}\right)\exp\left(-\frac{c_2\mathbf{s}^{\alpha_2}}{2c_0^2\,w}\right)\exp\left(-\frac{c_1\mathbf{s}^{\alpha_1}}{2c_0^2\,w}\right)dw.\end{aligned} \tag{101}$$

Now, we analyse each term in (101), by making an interpretation via convolution with known density functions. From (91) we have that

$$\exp\left(-\frac{c_2\mathbf{s}^{\alpha_2}}{2c_0^2\, w}\right)=E\left[e^{-\mathbf{s}X_2}\right], \qquad \text{where } X_2\sim S_{\alpha_2}\left(\frac{\sqrt{c_2}\,|x|}{c_0}\cos\left(\frac{\alpha_2\pi}{2}\right),1,0\right), \quad 1<\alpha_2<2, \tag{102}$$

$$\exp\left(-\frac{c_2\mathbf{s}^{\alpha_1}}{2c_0^2\, w}\right)=E\left[e^{-\mathbf{s}X_1}\right], \qquad \text{where } X_1\sim S_{\alpha_1}\left(\frac{\sqrt{c_2}\,|x|}{c_0}\cos\left(\frac{\alpha_1\pi}{2}\right),1,0\right), \quad 0<\alpha_1<1. \tag{103}$$

The probability density functions connected with (102) and (103) will be denoted by $q_{\alpha_2}(w,t)$ and $q_{\alpha_1}(w,t)$, respectively, and satisfy

$$\mathcal{L}_\psi^{-1}\left\{\exp\left(-\frac{c_2}{2c_0^2\, w}\,\mathbf{s}^{\alpha_2}\right)\right\}(t)\;=q_{\alpha_2}\left(w,\psi\left(t\right)\right) \quad \text{and} \quad \mathcal{L}_\psi^{-1}\left\{\exp\left(-\frac{c_2}{2c_0^2\, w}\,\mathbf{s}^{\alpha_1}\right)\right\}(t)\;=q_{\alpha_1}\left(w,\psi\left(t\right)\right). \tag{104}$$

Now, since

$$\mathcal{L}_\psi^{-1}\left\{\mathbf{s}^{\alpha_2-1}\right\}(t)\;=\frac{\psi\left(t\right)^{-\alpha_2}}{\Gamma\left(1-2\alpha_1\right)}$$

is only valid for $\alpha_2<1$, which does not happen in our case, we can only perform the inversion directly for the second term of (101). Hence, since

$$\mathcal{L}_\psi^{-1}\left\{\mathbf{s}^{\alpha_1-1}\right\}(t)\;=\frac{\psi\left(t\right)^{-\alpha_1}}{\Gamma\left(1-\alpha_1\right)}, \qquad \alpha_1<1 \tag{105}$$

and

$$\mathcal{F}^{-1}\left\{\frac{1}{w^2}\exp\left(-\frac{\kappa^2}{2w}\right)\right\}(x)\;=w^{-\frac{3}{2}}\exp\left(-\frac{wx^2}{2}\right), \tag{106}$$

by (20) and (26), we have for this term

$$\begin{aligned}&\frac{c_1}{2c_0^2}\,\mathbf{s}^{\alpha_1-1}\int_0^{+\infty}\frac{1}{w^2}\exp\left(-\frac{\kappa^2}{2w}\right)\exp\left(-\frac{c_2\mathbf{s}^{\alpha_2}}{2c_0^2\, w}\right)\exp\left(-\frac{c_1\mathbf{s}^{\alpha_1}}{2c_0^2\, w}\right)dw\\&=\frac{c_1}{2c_0^2}\int_0^{+\infty}w^{-\frac{3}{2}}\exp\left(-\frac{wx^2}{2}\right)\frac{\psi\left(t\right)^{-\alpha_1}}{\Gamma\left(1-\alpha_1\right)}*_\psi\left[q_{\alpha_2}\left(w,\psi\left(t\right)\right)*_\psi q_{\alpha_1}\left(w,\psi\left(t\right)\right)\right]dw.\end{aligned} \tag{107}$$

The second member in (107) corresponds to a combination of ψ-convolutions involving the probability density functions q_{α_2} and q_{α_1}, which are nonnegative functions by definition, and the power function $\psi\left(t\right)^{-\alpha_1}$, which is also a nonnegative function. Hence, for $0<\alpha_1<1$, we can ensure that expression (107) corresponds to a nonnegative function. Let us analyse the first term in (101). By the mean-value theorem and (100), we have

$$\begin{aligned}&\frac{c_2}{2c_0^2}\,\mathbf{s}^{\alpha_2-1}\int_0^{+\infty}\frac{1}{w^2}\exp\left(-\frac{x^2}{2w}\right)\exp\left(-\frac{c_2\mathbf{s}^{\alpha_2}}{2c_0^2\, w}\right)\exp\left(-\frac{c_1\mathbf{s}^{\alpha_1}}{2c_0^2\, w}\right)dw\\&=\exp\left(-\frac{c_1\mathbf{s}^{\alpha_1}}{2c_0^2\overline{w}}\right)\left[\frac{c_2}{2c_0^2}\,\mathbf{s}^{\alpha_2-1}\int_0^{+\infty}\frac{1}{w^2}\exp\left(-\frac{\kappa^2}{2w}\right)\exp\left(-\frac{c_2\mathbf{s}^{\alpha_2}}{2c_0^2\, w}\right)dw\right]\\&=\exp\left(-\frac{c_1\mathbf{s}^{\alpha_1}}{2c_0^2\overline{w}}\right)\frac{\mathbf{s}^{\alpha_2-1}}{\mathbf{s}^{\alpha_2}+\frac{c_0^2}{c_2}\,\kappa^2},\end{aligned} \tag{108}$$

for some $\overline{w}\in\,]0,+\infty[$ and $1\leq\alpha_2<2$. While the first factor in (108) corresponds via (103) to a stable distribution of order α_1, the second factor represents the ψ-Laplace Fourier transform of the solution of the generalized time-fractional wave equation with ψ-Hilfer derivative of Caputo type, which we already prove to be a nonnegative function by means of the representation (76) with $\mu_1=\mu_2=0$ and $c_1=0$. Hence, we finally conclude that the first fundamental solution of (35) can be considered as a probability density function for all $0<\alpha_1<1$ and $1\leq\alpha_2<2$ if and only if $\mu_1=\mu_2=1$.

■

Corollary 6.2 *For $n=1$ and $c_1=0$, the first fundamental solution $\mathbf{G_1}$ of the generalized time-fractional wave equation corresponds to a true probability density function for all $1\leq\alpha_2<2$ if and only if $\mu_2=1$.*

Let us analyse the limit cases of $\alpha_1 = 1$ and $\alpha_2 = 2$ and investigate the connections with the telegraph process. Denoting by $X_{\alpha_2,\alpha_1} = X_{\alpha_2,\alpha_1}(\psi(t))$ the process whose distribution at ψ coincide with $\mathbf{G_1}$, the limit cases of $\alpha_1 = 1$ and $\alpha_2 = 2$ are related to the classical telegraph process, which is defined using the function ψ as

$$T(\psi(t)) = V(a)\int_a^{\psi(t)} (-1)^{N(s)}\, ds,$$

where $V(a)$ is a two vector-valued random variable (with values $\pm c_0$ taken with probability $\frac{1}{2}$) and $N(\psi(t))$ is the number of events in $[a, \psi(t)]$ of a homogeneous Poisson process, independent of $V(a)$ (cf. [36]).

6.1 The case of the law process with Brownian time

We consider here the particular case when $n = 1$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 1$, and $\mu_1 = \mu_2 = 1$, which allows to obtain an interpretation of the fundamental solution in terms of the telegraph process and the Brownian motion. We consider the equation

$$c_2{}^H\partial_{t,a^+}^{1,1;\psi}u(x,t) + c_1{}^H\partial_{t,a^+}^{\frac{1}{2},1;\psi}u(x,t) - c_0^2\,\partial_x^2 u(x,t) = 0 \tag{109}$$

subject to (63), where the time-fractional derivatives coincide with Caputo fractional derivatives. From (41) we obtain the following expression for $\widehat{\mathbf{G_1}}$:

$$\widehat{\mathbf{G_1}}(\kappa,t) = \frac{c_1}{c_2}\psi(t)^{\frac{1}{2}} E_{(\frac{1}{2},1),\frac{3}{2}}\left(-\frac{c_1}{c_2}\psi(t)^{\frac{1}{2}}, -\frac{c_0^2|\kappa|^2}{c_2}\psi(t)\right) + E_{(\frac{1}{2},1),1}\left(-\frac{c_1}{c_2}\psi(t)^{\frac{1}{2}}, -\frac{c_0^2|\kappa|^2}{c_2}\psi(t)\right)$$

and for $\mathbf{G_1}$ (see Theorem 3.3):

$$\begin{aligned}\mathbf{G_1}(x,t) = {}& \frac{c_1\psi(t)^{\frac{1}{2}}}{c_2\sqrt{\pi}\,|x|} H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c|c} \dfrac{4c_0^2\psi(t)}{c_2|x|^2} & (0;1,1);\left(\dfrac{1}{2},1\right);-- \\ \dfrac{c_1}{c_2}\psi(t)^{\frac{1}{2}} & \left(-\dfrac{1}{2};1,\dfrac{1}{2}\right);--;(0,1)\end{array}\right] \\ & + \frac{1}{\sqrt{\pi}\,|x|} H^{0,1;0,1;1,0}_{1,1;1,0;0,1}\left[\begin{array}{c|c} \dfrac{4c_0^2\psi(t)}{c_2|x|^2} & (0;1,1);\left(\dfrac{1}{2},1\right);-- \\ \dfrac{c_1}{c_2}\psi(t)^{\frac{1}{2}} & \left(0;1,\dfrac{1}{2}\right);--;(0,1)\end{array}\right].\end{aligned} \tag{110}$$

Moreover, the double series representation associated to the previous expression is (see (75) with $\mu_1 = \mu_2 = 1$, $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = 1$):

$$\begin{aligned}\mathbf{G_1}(x,t) = {}& \frac{c_1}{2c_0\sqrt{c_2}}\sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty}\frac{\left(\frac{1-r}{2}\right)_p}{\Gamma\left(1+\frac{p-r}{2}\right)p!\,r!}\left(-\frac{c_1}{c_2}\psi(t)^{\frac{1}{2}}\right)^p\left(-\frac{\sqrt{c_2}\,|x|}{c_0\psi(t)^{\frac{1}{2}}}\right)^r \\ & + \frac{\psi(t)^{-\frac{1}{2}}}{2c_0\sqrt{c_2}}\sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty}\frac{\left(\frac{1-r}{2}\right)_p}{\Gamma\left(\frac{1}{2}+\frac{p-r}{2}\right)p!\,r!}\left(-\frac{c_1}{c_2}\psi(t)^{\frac{1}{2}}\right)^p\left(-\frac{\sqrt{c_2}\,|x|}{c_0\psi(t)^{\frac{1}{2}}}\right)^r.\end{aligned}$$

The probability density function given by (110) corresponds to the composition of the telegraph process $T = T(t)$, where $t \in I$, with a reflecting Brownian motion $|B| = |B(\psi(t))|$, with $t \in I$ (independent of T). This means that the probability law given by (110) coincides with the distribution of the telegraph process with a Brownian process with nonlinear time varying, i.e., $W(t) = T(|B(\psi(t))|)$, $t \in I$. The process W can be understood as the random motion of a particle moving with alternating velocities $\pm c_0$ (changing Poisson times) during an interval of length $2c_0|B(\psi(t))|$. In other words, the particle is located at time t in the random space interval $(-c_0|B(\psi(t))|, c_0|B(\psi(t))|)$. This shows that the distribution related to the equation (109) covers the whole real line and differs from the classical telegraph process, where the distribution is concentrated on a finite interval (spreading as time passes) because of the finite velocity of motion. The case when $\psi(t) = t$ was already studied in [36].

6.2 Graphical representation of the probability density function

In this section we present and discuss some plots of the probability density function $\mathbf{G_1}$ for some values of the fractional parameters α_1 and α_2, and particular choices of the function ψ. Hence, considering $n = 1$ and $\mu_1 = \mu_2 = 1$ in (75), we have the following double series representation of $\mathbf{G_1}$:

$$\begin{aligned}
&\mathbf{G_1}(x,t) \\
&= \frac{c_1\,\psi(t)^{\frac{\alpha_2}{2}-\alpha_1}}{2c_0\sqrt{c_2}} \sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p}{\Gamma\left(1+\frac{\alpha_2}{2}-\alpha_1+(\alpha_2-\alpha_1)p-\frac{\alpha_2 r}{2}\right)p!\,r!} \left(-\frac{c_1}{c_2}\psi(t)^{\alpha_2-\alpha_1}\right)^p \left(-\frac{\sqrt{c_2}\,|x|}{c_0\,\psi(t)^{\frac{\alpha_2}{2}}}\right)^r \\
&\quad + \frac{\sqrt{c_2}\,\psi(t)^{-\frac{\alpha_2}{2}}}{2c_0} \sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p}{\Gamma\left(1-\frac{\alpha_2}{2}+(\alpha_2-\alpha_1)p-\frac{\alpha_2 r}{2}\right)p!\,r!} \left(-\frac{c_1}{c_2}\psi(t)^{\alpha_2-\alpha_1}\right)^p \left(-\frac{\sqrt{c_2}\,|x|}{c_0\,\psi(t)^{\frac{\alpha_2}{2}}}\right)^r. \qquad (111)
\end{aligned}$$

The two double series that appear in (111) can be combined into a single one. In fact, since

$$\begin{aligned}
&\mathbf{G_1}(x,t) \\
&= \frac{c_1}{2c_0\sqrt{c_2}}\psi(t)^{\frac{\alpha_2}{2}-\alpha_1} \sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p}{\Gamma\left(1+\frac{\alpha_2}{2}-\alpha_1+(\alpha_2-\alpha_1)p-\frac{\alpha_2 r}{2}\right)p!\,r!} \left(-\frac{c_1}{c_2}\psi(t)^{\alpha_2-\alpha_1}\right)^p \left(-\frac{\sqrt{c_2}\,|x|}{c_0\,\psi(t)^{\frac{\alpha_2}{2}}}\right)^r \\
&+ \frac{\sqrt{c_2}}{2c_0}\psi(t)^{-\frac{\alpha_2}{2}} \sum_{p=-1}^{+\infty}\sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p\left(\frac{1-r}{2}+p\right)}{\Gamma\left(1+\frac{\alpha_2}{2}-\alpha_1+(\alpha_2-\alpha_1)p-\frac{\alpha_2 r}{2}\right)(p+1)!\,r!} \left(-\frac{c_1}{c_2}\psi(t)^{\alpha_2-\alpha_1}\right)^{p+1} \left(-\frac{\sqrt{c_2}\,|x|}{c_0\,\psi(t)^{\frac{\alpha_2}{2}}}\right)^r,
\end{aligned} \qquad (112)$$

then splitting the second double series into two double series in a convenient way, one of the resulting double series will cancel with the first double series. Hence, we finally obtain

$$\mathbf{G_1}(x,t) = \frac{\sqrt{c_2}}{2c_0}\psi(t)^{-\frac{\alpha_2}{2}} \sum_{p=0}^{+\infty}\sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_{p-1}\left(\frac{-1-r}{2}\right)}{\Gamma\left(1+\frac{\alpha_2}{2}-\alpha_1+(\alpha_2-\alpha_1)p-\frac{\alpha_2 r}{2}\right)p!\,r!} \left(-\frac{c_1}{c_2}\psi(t)^{\alpha_2-\alpha_1}\right)^p \left(-\frac{\sqrt{c_2}\,|x|}{c_0\,\psi(t)^{\frac{\alpha_2}{2}}}\right)^r. \qquad (113)$$

Expression (113) is more practical under a numerical point of view and it could also be used to obtain the graphical representations presented in Section 4.1.2.

Remark 6.3 *We observe that if we consider in* (113) $c_1 = 0$, *and* $\psi(t) = t$ *with* $t \in \mathbb{R}^+$, *we obtain the series representation of the first fundamental solution for the time-fractional wave equation with Caputo derivative deduced in [19] (see expression (50) with* $n = 1$*).*

In the following subsections we will consider $c_0 = c_1 = c_2 = 1$ in (113).

6.2.1 Caputo fractional derivative

The first case we present corresponds to $\psi(t) = t$, with $t \in \mathbb{R}^+$. In the following plots we present a graphical representation of $\mathbf{G_1}(x,t)$ (see expression (111)) for $\alpha_2 = 1.25, 1.50, 1.75$, $\alpha_1 = 0.25, 0.50, 0.75$, and different values of t.

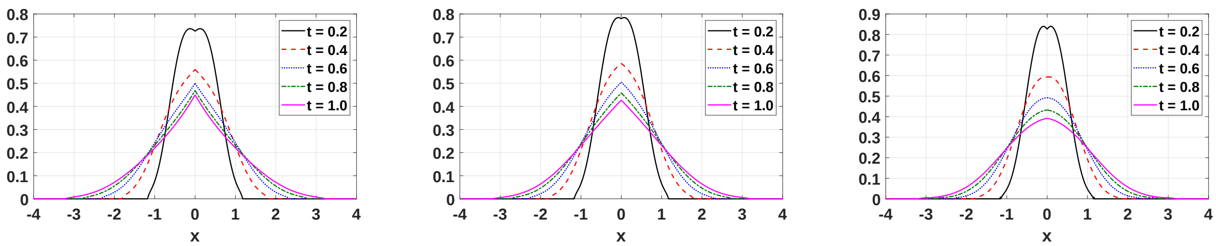

Figure 9: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = t$, $\alpha_2 = 1.25$ and $\alpha_1 = 0.25, 0.50, 0.75$ (from left).

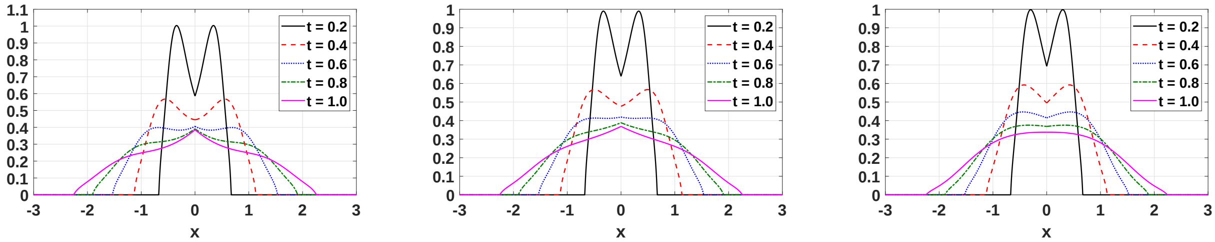

Figure 10: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = t$, $\alpha_2 = 1.50$ and $\alpha_1 = 0.25, 0.50, 0.75$ (from left).

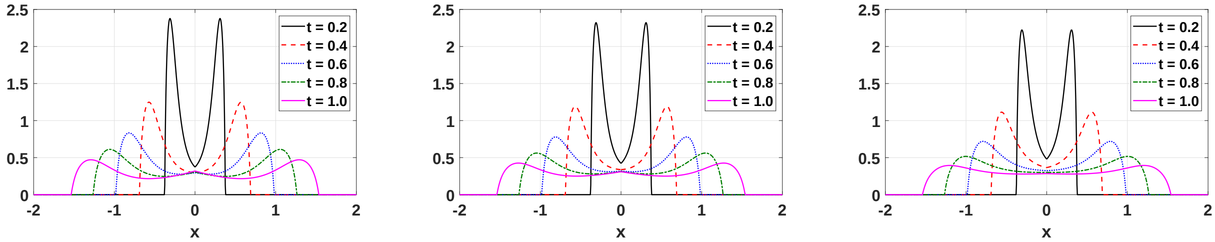

Figure 11: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = t$, $\alpha_2 = 1.75$ and $\alpha_1 = 0.25, 0.50, 0.75$ (from left).

The plots show that $\mathbf{G_1}$ is a nonnegative even function and corresponds to a fast perturbed wave phenomena due to the parameter α_1. In fact, these plots are deformations of those presented in Section 7.1 of [19], and they are also in agreement with those presented in [18, Sec. 6.1]. We can observe that with the increasing of time the behaviour of the fundamental solution changes in the origin and the decay becomes slower. Moreover, as α_1 increases the shape of the curve changes and the decay becomes slower. When α_2 increases the wave phenomena increases and the two symmetric maxima that appear move apart from the origin. Finally, it is clear from (113) that the there is a discontinuity of the first derivative at $x = 0$.

6.2.2 Caputo-Katugampola fractional derivative

Here we consider $\psi(t) = t^\rho$, with $\rho \in \mathbb{R}^+$, and $t \in \mathbb{R}^+$, which corresponds to the case when the time-fractional derivatives in (35) are in the Caputo-Katugampola sense and of orders α_1 and α_2 (see Table 1). In the following plots we present a graphical representation of $\mathbf{G_1}(x,t)$ for $\psi(t) = t^2$ and $\psi(t) = t^{\frac{1}{2}}$, $\alpha_1 = 0.25, 0.50, 0.75$, $\alpha_2 = 1.25, 1.50, 1.75$, and different values of t.

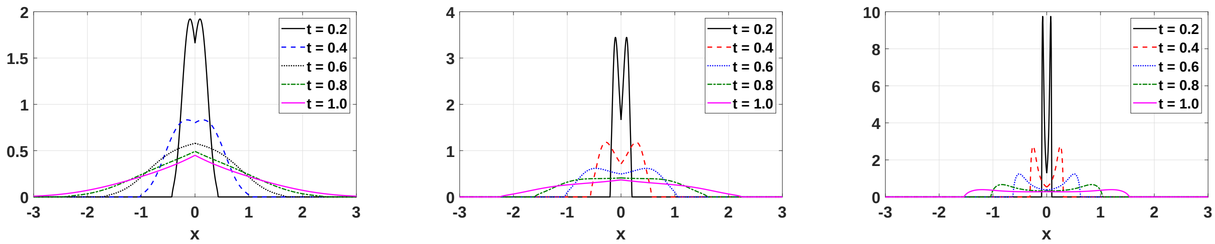

Figure 12: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = t^2$, $\alpha_1 = 0.25$ and $\alpha_2 = 1.25$, $\alpha_1 = 0.50$ and $\alpha_2 = 1.50$, and $\alpha_1 = 0.75$ and $\alpha_2 = 1.75$ (from left).

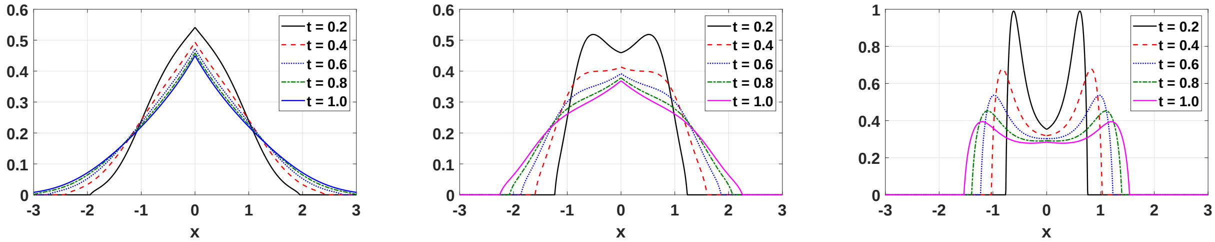

Figure 13: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = t^{\frac{1}{2}}$, $\alpha_1 = 0.25$ and $\alpha_2 = 1.25$, $\alpha_1 = 0.50$ and $\alpha_2 = 1.50$, and $\alpha_1 = 0.75$ and $\alpha_2 = 1.75$ (from left).

From the plots, we see that the behaviour of the fundamental solutions turns to be similar the one observed in Figures 9, 10, and 11 for the first, second, and third plots, respectively, however due to the nonlinearity of the functions ψ chosen there are differences in the behaviour of the fundamental solutions.

6.2.3 Caputo-Hadamard fractional derivative

Let us consider $\psi(t) = \ln t$, and $t \in]1, +\infty[$, which corresponds to the case when the time-fractional derivatives in (35) are in the Caputo-Hadamard sense and of orders α_1 and α_2 (see Table 1). We present a graphical representation of $\mathbf{G_1}(x,t)$ for $\psi(t) = \ln t$ (see expression (111)) for $x \in [-5, 5]$, $\alpha_1 = 0.25, 0.50, 0.75$, $\alpha_2 = 1.25, 1.50, 1.75$, and different values of t.

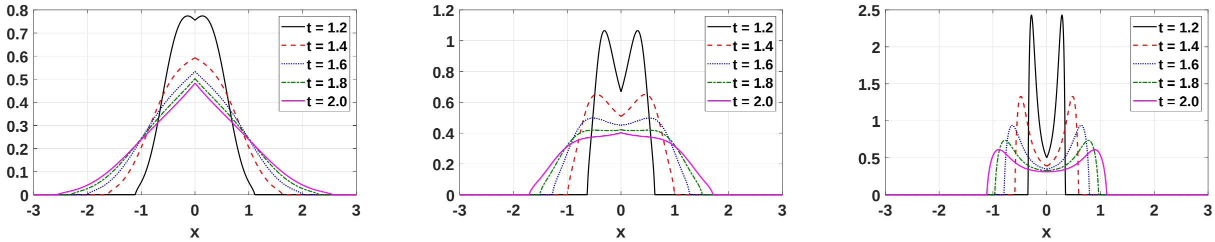

Figure 14: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = \ln t$, $\alpha_1 = 0.25$ and $\alpha_2 = 1.25$, $\alpha_1 = 0.50$ and $\alpha_2 = 1.50$, and $\alpha_1 = 0.75$ and $\alpha_2 = 1.75$ (from left).

Despite of the different range of the time variable $t \in]1, +\infty[$, the behaviour of the fundamental solutions is similar to the one observed in Figures 9, 10, and 11. The difference in the range of the plots is due to the slower time varying induced by the function $\psi(t) = \ln t$.

6.3 Caputo-Exponential type fractional derivative

In this subsection, we consider $\psi(t) = te^t$, with $t \in \mathbb{R}^+$, which corresponds to the case when the time-fractional derivative in (35) are in the ψ-Caputo sense and of orders α_1 and α_2 (see Table 1). In the following plots we present a graphical representation of $\mathbf{G_1}(x,t)$ (see expression (111)) for $\psi(t) = te^t$, $x \in [-5, 5]$, $\alpha_1 = 0.25, 0.50, 0.75$, $\alpha_2 = 1.25, 1.50, 1.75$, and different values of t.

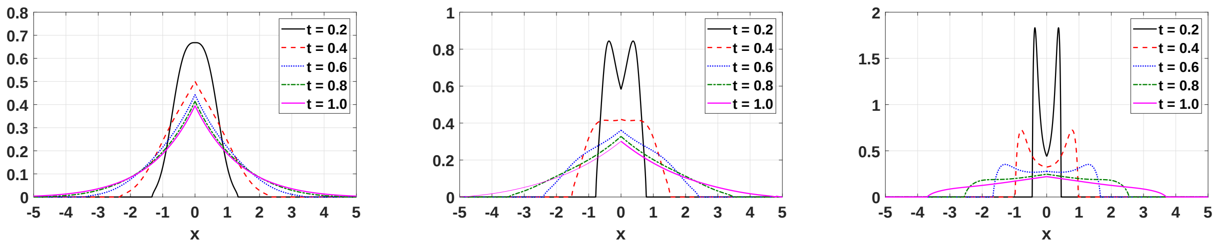

Figure 15: Plots of $\mathbf{G_1}(x,t)$ for $\psi(t) = te^t$, $\alpha_1 = 0.25$ and $\alpha_2 = 1.25$, $\alpha_1 = 0.50$ and $\alpha_2 = 1.50$, and $\alpha_1 = 0.75$ and $\alpha_2 = 1.75$ (from left).

We see that the behaviour of the fundamental solutions is similar to the one observed in Figures 9, 10, and 11, but in this case the time is varying faster than in the other cases. This explains the differences between these plots and the plots presented previously.

7 Particular cases

In this section we present some particular cases of the previous results in order to show the consistency of our work.

7.1 Time-fractional telegraph equation with Caputo fractional derivatives

If we consider in (35)-(37), $\psi(t) = t$ with $t \in \mathbb{R}^+$, and

$$c_2 = 1, \quad d = 0, \quad q(x,t) = 0, \quad \mu_1 = \mu_2 = 1, \quad f(x) = g_1(x) = g_2(x) = \delta(x) = \prod_{i=1}^{n} \delta(x_i), \tag{114}$$

our initial value problem reduces to

$$\begin{cases} \left({}^C\partial^{\alpha_2}_{0^+,t} + c_1\, {}^C\partial^{\alpha_1}_{0^+,t} - c_0^2\, \Delta_x\right) u(x,t) = 0 \\ u(x,0) = \delta(x) \\ \dfrac{\partial u}{\partial t}(x,0) = \delta(x) \end{cases}, \tag{115}$$

where the time-fractional derivatives are in the Caputo sense. This problem was already studied in [16]. In these conditions, the solution of (115) corresponds in the Fourier domain, via expression (40), to

$$\begin{aligned} \widehat{u}(\kappa,t) &= c_1 t^{\alpha_2-\alpha_1} E_{(\alpha_2-\alpha_1,\alpha_2),1+\alpha_2-\alpha_1}\left(-c_1\, t^{\alpha_2-\alpha_1}, -c_0^2\, |\kappa|^2 t^{\alpha_2}\right) + E_{(\alpha_2-\alpha_1,\alpha_2),1}\left(-c_1\, t^{\alpha_2-\alpha_1}, -c_0^2\, |\kappa|^2 t^{\alpha_2}\right) \\ &\quad + t\, E_{(\alpha_2-\alpha_1,\alpha_2),2}\left(-c_1\, t^{\alpha_2-\alpha_1}, -c_0^2\, |\kappa|^2 t^{\alpha_2}\right), \end{aligned}$$

which agrees with expression (3.3) in [16]. Furthermore, the representations of the solution in terms of Fox H-functions of two variables and double series coincide with the correspondent ones presented in [16,18].

7.2 Time-fractional neutral telegraph equation

The Cauchy problem associated to the time-fractional neutral telegraph equation with Caputo fractional derivatives is obtained from (115) by considering $n = 1$, $\alpha_2 = 2\alpha$ and $\alpha_1 = \alpha$, with $\frac{1}{2} < \alpha \leq 1$, and the second initial condition is replaced by $\frac{\partial u}{\partial t}(x,0) = 0$. This problem was already studied in [36] and is a particular case of the work we present here. In this case, we have the following representation of $u(x,t)$, in the Fourier domain, in terms of bivariate Mittag-Leffler functions

$$\widehat{u}(\kappa,t) = c_1\, t^{\alpha} E_{(\alpha,2\alpha),\alpha+1}\left(-c_1\, t^{\alpha}, -c_0^2\, |\kappa|^2 t^{2\alpha}\right) + E_{(\alpha,2\alpha),1}\left(-c_1\, t^{\alpha}, -c_0^2\, |\kappa|^2 t^{2\alpha}\right). \tag{116}$$

An equivalent representation in terms of one parameter Mittag-Leffler functions is presented in Theorem 2.1 of [36], however, expression (116) is simpler from our point of view. Moreover, we have the following double series representation of the solution

$$u(x,t) = \frac{c_1}{2c_0} \sum_{p=0}^{+\infty} \sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p (-c_1 t^{\alpha})^p}{\Gamma(1+\alpha(p-r))\, p!\, r!} \left(-\frac{|x|}{c_0 t^{\alpha}}\right)^r + \frac{1}{2c_0 t^{\alpha}} \sum_{p=0}^{+\infty} \sum_{r=0}^{+\infty} \frac{\left(\frac{1-r}{2}\right)_p (-c_1 t^{\alpha})^p}{\Gamma(1-\alpha+\alpha(p-r))\, p!\, r!} \left(-\frac{|x|}{c_0 t^{\alpha}}\right)^r,$$

which complements the results presented in [36] because the representation of the solution in the space-time domain was not given there. Making use of (10) we have from (81), with $\alpha_2 = 2\alpha$, $\alpha_1 = \alpha$, and $c_2 = \mu_1 = \mu_2 = 1$, the following general expression for the moments of arbitrary order $\gamma > 0$ of $\mathbf{G_1}$

$$\mathbf{M}^{\gamma}(t) = c_0^{\gamma}\, \Gamma(\gamma+1)\, t^{\alpha\gamma}\, E^{\frac{\gamma}{2}}_{\alpha,\alpha\gamma+1}(-c_1\, t^{\alpha}). \tag{117}$$

This formula generalizes the one presented in [36] for an arbitrary order $\gamma > 0$. For the particular case of $\gamma = 2$, we have the following expression for the variance

$$\mathbf{M}^2(t) = 2c_0^2\, t^{2\alpha}\, E_{\alpha,2\alpha+1}(-c_1\, t^{\alpha}), \tag{118}$$

which coincides with expression (5.3) in [36].

8 Conclusions

The telegraph equation containing fractional derivatives in time and/or in space is usually adopted to describe anomalous diffusive phenomena. In this work, we focused on the time-fractional telegraph equation in $\mathbb{R}^n \times \mathbb{R}^+$ where the time-fractional derivatives are the ψ-Hilfer derivatives of orders $\alpha_1 \in]0,1]$ and $\alpha_2 \in]1,2]$, and types $\mu_1, \mu_2 \in [0,1]$. The ψ-Hilfer derivative englobes several fractional derivatives proposed in the literature for particular choices of the function ψ and the type of derivative μ. In view of this, several particular cases can be obtained from our results.

Using integral transform techniques we were able to express the solution of the Cauchy problem associated with the time-fractional telegraph equation in closed form in terms of bivariate Mittag-Leffler functions in the Fourier domain and in terms of convolution integrals involving Fox H-functions of two-variables in the space-time domain. In the one-dimensional case, we made a detailed study of the first fundamental solution and we showed that it corresponds to a probability density function when the TFTE contains ψ-Caputo fractional derivatives of arbitrary orders $\alpha_1 \in]0,1]$ and $\alpha_2 \in]1,2]$.

Besides the case of the time-fractional telegraph equation with two-fractional derivatives, it could be also interesting to study the case of diffusive equations with multi-term or distributive-order time-fractional ψ-Hilfer derivatives.

A Appendix

This appendix presents a short overview of Horn's technique to study the convergence of double power series. For more details we refer the interested reader to [44] and [14] (pp. 223–229). Let the two positive quantities R_1 and R_2 be called the associated radii of convergence of the double power series

$$\sum_{p=0}^{+\infty}\sum_{q=0}^{+\infty} A_{p,q}\, z_1^p\, z_2^q, \tag{119}$$

such that it converges absolutely for $|z_1| < R_1$, $|z_2| < R_2$, and diverges when $|z_1| > R_1$, $|z_2| > R_2$. In the (r_1, r_2)-system of coordinates, the points representing the associated radii of convergence would lie on a certain curve ξ which, in turn, lies entirely within the rectangle $0 < r_1 < R_1$, $0 < r_2 < R_2$. The region between the curve ξ and the part of the rectangle containing $r_1 = r_2 = 0$ is the two-dimensional representation of the domain of convergence of the double power series. If we let

$$u_1(\nu_1,\nu_2) = \lim_{t\to+\infty} f_1(t\nu_1, t\nu_2), \qquad u_2(\nu_1,\nu_2) = \lim_{t\to+\infty} f_2(t\nu_1, t\nu_2), \tag{120}$$

where

$$f_1(p,q) = \frac{A_{p+1,q}}{A_{p,q}}, \qquad f_2(p,q) = \frac{A_{p,q+1}}{A_{p,q}}, \tag{121}$$

are rational functions of p and q satisfying the functional equation

$$f_1(p,q)\; f_2(p+1,q) = f_1(p,q+1)\; f_2(p,q); \qquad p,q = 0,1,2,\ldots$$

then we have that $R_1 = |u_1(1,0)|^{-1}$, $R_2 = |u_2(1,0)|^{-1}$, and ξ has the parametric representation

$$r_1 = |u_1(\nu_1,\nu_2)|^{-1}, \qquad r_2 = |u_2(\nu_1,\nu_2)|^{-1}, \tag{122}$$

where $\mu_1, \mu_2 > 0$. From (120) there are three cases worthy of note:

- **Case I.** If $u_1(\nu_1,\nu_2) = u_2(\nu_1,\nu_2) = 0$ then the radii of convergence R_1 and R_2, given by (122), are infinitely large. Hence the double power series (119) converges absolutely for all complex z_1 and z_2.

- **Case II.** If $u_1(\nu_1,\nu_2) = 1/\rho_1$ and $u_2(\nu_1,\nu_2) = 1/\rho_2$, with $\rho_1, \rho_2 > 0$, then from (122) we have that $R_1 = \rho_1$ and $R_2 = \rho_2$. Therefore, (119) converges absolutely for all complex z_1 and z_2 such that $|z_1| < \rho_1$ and $|z_2| < \rho_2$.

- **Case III.** If u_1 and u_2 become infinitely large as $t \to +\infty$, then $R_1 = R_2 = 0$, which shows that the double power series would diverge except when $z_1 = z_2 = 0$.

Taking into account this technique it is possible to prove the following auxiliary lemma.

Lemma A.1 *Let $a_1, a_2, a_3, a_4, b_1, b_2, b_3, m$ be reals numbers such that $a_2, a_4 \neq 0$, $b_2 > 0$, $b_3 > -1$, and $m \geq 0$. Then the following double power series*

$$\sum_{p=0}^{+\infty} \sum_{q=0}^{+\infty} \frac{(a_1 + a_2 q)_p}{(a_3 + a_4 q)_m \, \Gamma(b_1 + b_2 p + b_3 q) \, p! \, q!} \, z_1^p \, z_2^q, \tag{123}$$

is absolutely convergent for all complex z_1 and z_2.

Proof: Let us determine the functions f_1 and f_2 in (120), which are given by (121). From (123) and taking into account the definition of the Pochhammer symbol, we have that

$$\begin{aligned} f_1(t\nu_1, t\nu_2) &= \left. \frac{A_{p+1,q}}{A_{p,q}} \right|_{p=t\nu_1, q=t\nu_2} \\ &= \left. \frac{\Gamma(a_1 + a_2 q + p + 1)}{\Gamma(a_1 + a_2 q + p)} \, \frac{\Gamma(b_1 + b_2 p + b_3 q)}{\Gamma(b_1 + b_2 p + b_3 q + b_2)} \, \frac{1}{p+1} \right|_{p=t\nu_1, q=t\nu_2} \\ &\sim \left. (a_1 + a_2 q + p) \, (b_1 + b_2 p + b_3 q)^{-b_2} \, \frac{1}{p+1} \right|_{p=t\nu_1, q=t\nu_2} \end{aligned}$$

where we use the asymptotic formula $\Gamma(x+\alpha)/\Gamma(x+\beta) \sim x^{\alpha-\beta}$ when $x \to +\infty$ for $\alpha, \beta \in \mathbb{R}$ (see Formula (1.4) in [44]). When $p \to +\infty$ we have that

$$f_1(t\nu_1, t\nu_2) \sim p \, (b_1 + b_2 p + b_3 q)^{-b_2} \left. \frac{1}{p} \right|_{p=t\nu_1, q=t\nu_2} = (b_2 \nu_1 + b_3 \nu_2)^{-b_2} \, t^{-b_2}, \tag{124}$$

which implies by (120) that

$$u_1(\nu_1, \nu_2) = \lim_{t \to +\infty} f_1(t\nu_1, t\nu_2) = 0, \tag{125}$$

since $b_2 > 0$. Proceeding in a similar way for the function f_2, we get

$$f_2(t\nu_1, t\nu_2) \sim (\nu_1 + a_2 \nu_2)^{a_2} \, (a_2 \nu_2)^{-a_2} \, (b_2 \nu_1 + b_3 \nu_2)^{-b_3} \, t^{-b_3 - 1}, \tag{126}$$

and, hence,

$$u_2(\nu_1, \nu_2) = \lim_{t \to +\infty} f_2(t\nu_1, t\nu_2) = 0, \tag{127}$$

since $b_3 > -1$. From (125) and (127) we conclude that the radii of convergence R_1 and R_2, given by (122), are infinitely large, and therefore the double power series (123) converges absolutely for all complex z_1 and z_2.

■

Taking into account the definition of the Pochhammer symbol we can rewrite the double power series (123) as follows

$$\frac{\Gamma(a_3)}{\Gamma(a_3 + m) \, \Gamma(b_1)} \sum_{p=0}^{+\infty} \sum_{q=0}^{+\infty} \frac{(a_1)_{p + a_2 q} \, (a_3)_{a_4 q}}{(b_1)_{b_2 p + b_3 q} \, (a_1)_{a_2 q}} \, \frac{z_1^p}{p!} \, \frac{z_2^q}{q!},$$

where all the parameters are such that the gamma functions and the Pochhammer symbols are well-defined. In these cases, the previous double power series corresponds to the following generalised Lauricella series (see [43])

$$\frac{\Gamma(a_3)}{\Gamma(a_3 + m) \, \Gamma(b_1)} \, F_{1:0;1}^{1:0;1} \left[\begin{array}{c|c} [a_1 : 1, a_2] : -; [a_3 : a_4] & \\ & z_1, \, z_2 \\ [b_1 : b_2, b_3] : -; [a_1 : a_2] & \end{array} \right], \tag{128}$$

where not all coefficients are positive as it is considered in [43].

Acknowledgements

The work of the authors was supported by Portuguese funds through CIDMA–Center for Research and Development in Mathematics and Applications, and FCT–Fundação para a Ciência e a Tecnologia, within projects UIDB/04106/2020 and UIDP/04106/2020.

N. Vieira was also supported by FCT via the 2018 FCT program of Stimulus of Scientific Employment - Individual Support (Ref: CEECIND/01131/2018).

The authors thank the anonymous reviewers for the careful reading of the manuscript, their suggestions, and additional references that have improved the article.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables. 10th printing*, National Bureau of Standards, Wiley-Interscience Publication, John Wiley & Sons, New York etc., 1972.

[2] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul. **44** (2017), 460–81.

[3] T.M. Atanackovic, S. Pilipovic, and D. Zorica, *A diffusion wave equation with two fractional derivatives of different order*, J. Phys. A Math. Theor. **40**-No.20 (2007), 5319–5333.

[4] J. Banasiak and R. Mika, *Singular perturbed telegraph equations with applications in random walk theory*, J. Appl. Stoch. Anal. **11**-No.1 (1998), 9–28.

[5] L. Beghin and E. Orsingher, *The telegraph process stopped at stable-distributed times and its connection with the fractional telegraph equation*, Fract. Calc. Appl. Anal. **6**-No.2 (2003), 187–204.

[6] L. Boyadjiev and Yu. Luchko, *The neutral-fractional telegraph equation*, Math. Model. Nat. Phenom. **12**-No.6 (2017), 51–67.

[7] R.G. Buschman, *H-functions of two variables I*, Indian J. Math., **20**, (1978), 132–153.

[8] R.F. Camargo, A.O. Chiacchio and E.C. de Oliveira, *Differentiation to fractional orders and the fractional telegraph equation*, J. Math. Phys., **49**-No.3, (2008), Article-No. 033505 (12pp).

[9] R.C. Cascaval, E.C. Eckstein, L.C. Frota and J.A. Goldstein, *Fractional telegraph equations*, J. Math. Anal. Appl., **276**-No.1, (2002), 145–159.

[10] C. Cattaneo, *Sur une forme de l'équation de la chaleur éliminant le paradoxe d'une propagation instantanée*, C. R. Acad. Sci., Paris **246**, (1958), 431–433.

[11] J. Chen, F. Liu, V. Anh, *Analytical solution for the time-fractional telegraph equation by the method of separating variables*, J. Math. Anal. Appl., **338**-No.2 (2008), 1364—1377.

[12] F.S. Costa, E.C. Oliveira, and A.R.G. Plata, *Fractional diffusion with time-dependent diffusion coefficient*, Rep. Math. Phys. **87**-No.1 (2021), 59–79.

[13] F. Effenberger and Yu. Litvinenko, *The diffusion approximation versus the telegraph equation for modeling solar energetic particle transport with adiabatic focusing*, Astrophys. J. **783**-No.1 (2014), Article-No. 15 (7pp).

[14] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher transcendental functions. Vol. I*, Bateman Manuscript Project, McGraw-Hill Book Co, New York, 1953.

[15] M. Ferreira, M.M. Rodrigues and N. Vieira, *Application of the fractional Sturm-Liouville theory to a fractional Sturm-Liouville telegraph equation*, Complex Anal. Oper. Theory **15**-No.5 (2021), Article ID: 87, 36 pp.

[16] M. Ferreira, M.M. Rodrigues, and N. Vieira, *First and second fundamental solutions of the time-fractional telegraph equation with Laplace or Dirac operators*, Adv. Appl. Clifford Algebr. **28**-No.2 (2018), Article No.42 (14pp).

[17] M. Ferreira, M.M. Rodrigues and N. Vieira, *Fundamental solution of the time-fractional telegraph Dirac operator*, Math. Methods Appl. Sci. **40**-No.18 (2017), 7033–7050.

[18] M. Ferreira, M.M. Rodrigues, and N. Vieira, *Fundamental solution of the multi-dimensional time fractional telegraph equation*, Fract. Calc. Appl. Anal. **20**-No.4 (2017), 868–894.

[19] M. Ferreira and N. Vieira, *Fundamental solutions of the time fractional diffusion-wave and parabolic Dirac operators*, J. Math. Anal. Appl. **447**-No.1 (2017), 329–353.

[20] R. Gorenflo, A.A. Kilbas, F. Mainardi, and S.V. Rogosin, *Mittag-Leffler functions, related topics and applications.* 2nd extended and updated edition, Springer Monographs in Mathematics, Springer, Berlin, 2020.

[21] R. Gorenflo, Yu. Luchko, and F. Mainardi, *Wright functions as scale-invariant solutions of the diffusion-wave equation*, J. Comput. Appl. Math. **118**-No.1-2 (2000), 175–191.

[22] K. Górska, A. Horzela, E.K. Lenzi, G. Pagnini, and T. Sandev, *Generalized Cattaneo (telegrapher's) equations in modeling anomalous diffusion phenomena*, Physical Review E **102**-No.2 (2020), Article No.022128 (13pp.).

[23] R. Hilfer, *Experimental evidence for fractional time evolution in glass forming materials*, Chem. Phys. **284** (2002), 399–408.

[24] A. Giusti, I. Colombaro, R. Garra, R. Garrapa, F. Polito, M. Popolizio, F. Mainardi, *A practical guide to Prabhakar fractional calculus*, Fract. Calc. Appl. Anal. **23**-No.1 (2020), 9–54.

[25] N.T. Hai and S. Yakubovich, *The double Mellin-Barnes integrals and their applications to convolution theory*, World Scientific, Singapore, 1992.

[26] H.J. Haubold, A.M. Mathai, and R.K. Saxena, *Mittag-Leffler functions and their applications*, J. Appl. Math. **2011** (2011), Article No.298628 (51pp).

[27] W. Hayt, *Engineering Electromagnetics*, 5th ed. McGraw-Hill, New York, 1989.

[28] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies-Vol.204, Elsevier, Amsterdam, 2006.

[29] A. Kilbas and M. Saigo, *H-transforms. Theory and applications*, Analytical Methods and Special Functions-Vol.9, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[30] K.B. Lima, J.V.C. Sousa, and E.C. Oliveira, *Ulam-Hyers type stability for ψ-Hilfer fractional differential equations with impulses and delay*, Comput. Appl. Math. **40**-No. 8 (2021), Article No. 293 (20pp).

[31] Y. Luchko and R. Gorenflo, *An operational method for solving fractional differential equations with the Caputo derivatives*, Acta Math. Vietnam. **24**-No.2, (1999), 207–233.

[32] M.O. Mamchuev, *Solutions of the main boundary value problems for the time-fractional telegraph equation by the Green function method*, Fract. Calc. Appl. Anal. **20**-No.1, (2017), 190–211

[33] J. Masoliver, *Telegraphic transport processes and their fractional generalization: A review and some extensions*, Entropy **23**-No.3 (2021), Article No.364 (32pp.).

[34] A. Okuko, *Application of the telegraph equation to oceanic diffusion: another mathematical model*, Technical Report No.69, Chesapeake Bay Institute, Johns Hopkins University, Baltimore, 1971.

[35] E. Orsingher and B. Toaldo, *Space-time fractional equations and the related stable processes at random time*, J. Theor. Probab. **30**-No.1 (2017), 1–26.

[36] E. Orsingher and L. Beghin, *Time-fractional telegraph equations and telegraph processes with Brownian time*, Probab. Theory Relat. Fields **128**- No.1 (2004), 141–160.

[37] M. D'Ovidio, E. Orsingher, and B. Toaldo, *Time-changed processes governed by space-time fractional telegraph equations*, Stoch. Anal. Appl. **32**-No.6 (2014), 1009–1045.

[38] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and series. Volume 5: Direct Laplace transforms*, Gordon and Breach Science Publishers, New York etc., 1992.

[39] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, New York, NY, 1993.

[40] G. Samorodnitsky and M.S. Taqqu, *Stable non-Gaussian random processes: stochastic models with infinite variance: Stochastic Modeling*, Chapman & Hall, New York, 1994.

[41] R.K. Saxena, R. Garra, and E. Orsingher, *Analytical solution of space-time fractional telegraph-type equations involving Hilfer and Hadamard derivatives*, Integral Transform. Spec. Funct. **27**-No.1 (2016), 30–42.

[42] J. Singh, *Analysis of fractional blood alcohol model with composite fractional derivative*, Chaos Solit. Fractals **140** (2020), Article No.110127 (7pp.).

[43] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian hypergeometric series*, Ellis Horwood Series in Mathematics and Its Applications, Halsted Press: a division of John Wiley & Sons, Chichester, New York etc., 1985.

[44] H.M. Srivastava and M.C. Daoust, *A note on the convergence of Kampé de Fériet's double hypergeometrics series*, Math. Nachr. **53** (1972), 151–159.

[45] J.V.C. Sousa, R.F. Camargo, E.C. Oliveira, and G.S.F. Frederico, *Pseudo-fractional differential equations and generalized g-Laplace transform*, J. Pseudo-Differ. Oper. Appl. **12** (2021), Article No.44 (27pp.).

[46] J.V.C. Sousa and E. C. Oliveira, *On the ψ-Hilfer derivative*, Commun. Nonlinear Sci. Numer. Simulat. **60** (2018), 72–91.

[47] W. Thomson, *On the theory of the electric telegraph*, Proc. R. Soc. Lond, Ser. I **7** (1854), 382–399.

[48] Z. Tomovski, *Generalized Cauchy type problems for nonlinear fractional differential equations with composite fractional derivative operator*, Nonlinear Anal. **75** (2012), 3364–3384.

[49] N. Vieira, M.M. Rodrigues, and M. Ferreira, *Time-fractional telegraph equation of distributed order in higher dimensions*, Commun. Nonlinear Sci. Numer. Simulat. **102** (2021), Article No.105925 (32pp).

[50] V.H. Weston and S. He, *Wave splitting of telegraph equation in $\mathbb{R}$ and its application to inverse scattering*, Inverse Probl. **9**-No.6 (1993), 789–812.