

# Eigenfunctions of the time-fractional diffusion-wave operator\*

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## Abstract

In this paper, we present some new integral and series representations for the eigenfunctions of the multidimensional time-fractional diffusion-wave operator with the time-fractional derivative of order  $\beta \in ]1, 2[$  defined in the Caputo sense. The integral representations are obtained in form of the inverse Fourier-Bessel transform and as a double contour integrals of the Mellin-Barnes type. Concerning series expansions, the eigenfunctions are expressed as the double generalized hypergeometric series for any  $\beta \in ]1, 2[$  and as Kampé de Fériet and Lauricella series in two variables for the rational values of  $\beta$ . The limit cases  $\beta = 1$  (diffusion operator) and  $\beta = 2$  (wave operator) as well as an intermediate case  $\beta = \frac{3}{2}$  are studied in detail. Finally, we provide several plots of the eigenfunctions to some selected eigenvalues for different particular values of the fractional derivative order  $\beta$  and the spatial dimension  $n$ .

**Keywords:** Time-fractional diffusion-wave operator; Eigenfunctions; Caputo fractional derivatives; Generalized hypergeometric series.

**MSC 2010:** 35R11; 26A33; 47F05; 35C15; 33C65.

## 1 Introduction

In the last decades, there is an increased interest in fractional calculus, in particular in the study of fractional ordinary and partial differential equations. The introduction of fractional derivatives allows representing the physical reality more accurately by introducing a memory mechanism in the process (see [5]), e.g., if we consider the case of a time-fractional derivative, its calculation at a given time requires its knowledge at all previous times. Time-fractional diffusion-wave equations are obtained from the standard diffusion and wave equations by replacing the time derivative by a fractional derivative of order  $\beta \in ]0, 2[$ . These equations represent anomalous diffusion ( $0 < \beta < 1$ ) or anomalous wave propagation ( $1 < \beta < 2$ ) and have been studied over the last years

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\*The final version is published in *Mathematical Methods in the Applied Science*, **44**-No.2, (2021), 1713 – 1743. It is available via the website: <http://dx.doi.org/10.1002/mma.6874>

by several authors. In the one-dimensional case the time-fractional diffusion-wave equations have been studied comprehensively in several papers (see, for example, [10,23,24,26–29,31,35]). For the multidimensional case there are some works in this direction (see e.g. [7–9,13,14,18,19,22,31]). Still, the list of works on the multidimensional case is shorter when compared with the one of the one-dimensional case, and thus further investigations of the multidimensional case are required. One interesting subject for the multidimensional case is the study of the eigenvalues and the correspondent eigenfunctions of the time-fractional diffusion-wave operator. The knowledge of eigenvalues and eigenfunctions plays an important role in the study of the existence of solutions of nonlinear perturbations of this operator.

In this work, we consider the eigenfunction equation for the time-fractional diffusion-wave operator, i.e. we consider the following fractional partial differential equation

$$\left(\partial_t^\beta - c^2 \Delta_x\right) u(x, t) = \lambda u(x, t), \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $t > 0$ ,  $1 < \beta < 2$ ,  $\lambda \in \mathbb{C}$ ,  $c > 0$ , and  $\Delta_x$  is the Laplace operator in  $\mathbb{R}^n$ . For  $\gamma > 0$ ,  $\partial_t^\gamma$  is the Caputo fractional derivative of order  $\gamma$ .

The aim of this paper is to obtain explicit representations of the solutions of (1) in the form of double Mellin-Barnes contour integrals and series expansions in terms of generalized hypergeometric series of two variables. For rational  $\beta$  we were able to deduce series expansions in terms of Kampé de Fériet and Lauricella series of two variables. These series were first introduced by Appell in 1880 (see [3]) and later studied by himself and Kampé de Fériet (see [2,15]). It was Lauricella in [20] who proposed, in a straightforward way, the  $n$ -variable extension of the Appell series. The Lauricella series has special importance for applied mathematics and mathematical physics because elliptic integrals are hypergeometric functions of this type (see [6,21]). Elliptic integrals arise in several physical contexts, for example, the study of radiation field problems, the theory of scattering of acoustics electromagnetic waves by means of an elliptic disk, just to mention some (see [21] and the references therein indicated).

The structure of the papers reads as follows. In the preliminaries section we recall some basic concepts about fractional calculus and special functions needed for this work. In Section 3.1 we obtain a representation of the solutions of (1) in terms of contour integrals of two variables. In Section 3.1 we deduce, for the case of non-coincident sequence of poles corresponding to odd  $n$ , the series expansion of the eigenfunctions in terms of hypergeometric series of two variables. In Section 3.2 we restrict  $\beta$  to a rational number and obtain a representation of the eigenfunctions in terms of generalized hypergeometric series of two variables, more precisely in terms of Kampé de Fériet series and Lauricella series. We observe that, due to the extension and complexity of the involved expressions, the case of even  $n$  (corresponding to coincident sequence of poles) is only considered in the limit cases presented in Section 4.2 ( $\beta = 1$  - diffusion operator) and Section 5.2 ( $\beta = 2$  - wave operator). Moreover, in Sections 4.2 and 5.2 we explain how to deal with the singularities that appear in the series representation for even  $n$ . Another interesting and important particular case of the problem ( $\beta = \frac{3}{2}$ ) is considered in Section 6. Finally, several plots of the eigenfunctions to certain eigenvalues and for the dimensions  $n = 1, 2, 3$  are presented for all values of  $\beta$  mentioned above.

## 2 Preliminaries

In this section we present the main tools concerning fractional derivatives and special functions that we will use in our work. Let  $a, b \in \mathbb{R}$  with  $a < b$  let  $\alpha > 0$ . The left Riemann-Liouville fractional integral  $I_{a+}^\alpha$  of order  $\alpha$  is given by (see [17])

$$(I_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > a \quad (2)$$

Let  ${}^CD_{a+}^\alpha$  denote the left Caputo fractional derivative of order  $\alpha > 0$  on  $[a, b] \subset \mathbb{R}$ , which is defined by (see [17])

$$({}^CD_{a+}^\alpha f)(x) = (I_{a+}^{m-\alpha} D^m f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad x > a \quad (3)$$

where  $m = [\alpha] + 1$  and  $[\alpha]$  means the integer part of  $\alpha$ . We recall the Gauss-Legendre formula associated to the gamma function  $\Gamma(z)$  for later use (see [1])

$$\Gamma(mz) = m^{mz - \frac{1}{2}} (2\pi)^{\frac{1-m}{2}} \prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right), \quad m \in \mathbb{N}. \quad (4)$$

Another special function that we use in our work is the multivariate Mittag-Leffler function (see [25, 30]).

**Definition 2.1** (see [25]) *The multivariate Mittag-Leffler function  $E_{(a_1, \dots, a_r), b}(z_1, \dots, z_r)$  of  $r$  complex variables  $z_1, \dots, z_r \in \mathbb{C}$  with complex parameters  $a_1, \dots, a_r \in \mathbb{C}$  (with positive real parts) is defined by*

$$E_{(a_1, \dots, a_r), b}(z_1, \dots, z_r) = \sum_{k=0}^{+\infty} \sum_{\substack{l_1 + \dots + l_r = k \\ l_1, \dots, l_r \geq 0}} \binom{k}{l_1, \dots, l_r} \frac{\prod_{i=1}^r z_i^{l_i}}{\Gamma(b + \sum_{i=1}^r a_i l_i)}, \quad (5)$$

where the multinomial coefficients are given by

$$\binom{k}{l_1, \dots, l_r} := \frac{k!}{l_1! \times \dots \times l_r!}.$$

In particular, when  $r = 2$ , the multivariate Mittag-Leffler function (5) can be written as

$$E_{(a_1, a_2), b}(z_1, z_2) = \sum_{k=0}^{+\infty} \sum_{\substack{l_1 + l_2 = k \\ l_1, l_2 \geq 0}} \frac{k!}{l_1! l_2!} \frac{z_1^{l_1} z_2^{l_2}}{\Gamma(b + a_1 l_1 + a_2 l_2)} \quad (6)$$

$$= \sum_{l_1=0}^{+\infty} \sum_{l_2=0}^{+\infty} \frac{(l_1 + l_2)!}{l_1! l_2!} \frac{z_1^{l_1} z_2^{l_2}}{\Gamma(b + a_1 l_1 + a_2 l_2)}. \quad (7)$$

For  $r = 1$ , the multivariate Mittag-Leffler function (5) reduces to the two-parameter Mittag-Leffler function

$$E_{a_1, b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(b + k a_1)}, \quad a_1, b, z \in \mathbb{C}; \quad \Re(a_1) > 0.$$

For general properties of the Mittag-Leffler function see [12, 25]. We present now the definition of the Fox H-function in terms of a contour integral.

**Definition 2.2** (see [16]) *The Fox H-function  $H_{p,q}^{m,n}$  of a complex variable  $z$  is defined via a Mellin-Barnes type integral of the form:*

$$H_{p,q}^{m,n} \left[ z \left| \begin{array}{c} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{array} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)} z^{-s} ds, \quad (8)$$

where  $m, n, p, q \in \mathbb{N}$  such that  $0 \leq m \leq q$ ,  $0 \leq n \leq p$ ,  $a_i, b_j \in \mathbb{C}$ , and  $\alpha_i, \beta_j \in \mathbb{R}^+$  ( $i = 1, 2, \dots, p$ ;  $j = 1, 2, \dots, q$ ).

When all  $\alpha_i$  and  $\beta_j$  become equal to one the H-function reduces to the Meijer G-function. The conditions for the existence of the Fox H-function and the orientation of the contour  $\mathcal{L}$  are given by Theorem 1.1 in [16]. The previous definition can be generalized for the case of  $r$  complex variables, however, due to the purposes of our work we present only the definition for the case  $r = 2$ .

**Definition 2.3** (cf. [4]) *The H-function of two complex variables is defined via a double Mellin-Barnes type integral of the form*

$$\begin{aligned} H[z_1, z_2] &= H_{p_1, q_1; p_2, q_2; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} \left[ \begin{array}{c} z_1 \\ z_2 \end{array} \left| \begin{array}{c} (a_j; \alpha_j, A_j)_{1, p_1}; (c_j, \gamma_j)_{1, p_2}; (e_j, E_j)_{1, p_3} \\ (b_j; \beta_j, B_j)_{1, q_1}; (d_j, \delta_j)_{1, q_2}; (f_j, F_j)_{1, q_3} \end{array} \right. \right] \\ &= \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_2} \int_{\mathcal{L}_1} \phi(s, t) \phi_1(s) \phi_2(w) z_1^{-s} z_2^{-w} ds dw, \end{aligned}$$

where

$$\phi(s, w) = \frac{\prod_{j=1}^{m_1} \Gamma(b_j - \beta_j s - B_j w) \prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j s + A_j w)}{\prod_{j=m_1+1}^{q_1} \Gamma(1 - b_j + \beta_j s + B_j w) \prod_{j=n_1+1}^{p_1} \Gamma(a_j - \alpha_j s - A_j w)},$$

$$\phi_1(s) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - \delta_j s) \prod_{j=1}^{n_2} \Gamma(1 - c_j + \gamma_j s)}{\prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + \delta_j s) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - \gamma_j s)},$$

$$\phi_2(w) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j w) \prod_{j=1}^{n_3} \Gamma(1 - e_j + E_j w)}{\prod_{j=m_3+1}^{q_3} \Gamma(1 - f_j + F_j w) \prod_{j=n_3+1}^{p_3} \Gamma(e_j - E_j w)},$$

with  $z_1, z_2 \in \mathbb{C}$ ,  $m_i, n_i, p_i, q_i \in \mathbb{N}_0$  such that  $0 \leq m_i \leq q_i$ ,  $0 \leq n_i \leq p_i$  ( $i = 1, 2, 3$ ),  $a_j, b_j, c_j, d_j, e_j, f_j \in \mathbb{C}$ , and  $\alpha_j, A_j, \beta_j, B_j, \gamma_j, \delta_j, E_j, F_j \in \mathbb{R}^+$ .

The conditions for the analyticity and convergence of this special function, its general properties, and the orientation of the contours  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are studied e.g. in [4]. The H-function contains a vast number of special functions as particular cases. For example, when  $\alpha_j, A_j, \beta_j, B_j, \gamma_j, \delta_j, E_j, F_j$  are all equal to one it reduces to the Meijer G-function of two variables, i.e.,

$$\begin{aligned} & G_{p_1, q_1; p_2, q_2; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} \left[ \begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} (a_j)_{1, p_1}; (c_j)_{1, p_2}; (e_j)_{1, p_3} \\ (b_j)_{1, q_1}; (d_j)_{1, q_2}; (f_j)_{1, q_3} \end{matrix} \right] \\ &= H_{p_1, q_1; p_2, q_2; p_3, q_3}^{m_1, n_1; m_2, n_2; m_3, n_3} \left[ \begin{matrix} z_1 \\ z_2 \end{matrix} \middle| \begin{matrix} (a_j; 1)_{1, p_1}; (c_j, 1)_{1, p_2}; (e_j, 1)_{1, p_3} \\ (b_j; 1)_{1, q_1}; (d_j, 1)_{1, q_2}; (f_j, 1)_{1, q_3} \end{matrix} \right]. \end{aligned} \quad (9)$$

An interesting generalization of the hypergeometric series  ${}_pF_q$  is due to Fox (1928) and Wright (1935) who studied the asymptotic expansion of the generalized hypergeometric function  ${}_p\Psi_q$ , which is defined by (see [32])

$${}_p\Psi_q \left[ \begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix} \middle| z \right] = \sum_{k=0}^{+\infty} \frac{\prod_{j=1}^p \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^q \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}, \quad (10)$$

where  $\alpha_j, \beta_i \in \mathbb{R}$ , with  $j = 1, \dots, p$  and  $i = 1, \dots, q$ ,  $z \in \mathbb{C}$  and the coefficients  $A_1, \dots, A_p$  and  $B_1, \dots, B_q$  are positive real numbers such that  $1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j \geq 0$ . Moreover, when all  $A_j$  and  $B_j$  are equal to one we obtain

$${}_p\Psi_q \left[ \begin{matrix} (\alpha_1, 1), \dots, (\alpha_p, 1) \\ (\beta_1, 1), \dots, (\beta_q, 1) \end{matrix} \middle| z \right] = \frac{\prod_{j=1}^p \Gamma(\alpha_j)}{\prod_{j=1}^q \Gamma(\beta_j)} {}_pF_q \left[ \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \middle| z \right]. \quad (11)$$

In [34] the authors considered a multivariate extension of the series  ${}_p\Psi_q$  defined in (10). Their multiple hypergeometric series, known as the generalized Lauricella series in several variables, is defined by

$$\begin{aligned} & F_{C:D'; \dots; D^{(r)}}^{A:B'; \dots; B^{(r)}}(z_1, \dots, z_r) \equiv F_{C:D'; \dots; D^{(r)}}^{A:B'; \dots; B^{(r)}} \left( \begin{matrix} [(a) : \theta', \dots, \theta^r] : [(b') : \phi']; \dots; [(b^{(r)}) : \phi^{(r)}]; \\ [(c) : \psi', \dots, \psi^r] : [(d') : \delta']; \dots; [(d^{(r)}) : \delta^{(r)}]; \end{matrix} \middle| z_1, \dots, z_r \right) \\ &= \sum_{m_1, \dots, m_r=0}^{+\infty} \Omega(m_1, \dots, m_r) \frac{z_1^{m_1}}{m_1!} \dots \frac{z_r^{m_r}}{m_r!}, \end{aligned} \quad (12)$$

where

$$\Omega(m_1, \dots, m_r) = \frac{\prod_{j=1}^A (a_j)_{m_1 \theta'_j + \dots + m_r \theta_j^{(r)}} \prod_{j=1}^{B'} (b'_j)_{m_1 \phi'_j} \prod_{j=1}^{B^{(r)}} (b_j^{(r)})_{m_r \phi_j^{(r)}}}{\prod_{j=1}^C (c_j)_{m_1 \psi'_j + \dots + m_r \psi_j^{(r)}} \prod_{j=1}^{D'} (d'_j)_{m_1 \delta'_j} \prod_{j=1}^{D^{(r)}} (d_j^{(r)})_{m_r \delta_j^{(r)}}},$$

and the coefficients

$$\begin{cases} \theta_j^{(k)}, j = 1, \dots, A; & \phi_j^{(k)}, j = 1, \dots, B^{(k)}; & \psi_j^{(k)}, j = 1, \dots, C; \\ \delta_j^{(k)}, j = 1, \dots, D^{(k)}; & \forall k \in \{1, \dots, n\} \end{cases} \quad (13)$$

are real,  $(a)$  abbreviates the array of  $A$  parameters  $a_1, \dots, a_A$ ,  $(b^{(k)})$  abbreviates the array of  $B^{(k)}$  parameters  $b_j^{(k)}, j = 1, \dots, B^{(k)}; \forall k \in \{1, \dots, r\}$ , with similar interpretation for  $(c)$  and  $(d^{(k)})$ ,  $k = 1, \dots, r$ ; etc. For the precise conditions under which the multiple series (12) and the particular case of  $n = 2$  converge absolutely we refer to [33]. When all the coefficients in (13) are equal to 1 and  $r = 2$ , the generalized Lauricella series (12) reduces to the Kampé de Fériet series of two variables (see [34]):

$$F_{C:D';D''}^{A:B';B''}(z_1, z_2) \equiv F_{C:D';D''}^{A:B';B''} \left( \begin{matrix} (a) : (b'); b''; \\ (c) : (d'); d''; \end{matrix} \middle| z_1, z_2 \right). \quad (14)$$

### 3 Eigenfunctions of the time-fractional diffusion-wave-operator

#### 3.1 Representation in terms of contour integrals

In this section, we deduce the representations of the eigenfunctions of the time-fractional diffusion-wave operator in terms of the double Mellin-Barnes integrals, the H-functions of two variables, and the Meijer G-function of two variables.

We look for a function  $u_\lambda^\beta$  such that

$$\left( \partial_t^\beta - c^2 \Delta_x \right) u_\lambda^\beta(x, t) = \lambda u_\lambda^\beta(x, t), \quad (15)$$

where  $x \in \mathbb{R}^n$ ,  $t > 0$ ,  $\lambda \in \mathbb{C}$ ,  $\Delta_x$  is the Laplace operator in  $\mathbb{R}^n$ , and  $\partial_t^\beta$  is the left Caputo fraction derivative (3) of order  $1 < \beta < 2$  with respect to the variable  $t$ . We additionally assume that  $u_\lambda^\beta$  satisfies that following initial conditions:

$$u_\lambda^\beta(x, 0) = \delta(x) \quad \text{and} \quad \frac{\partial u_\lambda^\beta}{\partial t}(x, 0) = 0, \quad (16)$$

where  $\delta(x) = \prod_{j=1}^n \delta(x_j)$  is the distributional Dirac delta function in  $\mathbb{R}^n$ . Problem (15)-(16) can be seen as a particular case of the problem studied in [8] where the authors deduced the first fundamental solution, denoted by  $G_1^{\alpha, \beta}$ , of the time-fractional telegraph equation:

$$\begin{cases} \left( \partial_t^\beta + a \partial_t^\alpha - c^2 \Delta_x \right) G_1^{\alpha, \beta}(x, t) = 0 \\ G_1^{\alpha, \beta}(x, 0) = \delta(x) \\ \frac{\partial G_1^{\alpha, \beta}}{\partial t}(x, 0) = 0 \end{cases}, \quad (17)$$

with  $a > 0$  and  $0 \leq \alpha < 1$ . Therefore, making  $\alpha = 0$  and  $a = -\lambda$  in the integral representation of the solution of (17) (expressions (3.15) and (3.16) in [8]), we obtain the following representation of the eigenfunctions of the time-fractional diffusion-wave equation in terms of Fourier-Bessel integral (18), double Mellin-Barnes integrals

(19) and H-function of two variables (20):

$$u_\lambda^\beta(x, t) = \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^{+\infty} \tau^{\frac{n}{2}} E_{(\beta, \beta), 1}(\lambda t^\beta, -c^2 \tau^2 t^\beta) J_{\frac{n}{2}-1}(\tau|x|) d\tau$$

$$- \frac{\lambda t^\beta |x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^{+\infty} \tau^{\frac{n}{2}} E_{(\beta, \beta), 1+\beta}(\lambda t^\beta, -c^2 \tau^2 t^\beta) J_{\frac{n}{2}-1}(\tau|x|) d\tau \quad (18)$$

$$= \frac{1}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma(1+w+s) \Gamma(\frac{n}{2}+s) \Gamma(-w)}{\Gamma(1+\beta w+\beta s)} (-\lambda t^\beta)^w \left(\frac{4c^2 t^\beta}{|x|^2}\right)^s dw ds$$

$$- \frac{\lambda t^\beta}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma(1+w+s) \Gamma(\frac{n}{2}+s) \Gamma(-w)}{\Gamma(1+\beta+\beta w+\beta s)} (-\lambda t^\beta)^w \left(\frac{4c^2 t^\beta}{|x|^2}\right)^s dw ds \quad (19)$$

$$= \frac{1}{\pi^{\frac{n}{2}} |x|^n} H_{1,1;1,0;0,1}^{0,1;0,1;1,0} \left[ \begin{matrix} 4c^2 t^\beta |x|^{-2} \\ -\lambda t^\beta \end{matrix} \middle| \begin{matrix} (0; 1, 1); (1 - \frac{n}{2}, 1); - \\ (0; \beta, \beta); -; (0, 1) \end{matrix} \right]$$

$$- \frac{\lambda t^\beta}{\pi^{\frac{n}{2}} |x|^n} H_{1,1;1,0;0,1}^{0,1;0,1;1,0} \left[ \begin{matrix} 4c^2 t^\beta |x|^{-2} \\ -\lambda t^\beta \end{matrix} \middle| \begin{matrix} (0; 1, 1); (1 - \frac{n}{2}, 1); - \\ (-\beta; \beta, \beta); -; (0, 1) \end{matrix} \right], \quad (20)$$

where  $|x|^2 = x_1^2 + \dots + x_n^2$  is the length squared of a vector  $x \in \mathbb{R}^n$ ,  $J_\nu$  represents the Bessel function of first kind with index  $\nu$  (see [1]), and  $E_{(a_1, a_2), b}$  is the bivariate Mittag-Leffler function (7).

### 3.2 Representation in terms of generalized hypergeometric series for real $\beta$

In this section, we obtain a representation of the eigenfunctions of the time-fractional diffusion-wave-operator in terms of double series when  $\beta$  is a real number in  $]1, 2[$ . Considering the reflections  $s \mapsto -s$  and  $w \mapsto -w$  in (19) we obtain

$$u_\lambda^\beta(x, t) = \frac{1}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma(1-w-s) \Gamma(\frac{n}{2}-s) \Gamma(w)}{\Gamma(1-\beta w-\beta s)} (-\lambda t^\beta)^{-w} \left(\frac{4c^2 t^\beta}{|x|^2}\right)^{-s} dw ds$$

$$- \frac{\lambda t^\beta}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma(1-w-s) \Gamma(\frac{n}{2}-s) \Gamma(w)}{\Gamma(1+\beta-\beta w-\beta s)} (-\lambda t^\beta)^{-w} \left(\frac{4c^2 t^\beta}{|x|^2}\right)^{-s} dw ds. \quad (21)$$

Applying conditions (3.5) of Theorem 3.1 in [11], the above double Mellin-Barnes integrals is convergent for all  $|x|, t \in \mathbb{R}^+$ . Moreover, by Theorem 1.1 in [16] we can choose the contour  $\mathcal{L}_2$  as  $\mathcal{L}_{-\infty}$ . To compute the above integrals we need to apply Residue Theory. As the gamma functions  $\Gamma(w)$  have simple poles at  $w = -m_1$ , with  $m_1 \in \mathbb{N}_0$ , then applying the Residue Theorem to (21) we obtain

$$u_\lambda^\beta(x, t) = \frac{1}{\pi^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \frac{(\lambda t^\beta)^{m_1}}{m_1!} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \frac{\Gamma(1+m_1-s) \Gamma(\frac{n}{2}-s)}{\Gamma(1+\beta m_1-\beta s)} \left(\frac{4c^2 t^\beta}{|x|^2}\right)^{-s} ds$$

$$- \frac{\lambda t^\beta}{\pi^{\frac{n}{2}} |x|^n} \sum_{m_1=0}^{+\infty} \frac{(\lambda t^\beta)^{m_1}}{m_1!} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \frac{\Gamma(1+m_1-s) \Gamma(\frac{n}{2}-s)}{\Gamma(1+\beta+\beta m_1-\beta s)} \left(\frac{4c^2 t^\beta}{|x|^2}\right)^{-s} ds. \quad (22)$$

To compute the integral with respect to  $s$  the parity of the space dimension needs to be taken into account. When  $n$  is odd we obtain non-coincident sequences of simple poles and when  $n$  is even we obtain a sequence with simple and double poles. Due to the extension and complexity of the involved expressions, we will consider at this moment only the case of simple poles, i.e., the case of odd  $n$ . Hence, for  $n$  odd in (22) the integrals with respect to  $s$  are convergent and  $\mathcal{L}_1 = \mathcal{L}_{+\infty}$  (cf. [16, Thm. 1.1]). The gamma functions  $\Gamma(1+m_1-s)$  and  $\Gamma(\frac{n}{2}-s)$  have the following sequence of simple poles, respectively:

- $s = 1 + m_1 + m_2$ , with  $m_2 \in \mathbb{N}_0$ ;
- $s = \frac{n}{2} + m_2$ , with  $m_2 \in \mathbb{N}_0$ .

Hence, applying the Residue Theorem to (22) we get

$$\begin{aligned}
u_\lambda^\beta(x, t) = & \frac{|x|^{2-n}}{4c^2 \pi^{\frac{n}{2}} t^\beta} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma\left(\frac{n}{2} - 1 - m_1 - m_2\right)}{m_1! m_2! \Gamma(1 - \beta - \beta m_2)} \left(-\frac{\lambda |x|^2}{4c^2}\right)^{m_1} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^{m_2} \\
& + \frac{1}{(4c^2 \pi t^\beta)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma\left(1 - \frac{n}{2} + m_1 - m_2\right)}{m_1! m_2! \Gamma\left(1 - \frac{\beta n}{2} + \beta m_1 - \beta m_2\right)} (-\lambda t^\beta)^{m_1} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^{m_2} \\
& - \frac{\lambda |x|^{2-n}}{4c^2 \pi^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma\left(\frac{n}{2} - 1 - m_1 - m_2\right)}{m_1! m_2! \Gamma(1 - \beta m_2)} \left(-\frac{\lambda |x|^2}{4c^2}\right)^{m_1} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^{m_2} \\
& - \frac{\lambda t^{\beta(1-\frac{n}{2})}}{(4c^2 \pi)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma\left(1 - \frac{n}{2} + m_1 - m_2\right)}{m_1! m_2! \Gamma\left(1 + \beta - \frac{\beta n}{2} + \beta m_1 - \beta m_2\right)} (-\lambda t^\beta)^{m_1} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^{m_2}, \quad (23)
\end{aligned}$$

which corresponds to the representation in terms of double hypergeometric series of the eigenfunctions of the time-fractional diffusion-wave operator of order  $\beta \in ]1, 2[$ .

### 3.3 Representation in terms of hypergeometric series for rational $\beta$

In this section, we obtain a representation of the eigenfunctions of the time-fractional diffusion-wave-operator in terms of Kampé de Fériet and generalized Lauricella series when  $\beta$  is a rational number, i.e.  $\beta = \frac{p}{q}$  with  $p, q \in \mathbb{N}$ . Since  $1 < \beta < 2$  then  $q < p < 2q$ . Hence, from (19) we get

$$\begin{aligned}
u_\lambda^{\frac{p}{q}}(x, t) = & \frac{1}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma(1+w+s) \Gamma\left(\frac{n}{2} + s\right) \Gamma(-w)}{\Gamma\left(1 + \frac{p}{q}(w+s)\right)} \left(-\lambda t^{\frac{p}{q}}\right)^w \left(\frac{4c^2 t^{\frac{p}{q}}}{|x|^2}\right)^s dw ds \\
& - \frac{\lambda t^{\frac{p}{q}}}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma(1+w+s) \Gamma\left(\frac{n}{2} + s\right) \Gamma(-w)}{\Gamma\left(1 + \frac{p}{q}(1+w+s)\right)} \left(-\lambda t^{\frac{p}{q}}\right)^w \left(\frac{4c^2 t^{\frac{p}{q}}}{|x|^2}\right)^s dw ds. \quad (24)
\end{aligned}$$

Considering the change of variables  $s = q s_1$  and  $w = q w_1$  in the previous integrals, we get

$$\begin{aligned}
u_\lambda^{\frac{p}{q}}(x, t) = & \frac{q^2}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma\left(q\left(\frac{1}{q} + w_1 + s_1\right)\right) \Gamma\left(q\left(\frac{n}{2q} + s_1\right)\right) \Gamma(-q w_1)}{\Gamma\left(p\left(\frac{1}{p} + w_1 + s_1\right)\right)} \left(-\lambda t^{\frac{p}{q}}\right)^{q w_1} \left(\frac{4c^2 t^{\frac{p}{q}}}{|x|^2}\right)^{q s_1} dw_1 ds_1 \\
& - \frac{\lambda t^{\frac{p}{q}} q^2}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma\left(q\left(\frac{1}{q} + w_1 + s_1\right)\right) \Gamma\left(q\left(\frac{n}{2q} + s_1\right)\right) \Gamma(-q w_1)}{\Gamma\left(p\left(\frac{1}{p} + \frac{1}{q} + w_1 + s_1\right)\right)} \left(-\lambda t^{\frac{p}{q}}\right)^{q w_1} \left(\frac{4c^2 t^{\frac{p}{q}}}{|x|^2}\right)^{q s_1} dw_1 ds_1. \quad (25)
\end{aligned}$$

Applying now the Gauss-Legendre formula (4) for the gamma function and after straightforward calculations, we obtain

$$\begin{aligned}
u_\lambda^{\frac{p}{q}}(x, t) = & \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2}}} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\prod_{k=1}^{q-1} \Gamma\left(\frac{k}{q} + w_1 + s_1\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} + s_1\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} - w_1\right)}{\prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p} + w_1 + s_1\right)} \\
& \times \left(\frac{(-\lambda)^q t^p}{p^p}\right)^{w_1} \left(\frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}}\right)^{s_1} dw_1 ds_1
\end{aligned}$$

$$\begin{aligned}
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\prod_{k=1}^q \Gamma\left(\frac{k}{q} + w_1 + s_1\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} + s_1\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} - w_1\right)}{\prod_{k=1}^p \Gamma\left(\frac{k}{p} + \frac{1}{q} + w_1 + s_1\right)} \\
& \quad \times \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{w_1} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{s_1} dw_1 ds_1. \tag{26}
\end{aligned}$$

Expression (26) can be written in terms of a Meijer G-function of two variables (see (9)):

$$\begin{aligned}
u_{\lambda}^{\frac{p}{q}}(x, t) &= \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} G_{q,p; q,0; q,0}^{0,q; 0,q; q,0} \left[ \begin{matrix} (4c^2 q^2)^q t^p p^{-p} |x|^{-2q} \\ (-\lambda)^q t^p p^{-p} \end{matrix} \middle| \begin{matrix} \left(\frac{k}{q}\right)_{1,q}; \left(\frac{k-1}{q} + \frac{n}{2q}\right)_{1,q}; - \\ \left(\frac{k}{p}\right)_{1,p}; -; \left(\frac{k-1}{q}\right)_{1,q} \end{matrix} \right] \\
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} G_{q,p; q,0; q,0}^{0,q; 0,q; q,0} \left[ \begin{matrix} (4c^2 q^2)^q t^p p^{-p} |x|^{-2q} \\ (-\lambda)^q t^p p^{-p} \end{matrix} \middle| \begin{matrix} \left(\frac{k}{q}\right)_{1,q}; \left(\frac{k-1}{q} + \frac{n}{2q}\right)_{1,q}; - \\ \left(\frac{k}{p} + \frac{1}{q}\right)_{1,p}; -; \left(\frac{k-1}{q}\right)_{1,q} \end{matrix} \right]. \tag{27}
\end{aligned}$$

Making the change of variables  $s_1 = -s$  and  $w_1 = -w$  in (26) we obtain

$$\begin{aligned}
& u_{\lambda}^{\frac{p}{q}}(x, t) \\
&= \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\prod_{k=1}^{q-1} \Gamma\left(\frac{k}{q} - w - s\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - s\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + w\right)}{\prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p} - w - s\right)} \\
& \quad \times \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{-w} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-s} dw ds \\
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\prod_{k=1}^q \Gamma\left(\frac{k}{q} - w - s\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - s\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + w\right)}{\prod_{k=1}^p \Gamma\left(\frac{k}{p} + \frac{1}{q} - w - s\right)} \\
& \quad \times \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{-w} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-s} dw ds. \tag{28}
\end{aligned}$$

To compute now the above integrals we need to apply Residue Theory. The inner integrals with respect to  $w$  are convergent and  $\mathcal{L}_2 = \mathcal{L}_{-\infty}$  (cf. [16, Thm. 1.1]). As the gamma functions  $\prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + w\right)$  have simple poles at  $w = -\frac{k_1-1}{q} - m_1$ , with  $m_1 \in \mathbb{N}_0$ , for each fixed  $k_1$ , where  $k_1 = 1, \dots, q$ , then from (28) we obtain

$$\begin{aligned}
& u_{\lambda}^{\frac{p}{q}}(x, t) \\
&= \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \frac{1}{2\pi i} \sum_{k_1=1}^q \sum_{m_1=0}^{+\infty} \left\{ \frac{(-1)^{m_1}}{m_1!} \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q} + m_1} \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q} - m_1\right) \right. \\
& \quad \left. \int_{\mathcal{L}_1} \frac{\prod_{k=1}^{q-1} \Gamma\left(\frac{k}{q} + \frac{k_1-1}{q} + m_1 - s\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - s\right)}{\prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p} - s + \frac{k_1-1}{q} + m_1\right)} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-s} ds \right\}
\end{aligned}$$



$$\begin{aligned}
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \frac{1}{2\pi i} \sum_{k_1=1}^q \sum_{m_1=0}^{+\infty} \left\{ \frac{(-1)^{m_1}}{m_1!} \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q} + m_1} \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q} - m_1\right) \right. \\
& \left. \int_{\mathcal{L}_1} \frac{\prod_{k=1}^q \Gamma\left(\frac{k}{q} + \frac{k_1-1}{q} + m_1 - s\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - s\right)}{\prod_{k=1}^p \Gamma\left(\frac{k}{p} + \frac{1}{q} - s + \frac{k_1-1}{q} + m_1\right)} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-s} ds \right\}. \quad (29)
\end{aligned}$$

To compute the integral with respect to  $s$  the parity of the space dimension needs to be taken into account. When  $n$  is odd we obtain sequences of simple poles and when  $n$  is even we obtain sequences of double poles. Due to the extension and complexity of the involved expressions, we will consider at this moment only the case of simple poles, i.e., the case of  $n$  odd. The case of  $n$  even is only considered in the limit cases presented in Sections 4.2 and 5.2. Hence, for  $n$  odd in (26) the integrals with respect to  $s$  are convergent and  $\mathcal{L}_1 = \mathcal{L}_{+\infty}$  (cf. [16, Thm. 1.1]). The gamma functions  $\prod_{k=1}^{\theta} \Gamma\left(\frac{k}{q} - s + \frac{k_1-1}{q} + m_1\right)$ , with  $\theta \in \{q-1, q\}$ , and  $\prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - s\right)$  have the following sequence of simple poles, respectively:

- $s = \frac{k_2}{q} + \frac{k_1-1}{q} + m_1 + m_2$ , with  $m_2 \in \mathbb{N}_0$ , for each fixed  $k_2$ , where  $k_2 = 1, \dots, \theta$ ,  $\theta \in \{q-1, q\}$ ;
- $s = \frac{k_3-1}{q} + \frac{n}{2q} + m_3$ , with  $m_3 \in \mathbb{N}_0$ , for each fixed  $k_3$ , where  $k_3 = 1, \dots, q$ .

Hence, applying the Residue Theorem to (29) we obtain

$$\begin{aligned}
& u_{\lambda}^{\frac{p}{q}}(x, t) \\
& = \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2}}} \sum_{k_1=1}^q \sum_{m_1=0}^{+\infty} \sum_{k_2=1}^{q-1} \sum_{m_2=0}^{+\infty} \left\{ \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q} + m_1} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-\frac{k_2}{q} - \frac{k_1-1}{q} - m_1 - m_2} \right. \\
& \times \frac{\prod_{\substack{k=1 \\ k \neq k_2}}^{q-1} \Gamma\left(\frac{k}{q} - \frac{k_2}{q} - m_2\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - \frac{k_2}{q} - \frac{k_1-1}{q} - m_1 - m_2\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q} - m_1\right)}{\prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p} - \frac{k_2}{q} - m_2\right)} \Bigg\} \\
& + \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2}}} \sum_{k_1=1}^q \sum_{m_1=0}^{+\infty} \sum_{k_3=1}^q \sum_{m_3=0}^{+\infty} \left\{ \frac{(-1)^{m_1+m_3}}{m_1! m_3!} \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q} + m_1} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-\frac{k_3-1}{q} - \frac{n}{2q} - m_3} \right. \\
& \times \frac{\prod_{k=1}^{q-1} \Gamma\left(\frac{k}{q} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q} + m_1 - m_3\right) \prod_{\substack{k=1 \\ k \neq k_3}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_3-1}{q} - m_3\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q} - m_1\right)}{\prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q} + m_1 - m_3\right)} \Bigg\} \\
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \sum_{k_1=1}^q \sum_{m_1=0}^{+\infty} \sum_{k_2=1}^{q-1} \sum_{m_2=0}^{+\infty} \left\{ \frac{(-1)^{m_1+m_2}}{m_1! m_2!} \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q} + m_1} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-\frac{k_2}{q} - \frac{k_1-1}{q} - m_1 - m_2} \right. \\
& \times \frac{\prod_{\substack{k=1 \\ k \neq k_2}}^q \Gamma\left(\frac{k}{q} - \frac{k_2}{q} - m_2\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - \frac{k_2}{q} - \frac{k_1-1}{q} - m_1 - m_2\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q} - m_1\right)}{\prod_{k=1}^p \Gamma\left(\frac{k}{p} + \frac{1}{q} - \frac{k_2}{q} - m_2\right)} \Bigg\} \\
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \sum_{k_1=1}^q \sum_{m_1=0}^{+\infty} \sum_{k_3=1}^q \sum_{m_3=0}^{+\infty} \left\{ \frac{(-1)^{m_1+m_3}}{m_1! m_3!} \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q} + m_1} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-\frac{k_3-1}{q} - \frac{n}{2q} - m_3} \right. \\
& \times \frac{\prod_{k=1}^q \Gamma\left(\frac{k}{q} - \frac{k_3-1}{q} - \frac{n}{2q} - m_3 + \frac{k_1-1}{q} + m_1\right) \prod_{\substack{k=1 \\ k \neq k_3}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_3-1}{q} - m_3\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q} - m_1\right)}{\prod_{k=1}^p \Gamma\left(\frac{k}{p} + \frac{1}{q} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q} + m_1 - m_3\right)} \Bigg\}. \quad (30)
\end{aligned}$$

To obtain Kampé de Fériet and generalized Lauricella series we use now the relation  $\Gamma(z - m) = \frac{(-1)^m \Gamma(z)}{(1 - z)_m}$ , with  $m \in \mathbb{N}$ . Hence, we can rewrite (30) as

$$\begin{aligned}
u_{\lambda}^{\frac{p}{q}}(x, t) &= \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2}}} \sum_{k_1=1}^q \sum_{k_2=1}^{q-1} \left\{ \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q}} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-\frac{k_2}{q} - \frac{k_1-1}{q}} \right. \\
&\quad \times \frac{\prod_{\substack{k=1 \\ k \neq k_2}}^{q-1} \Gamma\left(\frac{k}{q} - \frac{k_2}{q}\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - \frac{k_2}{q} - \frac{k_1-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q}\right)}{\prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p} - \frac{k_2}{q}\right)} \\
&\quad \times \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \left[ \frac{1}{m_1! m_2!} \left( -\frac{(-\lambda)^q |x|^{2q}}{q^{2q} (4c^2)^q} \right)^{m_1} \left( \frac{p^p |x|^{2q}}{q^{2q} (4c^2)^q t^p} \right)^{m_2} \right. \\
&\quad \times \left. \frac{\prod_{k=1}^{p-1} \left(1 - \frac{k}{p} + \frac{k_2}{q}\right)_{m_2}}{\prod_{\substack{k=1 \\ k \neq k_2}}^{q-1} \left(1 - \frac{k}{q} + \frac{k_2}{q}\right)_{m_2} \prod_{k=1}^q \left(1 - \frac{k-1}{q} - \frac{n}{2q} + \frac{k_2}{q} + \frac{k_1-1}{q}\right)_{m_1+m_2} \prod_{\substack{k=1 \\ k \neq k_1}}^q \left(1 - \frac{k-1}{q} + \frac{k_1-1}{q}\right)_{m_1}} \right] \Big\} \\
&+ \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2}}} \sum_{k_1=1}^q \sum_{k_3=1}^q \left\{ \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q}} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-\frac{k_3-1}{q} - \frac{n}{2q}} \right. \\
&\quad \times \frac{\prod_{k=1}^{q-1} \Gamma\left(\frac{k}{q} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_3}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_3-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q}\right)}{\prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q}\right)} \Big\}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \left[ \frac{1}{m_1! m_3!} \left( -\frac{(-\lambda)^q t^p}{p^p} \right)^{m_1} \left( \frac{p^p |x|^{2q}}{q^{2q} (4c^2)^q t^p} \right)^{m_3} \right. \\
& \times \frac{\prod_{k=1}^{p-1} \left( 1 - \frac{k}{p} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q} \right)_{m_3-m_1}}{\prod_{k=1}^{q-1} \left( 1 - \frac{k}{q} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q} \right)_{m_2} \prod_{\substack{k=1 \\ k \neq k_3}}^q \left( 1 - \frac{k-1}{q} - \frac{n}{2q} + \frac{k_3-1}{q} \right)_{m_3} \prod_{\substack{k=1 \\ k \neq k_1}}^q \left( 1 - \frac{k-1}{q} + \frac{k_1-1}{q} \right)_{m_1}} \left. \right] \Bigg\} \\
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \sum_{k_1=1}^q \sum_{k_2=1}^q \left\{ \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q}} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^q - \frac{k_2}{q} - \frac{k_1-1}{q} \right. \\
& \times \frac{\prod_{\substack{k=1 \\ k \neq k_2}}^q \Gamma\left(\frac{k}{q} - \frac{k_2}{q}\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - \frac{k_2}{q} - \frac{k_1-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q}\right)}{\prod_{k=1}^p \Gamma\left(\frac{k}{p} + \frac{1}{q} - \frac{k_2}{q}\right)} \\
& \times \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \left[ \frac{1}{m_1! m_2!} \left( -\frac{(-\lambda)^q |x|^{2q}}{q^{2q} (4c^2)^q} \right)^{m_1} \left( \frac{p^p |x|^{2q}}{q^{2q} (4c^2)^q t^p} \right)^{m_2} \right. \\
& \times \frac{\prod_{k=1}^p \left( 1 - \frac{k}{p} - \frac{1}{q} + \frac{k_2}{q} \right)_{m_2}}{\prod_{\substack{k=1 \\ k \neq k_2}}^q \left( 1 - \frac{k}{q} + \frac{k_2}{q} \right)_{m_2} \prod_{k=1}^q \left( 1 - \frac{k-1}{q} - \frac{n}{2q} + \frac{k_2}{q} + \frac{k_1-1}{q} \right)_{m_1+m_2} \prod_{\substack{k=1 \\ k \neq k_1}}^q \left( 1 - \frac{k-1}{q} + \frac{k_1-1}{q} \right)_{m_1}} \left. \right] \Bigg\} \\
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \sum_{k_1=1}^q \sum_{k_3=1}^q \left\{ \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q}} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^q - \frac{k_3-1}{q} - \frac{n}{2q} \right. \\
& \times \frac{\prod_{k=1}^q \Gamma\left(\frac{k}{q} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_3}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_3-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q}\right)}{\prod_{k=1}^p \Gamma\left(\frac{k}{p} + \frac{1}{q} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q}\right)} \\
& \times \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \left[ \frac{1}{m_1! m_3!} \left( -\frac{(-\lambda)^q t^p}{p^p} \right)^{m_1} \left( \frac{p^p |x|^{2q}}{q^{2q} (4c^2)^q t^p} \right)^{m_3} \right. \\
& \times \frac{\prod_{k=1}^p \left( 1 - \frac{k}{p} - \frac{1}{q} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q} \right)_{m_3-m_1}}{\prod_{k=1}^q \left( 1 - \frac{k}{q} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q} \right)_{m_3-m_1} \prod_{\substack{k=1 \\ k \neq k_3}}^q \left( 1 - \frac{k-1}{q} - \frac{n}{2q} + \frac{k_3-1}{q} \right)_{m_3} \prod_{\substack{k=1 \\ k \neq k_1}}^q \left( 1 - \frac{k-1}{q} + \frac{k_1-1}{q} \right)_{m_1}} \left. \right] \Bigg\}. \tag{31}
\end{aligned}$$

Taking into account (14) and (12) with  $n = 2$ , we finally obtain a series representation of the eigenfunctions in terms of Kampé de Fériet and generalized Lauricella series:

$$\begin{aligned}
u_{\lambda}^{\frac{p}{q}}(x, t) &= \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2}}} \sum_{k_1=1}^q \sum_{k_2=1}^{q-1} \left\{ \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q}} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^q - \frac{k_2}{q} - \frac{k_1-1}{q} \right. \\
& \times \frac{\prod_{\substack{k=1 \\ k \neq k_2}}^{q-1} \Gamma\left(\frac{k}{q} - \frac{k_2}{q}\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - \frac{k_2}{q} - \frac{k_1-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q}\right)}{\prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p} - \frac{k_2}{q}\right)}
\end{aligned}$$

$$\begin{aligned}
& \times F_{q:q-1;q-2}^{0:0;p-1} \left( \overline{\left(1 - \frac{k-1}{q} - \frac{n}{2q} + \frac{k_2}{q} + \frac{k_1-1}{q}\right)_{1,q}} : \overline{\left(1 - \frac{k-1}{q} + \frac{k_1-1}{q}\right)_{1,q}}_{k \neq k_1} ; \right. \\
& \qquad \qquad \qquad \left(1 - \frac{k}{q} + \frac{k_2}{q}\right)_{1,p-1} ; \\
& \qquad \qquad \qquad \left(1 - \frac{k}{q} + \frac{k_2}{q}\right)_{1,q-1} ; \left. - \frac{(-\lambda)^q |x|^{2q}}{q^{2q} (4c^2)^q}, \frac{p^p |x|^{2q}}{q^{2q} (4c^2)^q t^p} \right) \\
& + \frac{q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2}}} \sum_{k_1=1}^q \sum_{k_3=1}^q \left\{ \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q}} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-\frac{k_3-1}{q} - \frac{n}{2q}} \right. \\
& \qquad \times \frac{\prod_{k=1}^{q-1} \Gamma\left(\frac{k}{q} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_3}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_3-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q}\right)}{\prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q}\right)} \\
& \times F_{q-1;q-1;q-1}^{p-1:0;0} \left( \left[ \left(1 - \frac{k}{p} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q}\right)_{1,p-1} : -1, 1 \right] : \overline{\phantom{\left(1 - \frac{k}{p} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q}\right)_{1,p-1}}} ; \right. \\
& \qquad \left[ \left(1 - \frac{k}{q} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q}\right)_{1,q-1} : -1, 1 \right] : \left[ \left(1 - \frac{k-1}{q} + \frac{k_1-1}{q}\right)_{1,q} : 1 \right] ; \\
& \qquad \qquad \qquad \overline{\phantom{\left(1 - \frac{k}{p} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q}\right)_{1,p-1}}} ; \\
& \qquad \qquad \qquad \left[ \left(1 - \frac{k-1}{q} + \frac{k_3-1}{q}\right)_{1,q} : 1 \right] ; \left. - \frac{(-\lambda)^q t^p}{p^p}, \frac{p^p |x|^{2q}}{q^{2q} (4c^2)^q t^p} \right) \\
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \sum_{k_1=1}^q \sum_{k_2=1}^q \left\{ \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q}} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-\frac{k_2}{q} - \frac{k_1-1}{q}} \right. \\
& \qquad \times \frac{\prod_{\substack{k=1 \\ k \neq k_2}}^q \Gamma\left(\frac{k}{q} - \frac{k_2}{q}\right) \prod_{k=1}^q \Gamma\left(\frac{k-1}{q} + \frac{n}{2q} - \frac{k_2}{q} - \frac{k_1-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q}\right)}{\prod_{k=1}^p \Gamma\left(\frac{k}{p} + \frac{1}{q} - \frac{k_2}{q}\right)} \\
& \times F_{q:q-1;q-1}^{0:0;p} \left( \overline{\left(1 - \frac{k-1}{q} - \frac{n}{2q} + \frac{k_2}{q} + \frac{k_1-1}{q}\right)_{1,q}} : \overline{\left(1 - \frac{k-1}{q} + \frac{k_1-1}{q}\right)_{1,q}}_{k \neq k_1} ; \right. \\
& \qquad \qquad \qquad \left(1 - \frac{k}{q} - \frac{1}{q} + \frac{k_2}{q}\right)_{1,p} ; \\
& \qquad \qquad \qquad \left(1 - \frac{k}{q} + \frac{k_2}{q}\right)_{1,q} ; \left. - \frac{(-\lambda)^q |x|^{2q}}{q^{2q} (4c^2)^q}, \frac{p^p |x|^{2q}}{q^{2q} (4c^2)^q t^p} \right) \\
& - \frac{\lambda t^{\frac{p}{q}} q^{\frac{n+3}{2}} (2\pi)^{\frac{2-3q+p}{2}}}{\pi^{\frac{n}{2}} |x|^n p^{\frac{1}{2} + \frac{p}{q}}} \sum_{k_1=1}^q \sum_{k_3=1}^q \left\{ \left( \frac{(-\lambda)^q t^p}{p^p} \right)^{\frac{k_1-1}{q}} \left( \frac{(4c^2 q^2 t^{\frac{p}{q}})^q}{p^p |x|^{2q}} \right)^{-\frac{k_3-1}{q} - \frac{n}{2q}} \right. \\
& \qquad \times \frac{\prod_{k=1}^q \Gamma\left(\frac{k}{q} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_3}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_3-1}{q}\right) \prod_{\substack{k=1 \\ k \neq k_1}}^q \Gamma\left(\frac{k-1}{q} - \frac{k_1-1}{q}\right)}{\prod_{k=1}^p \Gamma\left(\frac{k}{p} + \frac{1}{q} - \frac{k_3-1}{q} - \frac{n}{2q} + \frac{k_1-1}{q}\right)}
\end{aligned}$$

$$\begin{aligned}
& \times F_{q;q-1;q-1}^{p;0;0} \left( \begin{array}{c} \left[ \left( 1 - \frac{k}{p} - \frac{1}{q} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q} \right)_{1,p} : -1, 1 \right] : \text{---} ; \\ \left[ \left( 1 - \frac{k}{q} + \frac{k_3-1}{q} + \frac{n}{2q} - \frac{k_1-1}{q} \right)_{1,q} : -1, 1 \right] : \left[ \left( 1 - \frac{k-1}{q} + \frac{k_1-1}{q} \right)_{1,q} : 1 \right] ; \\ \text{---} ; \\ \left[ \left( 1 - \frac{k-1}{q} + \frac{k_3-1}{q} \right)_{1,q} : 1 \right] ; \end{array} - \frac{(-\lambda)^q t^p}{p^p}, \frac{p^p |x|^{2q}}{q^{2q} (4c^2)^q t^p} \right). \quad (32)
\end{aligned}$$

The previous expressions are not easy to handle. However, it is interesting to see that some very general hypergeometric special functions that were considered in the literature found an application in a mathematical physical equation in  $\mathbb{R}^n \times \mathbb{R}^+$ . In the following sections we consider particular values of  $\beta$  and study the corresponding eigenfunctions.

## 4 The diffusion case

In this section we obtain the representation, in terms of double Mellin-Barnes integrals and in terms of double hypergeometric series, of the eigenfunctions of the diffusion operator  $(\partial_t - c^2 \Delta_x)$  for arbitrary space dimension. In the end of the section we present some plots of the eigenfunctions for some particular values of the dimension  $n$  and the eigenvalue  $\lambda$ . We observe that if we consider  $\lambda = 0$  in the results presented in Subsections 4.1 and 4.2 we recover the fundamental solution of the diffusion operator already studied in [7].

Considering  $p = q = 1$ , ( $\beta = 1$ ) in (28) we obtain, after straightforward calculations, the following representation in terms of Mellin-Barnes integrals for the eigenfunctions of the diffusion operator

$$\begin{aligned}
u_\lambda^1(x, t) &= \frac{1}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \Gamma\left(\frac{n}{2} - s\right) \Gamma(w) (-\lambda t)^{-w} \left(\frac{4c^2 t}{|x|^2}\right)^{-s} dw ds \\
&\quad - \frac{\lambda t}{\pi^{\frac{n}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma(1 - w - s) \Gamma\left(\frac{n}{2} - s\right) \Gamma(w)}{\Gamma(2 - w - s)} (-\lambda t)^{-w} \left(\frac{4c^2 t}{|x|^2}\right)^{-s} dw ds. \quad (33)
\end{aligned}$$

Moreover, considering  $p = q = 1$  in (29) we get

$$\begin{aligned}
u_\lambda^1(x, t) &\frac{1}{\pi^{\frac{n}{2}} |x|^n} \sum_{m_1=0}^{\infty} \frac{(\lambda t)^{m_1}}{m_1!} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \Gamma\left(\frac{n}{2} - s\right) \left(\frac{4c^2 t}{|x|^2}\right)^{-s} ds \\
&\quad - \frac{\lambda t}{\pi^{\frac{n}{2}} |x|^n} \sum_{m_1=0}^{\infty} \frac{(\lambda t)^{m_1}}{m_1!} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \frac{\Gamma(1 + m_1 - s) \Gamma\left(\frac{n}{2} - s\right)}{\Gamma(2 + m_1 - s)} \left(\frac{4c^2 t}{|x|^2}\right)^{-s} ds \quad (34)
\end{aligned}$$

In order to continue with the application of the Residue Theorem, we need, as it was previously indicated, to take into account the parity of the space dimension. In the next two subsections we study the behavior of (34) for these two cases.

### 4.1 Series representation for odd dimension

Here, we consider the case of  $n$  odd in (34). In this case the integrals with respect to  $s$  are convergent and  $\mathcal{L}_1 = \mathcal{L}_{+\infty}$  (cf. [16, Thm. 1.1]). The gamma functions  $\Gamma\left(\frac{n}{2} - s\right)$  and  $\Gamma(1 + m_1 - s)$  have the following sequence of simple poles, respectively:

- $s = \frac{n}{2} + m_3$ , with  $m_3 \in \mathbb{N}_0$ ;
- $s = 1 + m_1 + m_2$ , with  $m_2 \in \mathbb{N}_0$ .

Hence, applying the Residue Theorem to (34) we obtain

$$\begin{aligned}
u_\lambda^1(x, t) &= \frac{1}{(4\pi c^2 t)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3!} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3} \\
&\quad - \frac{\lambda}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma(\frac{n}{2} - 1 - m_1 - m_2)}{m_1! m_2! \Gamma(1 - m_2)} \left( \frac{\lambda |x|^2}{4c^2} \right)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_2} \\
&\quad - \frac{\lambda t^{1-\frac{n}{2}}}{(4c^2 \pi)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{\Gamma(1 - \frac{n}{2} + m_1 - m_3)}{m_1! m_3! \Gamma(2 - \frac{n}{2} + m_1 - m_3)} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3}. \tag{35}
\end{aligned}$$

It follows from our general derivations that (35) is an eigenfunction of the diffusion operator. However, let us verify this independently by substituting the function (35) into the equation (1). Taking into account (35) and observing that the second term does not depend on  $t$ , we get

$$\begin{aligned}
\partial_t u_\lambda^1(x, t) &= -\frac{\lambda}{(4c^2 \pi t)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3!} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3} \\
&\quad + \frac{1}{(4c^2 \pi t)^{\frac{n}{2}} t} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{m_1 - m_3 - \frac{n}{2}}{m_1! m_3!} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3}. \tag{36}
\end{aligned}$$

Taking into account the following relations

$$\Delta(|x|^{2m_3}) = 2m_3(3m_3 + n - 2)|x|^{2m_3-2} \quad \text{and} \quad \Delta(|x|^{2m_1+2-n}) = 2m_1(2m_1 + 2 - n)|x|^{2m_1-2},$$

we obtain

$$\begin{aligned}
\Delta u_\lambda^1(x, t) &= -\frac{2\lambda t}{(4c^2 \pi t)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{2m_3 + n - 2}{m_1! (m_3 - 1)! (m_1 - m_3 + 1 - \frac{n}{2})} (\lambda t)^{m_1} (-4c^2 t)^{-m_3} |x|^{2m_3-2} \\
&\quad - \frac{\lambda}{2c^2 \pi^{\frac{n}{2}}} \sum_{m_1=1}^{+\infty} \frac{\Gamma(\frac{n}{2} - 1 - m_1) (2m_1 + 2 - n)}{(m_1 - 1)!} \left( \frac{\lambda}{4c^2} \right)^{m_1} |x|^{2m_1-n} \\
&\quad + \frac{2}{(4c^2 \pi t)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=1}^{+\infty} \frac{2m_3 + n - 2}{m_1! (m_3 - 1)!} (\lambda t)^{m_1} (-4c^2 t)^{-m_3} |x|^{2m_3-2}.
\end{aligned}$$

Rearranging the series and making some simplifications leads to

$$\begin{aligned}
\Delta u_\lambda^1(x, t) &= \frac{\lambda}{c^2 (4c^2 \pi t)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=1}^{+\infty} \frac{m_3 + \frac{n}{2}}{m_1! m_3! (m_1 - m_3 - \frac{n}{2})} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3} \\
&\quad + \lambda \frac{\lambda}{4c^4 \pi^{\frac{n}{2}} |x|^{n-2}} \sum_{m_1=0}^{+\infty} \frac{\Gamma(\frac{n}{2} - 1 - m_1)}{m_1!} \left( \frac{\lambda |x|^2}{4c^2} \right)^{m_1} \\
&\quad - \frac{1}{c^2 t (4c^2 \pi t)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{m_3 + \frac{n}{2}}{m_1! m_3!} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3}. \tag{37}
\end{aligned}$$

From (36) and (37) and after straightforward calculations we obtain

$$\begin{aligned}
(\partial_t - c^2 \Delta) u_\lambda^1(x, t) &= -\frac{\lambda}{(4c^2 \pi t)^{\frac{n}{2}}} \sum_{m_1=1}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{(m_1 - 1)! m_3! (m_1 - m_3 - \frac{n}{2})} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3} \\
&\quad + \lambda \frac{-\lambda}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2}} \sum_{m_1=0}^{+\infty} \frac{\Gamma(\frac{n}{2} - 1 - m_1)}{m_1!} \left( \frac{\lambda |x|^2}{4c^2} \right)^{m_1} \\
&\quad + \frac{1}{t (4c^2 \pi t)^{\frac{n}{2}}} \sum_{m_1=1}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{(m_1 - 1)! m_3!} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3}.
\end{aligned}$$

Rearranging the series and making some simplifications finally leads to

$$\begin{aligned}
(\partial_t - c^2 \Delta) u_\lambda^1(x, t) &= \lambda \left[ \frac{1}{(4\pi c^2 t)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3!} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3} \right. \\
&\quad - \frac{\lambda}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2}} \sum_{m_1=0}^{+\infty} \frac{\Gamma(\frac{n}{2} - 1 - m_1)}{m_1!} \left( \frac{\lambda |x|^2}{4c^2} \right)^{m_1} \\
&\quad \left. - \frac{\lambda t^{1-\frac{n}{2}}}{(4c^2 \pi)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{\Gamma(1 - \frac{n}{2} + m_1 - m_3)}{m_1! m_3! \Gamma(2 - \frac{n}{2} + m_1 - m_3)} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3} \right] \\
&= \lambda u_\lambda^1(x, t),
\end{aligned} \tag{38}$$

i.e.,  $u_\lambda^1$  is an eigenfunction of  $\partial_t - c^2 \Delta$ , for  $n$  odd.

## 4.2 Series representation for even dimension

Here, we consider the case of  $n$  even in (34). First, we perform the change of variable  $u = 1 - s + m_1$  in the second term in (34), which leads to

$$\begin{aligned}
u_\lambda^1(x, t) &= \frac{1}{\pi^{\frac{n}{2}} |x|^n} \sum_{m_1=0}^{+\infty} \frac{(\lambda t)^{m_1}}{m_1!} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \Gamma\left(\frac{n}{2} - s\right) \left( \frac{4c^2 t}{|x|^2} \right)^{-s} ds \\
&\quad - \frac{\lambda t}{\pi^{\frac{n}{2}} |x|^n} \sum_{m_1=0}^{+\infty} \frac{(\lambda t)^{m_1}}{m_1!} \frac{1}{2\pi i} \int_{\mathcal{L}_1^*} \frac{\Gamma(u) \Gamma(\frac{n}{2} - 1 - m_1 + u)}{\Gamma(1 + u)} \left( \frac{4c^2 t}{|x|^2} \right)^{u-1-m_1} du.
\end{aligned} \tag{39}$$

Concerning the first integral with respect to  $s$  in (39) it is convergent and  $\mathcal{L}_1 = \mathcal{L}_{-\infty}$  (cf. [16, Thm. 1.1]). The gamma function  $\Gamma(\frac{n}{2} - s)$  has the following sequence of simple poles

- $s = \frac{n}{2} + m_2$ , with  $m_2 \in \mathbb{N}_0$ .

Concerning the second integral with respect to  $s$  in (39) it is convergent and  $\mathcal{L}_1^* = \mathcal{L}_{-\infty}$  (cf. [16, Thm. 1.1]). The gamma function  $\Gamma(u)$  has poles at  $u = -m_3$ , for  $m_3 \in \mathbb{N}_0$ , and the gamma function  $\Gamma(\frac{n}{2} - 1 - m_1 + u)$  has poles at  $u = 1 - \frac{n}{2} + m_1 - m_4$ , for  $m_4 \in \mathbb{N}_0$ . Since  $n$  is even we can have simple and/or double poles in the following two cases:

- for  $m_1 - \frac{n}{2} + 1 < 0$  there are simple poles at  $u = -m_3$ , for  $m_3 = 0, \dots, -m_1 + \frac{n}{2} - 2$  and double poles at  $u = -m_3$ , for  $m_3 \geq -m_1 + \frac{n}{2} - 1$ ;
- for  $m_1 - \frac{n}{2} + 1 \geq 0$  there are simple poles at  $u = 1 - \frac{n}{2} + m_1 - m_4$ , for  $m_4 = 0, \dots, m_1 - \frac{n}{2}$  and double poles at  $u = 1 - \frac{n}{2} + m_1 - m_4$ , for  $m_4 \geq 1 - \frac{n}{2} + m_1$ .

Therefore, applying the Residue Theorem to (39) and after straightforward calculations we arrive to

$$\begin{aligned}
u_\lambda^1(x, t) &= \frac{1}{(4c^2 \pi t)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{1}{m_1! m_2!} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_2} \\
&\quad - \frac{\lambda |x|^{2-n}}{4c^2 \pi^{\frac{n}{2}}} \sum_{m_1=0}^{\frac{n}{2}-2} \sum_{m_3=0}^{\frac{n}{2}-m_1-2} \frac{\Gamma(\frac{n}{2} - 1 - m_1 - m_3)}{m_1! m_3! \Gamma(1 - m_3)} \left( \frac{\lambda |x|^2}{4c^2} \right)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3} \\
&\quad + \frac{(-1)^{\frac{n}{2}+1} \lambda |x|^{2-n}}{4c^2 \pi^{\frac{n}{2}}} \sum_{m_1=0}^{\frac{n}{2}-2} \sum_{m_3=\frac{n}{2}-m_1-1}^{+\infty} \frac{1}{m_1! m_3! \Gamma(2 - \frac{n}{2} + m_1 + m_3)} \left( -\frac{\lambda |x|^2}{4c^2} \right)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_3} \\
&\quad - \frac{\lambda t^{1-\frac{n}{2}}}{(4c^2 \pi)^{\frac{n}{2}}} \sum_{m_1=\frac{n}{2}-1}^{+\infty} \sum_{m_4=0}^{m_1-\frac{n}{2}} \frac{1}{m_1! m_4! (1 - \frac{n}{2} + m_1 - m_4)} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4c^2 t} \right)^{m_4}
\end{aligned}$$

$$\begin{aligned}
& + \frac{(-1)^{\frac{n}{2}} \lambda}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2}} \sum_{m_1=\frac{n}{2}-1}^{+\infty} \sum_{m_4=0}^{+\infty} \left\{ \frac{\ln\left(\frac{4c^2 t}{|x|^2}\right) - \psi(1-m_4) + \psi(1+m_4) + \psi\left(2-\frac{n}{2}+m_1+m_4\right)}{m_1! m_4! \Gamma\left(2-\frac{n}{2}+m_1+m_4\right) \Gamma(1-m_4)} \right. \\
& \quad \left. \times \left(-\frac{\lambda |x|^2}{4c^2}\right)^{m_1} \left(\frac{|x|^2}{4c^2 t}\right)^{m_4} \right\}. \tag{40}
\end{aligned}$$

**Remark 4.1** We observe that for some rational values of  $\beta$  we have an indetermination in the series coefficients due to the terms  $\psi(1-m_4)$  and  $\Gamma(1-m_4)$  when  $m_4 \geq 1$  in the inner series that appear in the last term of (40). These indeterminations can be removed after applying the following properties of the gamma function and the digamma function:

$$\Gamma(1+z) = z \Gamma(z), \quad \Gamma(z) \Gamma(-z) = -\frac{\pi \csc(\pi z)}{z}, \quad \Gamma\left(\frac{1}{2}-z\right) = \frac{\pi \sec(\pi z)}{\Gamma\left(\frac{1}{2}+z\right)}, \quad \psi(1-z) = \pi \cot(\pi z) + \psi(z). \tag{41}$$

In fact, for  $m_4 \geq 1$  we have

$$\frac{-\psi(1-m_4)}{\Gamma(1-m_4)} = \frac{-\pi \cot(m_4 \pi) - \psi(m_4)}{\Gamma(m_4) \frac{\pi}{\sin(m_4 \pi)}} = (-1)^{m_4} \Gamma(m_4) - \frac{\psi(m_4) \Gamma(m_4) \sin(m_4 \pi)}{\pi} = (-1)^{m_4} \Gamma(m_4), \tag{42}$$

where the last simplification is valid when  $m_4 \geq 1$ . From (42) we have that the last double series in (40) can be rewritten as

$$\begin{aligned}
& \frac{(-1)^{\frac{n}{2}} \lambda}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2}} \sum_{m_1=\frac{n}{2}-1}^{+\infty} \frac{\ln\left(\frac{4c^2 t}{|x|^2}\right) + \psi\left(2-\frac{n}{2}+m_1\right)}{m_1! \Gamma\left(2-\frac{n}{2}+m_1\right)} \left(-\frac{\lambda |x|^2}{4c^2}\right)^{m_1} \\
& + \frac{(-1)^{\frac{n}{2}} \lambda}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2}} \sum_{m_1=\frac{n}{2}-1}^{+\infty} \sum_{m_4=1}^{+\infty} \left\{ \frac{1}{m_1! \Gamma\left(2-\frac{n}{2}+m_1+m_4\right) m_4} \left(-\frac{\lambda |x|^2}{4c^2}\right)^{m_1} \left(-\frac{|x|^2}{4c^2 t}\right)^{m_4} \right\}. \tag{43}
\end{aligned}$$

In a similar way as it was done in the previous section it is possible to directly verify that  $u_\lambda^1$  given by (40) is an eigenfunction of  $\partial_t - c^2 \Delta$ , for  $n$  even.

### 4.3 Plots of the eigenfunctions

In this subsection we present some plots of the eigenfunctions of the diffusion operator.

#### 4.3.1 The case of $n = 1$

Considering  $c = 1$  and  $n = 1$  in (35) we obtain

$$\begin{aligned}
u_\lambda^1(x, t) &= \frac{1}{\sqrt{4\pi t}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3!} (\lambda t)^{m_1} \left(-\frac{|x|^2}{4c^2 t}\right)^{m_3} - \frac{\lambda |x|}{4\sqrt{\pi}} \sum_{m_1=0}^{+\infty} \frac{\Gamma\left(-\frac{1}{2}-m_1\right)}{m_1!} \left(\frac{\lambda |x|^2}{4}\right)^{m_1} \\
& - \frac{\lambda \sqrt{t}}{\sqrt{4\pi}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3! \left(\frac{1}{2}+m_1-m_3\right)} (\lambda t)^{m_1} \left(-\frac{|x|^2}{4t}\right)^{m_3}, \tag{44}
\end{aligned}$$

Now, we present some plots of (44) for some fixed values of  $t$  and  $\lambda$  and  $x \in [-5, 5]$ .

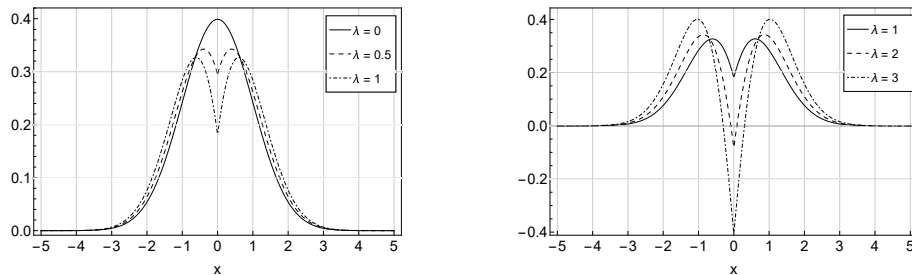


Figure 1: Plots of  $u_\lambda^1(x, t)$  for  $t = 0.5$  and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 0.5$  and  $\lambda = 1, 2, 3$  (2nd plot).



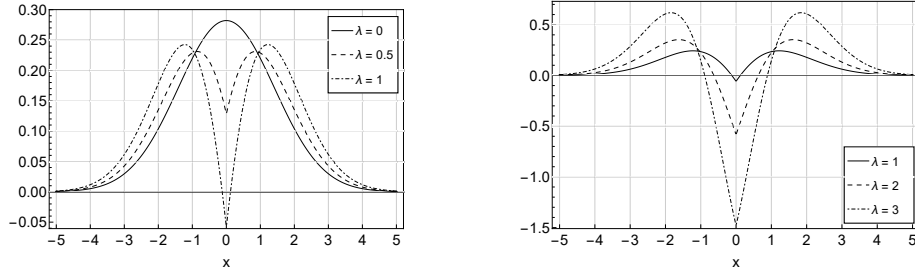


Figure 2: Plots of  $u_\lambda^1(x, t)$  for  $t = 1$  and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 1$  and  $\lambda = 1, 2, 3$  (2nd plot).

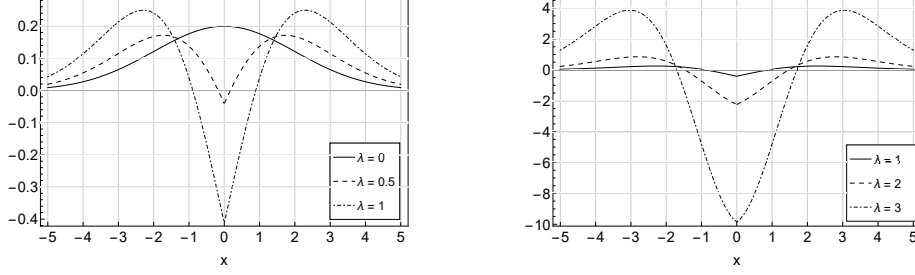


Figure 3: Plots of  $u_\lambda^1(x, t)$  for  $t = 2$  and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 2$  and  $\lambda = 1, 2, 3$  (2nd plot).

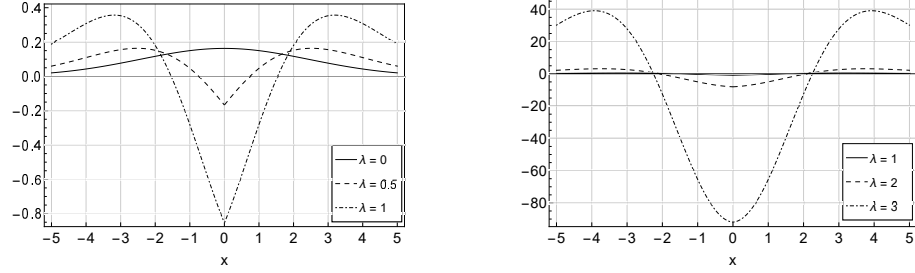


Figure 4: Plots of  $u_\lambda^1(x, t)$  for  $t = 3$  and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 3$  and  $\lambda = 1, 2, 3$  (2nd plot).

From the plots in Figures 1, 2, 3, and 4, we can see that they are similar to those presented in [7] for the fundamental solution of the diffusion operator. The eigenfunction  $u_\lambda^1$  attains two symmetric maxima that move apart from the origin with the increasing of the time and the eigenvalue  $\lambda$ , and attains a minimum at  $x = 0$  which decreases when the time and the parameter  $\lambda$  increase. The range near the origin increases with  $\lambda$ , and for all values of  $\lambda$  the eigenfunctions are continuous and differentiable in  $\mathbb{R} \times \mathbb{R}^+$ , even when  $x$  tends to zero. In fact, from (44) we have that

$$\lim_{x \rightarrow 0} u_\lambda^1(x, t) = \frac{1}{\sqrt{4\pi t}} \sum_{m_1=0}^{+\infty} \frac{(\lambda t)^{m_1}}{m_1!} - \frac{\lambda \sqrt{t}}{\sqrt{4\pi}} \sum_{m_1=0}^{\infty} \frac{(\lambda t)^{m_1}}{m_1! \left(\frac{1}{2} + m_1\right)} = \frac{1}{2\sqrt{\pi t}} e^{\lambda t} - \frac{\lambda \sqrt{t}}{2} \operatorname{erfi}(1),$$

which is a continuous and differentiable function in  $\mathbb{R}^+$ . In the following figure we present some plots of (44) for some fixed values of  $\lambda$  and  $(x, t) \in [-4, 4] \times [0.2, 4]$ .

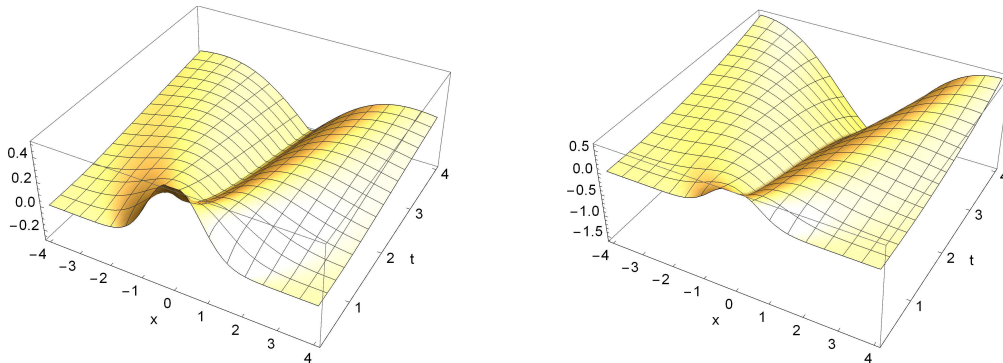


Figure 5: Plots of  $u_\lambda^1$  for  $\lambda = 0.5$  (1st plot) and  $\lambda = 1$  (2nd plot).

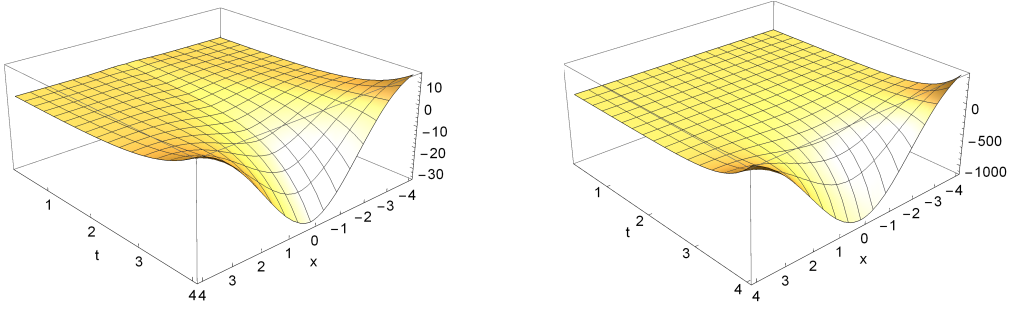


Figure 6: Plots of  $u_\lambda^1$  for  $\lambda = 2$  (1st plot) and  $\lambda = 3$  (2nd plot).

From the plots in Figures 5 and 6 the conclusions are the same as in the analysis of the plots in Figures 1, 2, 3, and 4. In these plots it is even more clear that the range near the origin increases with  $\lambda$ . It is possible to see that when  $|x|$  tends to zero the eigenfunction  $u_\lambda^1$  is continuous and differentiable in  $\mathbb{R}^+$ . We remark that

$$\lim_{t \rightarrow 0^+} u_\lambda^1(x, t) = +\infty.$$

Therefore, we did not plot the function near  $t = 0$  to see more clearly the behaviour of the eigenfunction.

#### 4.3.2 The case of $n = 2$

Considering  $c = 1$  and  $n = 2$  in (40) and (43) leads to

$$\begin{aligned} u_\lambda^1(x, t) = & \frac{1}{4\pi t} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{1}{m_1! m_2!} (\lambda t)^{m_1} \left(-\frac{|x|^2}{4t}\right)^{m_2} - \frac{\lambda}{4\pi} \sum_{m_1=0}^{+\infty} \sum_{m_4=0}^{m_1-1} \frac{1}{m_1! m_4! (m_1 - m_4)} (\lambda t)^{m_1} \left(-\frac{|x|^2}{4t}\right)^{m_4} \\ & - \frac{\lambda}{4\pi} \sum_{m_1=0}^{+\infty} \frac{\ln\left(\frac{4t}{|x|^2}\right) + \psi(1 + m_1)}{(m_1!)^2} \left(-\frac{\lambda |x|^2}{4}\right)^{m_1} + \frac{\lambda}{4\pi} \sum_{m_1=0}^{+\infty} \sum_{m_4=1}^{+\infty} \frac{1}{m_1! (m_1 + m_4)! m_4} \left(-\frac{\lambda |x|^2}{4}\right)^{m_1} \left(-\frac{|x|^2}{4t}\right)^{m_4}. \end{aligned} \quad (45)$$

Now, we present some plots of (45) for some fixed values of  $\lambda$  and  $(|x|, t) \in [0.1, 3] \times [0.4, 4]$ .

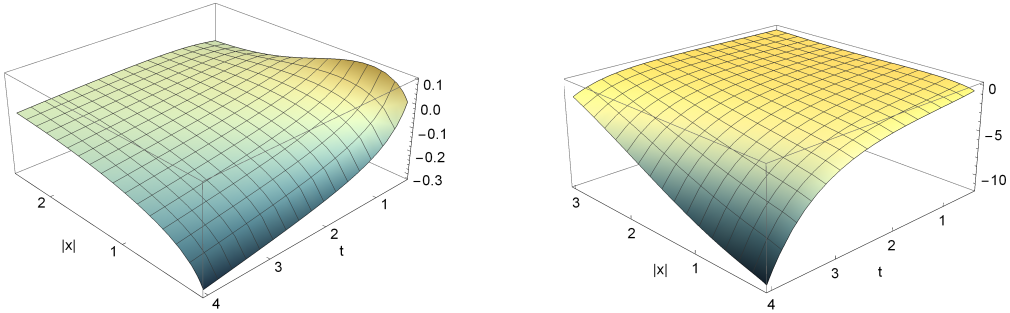


Figure 7: Plots of  $u_\lambda^1$  for  $\lambda = 0.5$  (1st plot) and  $\lambda = 1$  (2nd plot).

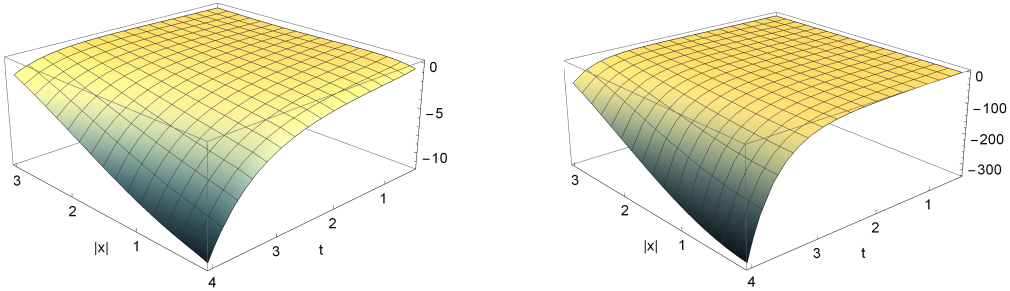


Figure 8: Plots of  $u_\lambda^1$  for  $\lambda = 2$  (1st plot) and  $\lambda = 3$  (2nd plot).

From the plots in Figures 7 and 8 we observe that the decay of  $u_\lambda^1$  is more pronounced in time than in space and the range of the plots increases with  $\lambda$ . Moreover, we observe that due to the logarithmic function that appears in the third series in (45) we have that

$$\lim_{|x| \rightarrow 0^+} u_\lambda^1(x, t) = -\infty, \quad \text{and} \quad \lim_{t \rightarrow 0^+} u_\lambda^1(x, t) = +\infty.$$

Therefore, we did not plot the functions near  $|x| = 0$  and  $t = 0$  to be able to see the behaviour of the eigenfunctions.

#### 4.3.3 The case of $n = 3$

Considering  $c = 1$  and  $n = 1$  in (35) we obtain

$$\begin{aligned} u_\lambda^1(x, t) = & \frac{1}{\sqrt{(4\pi t)^3}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3!} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4t} \right)^{m_3} - \frac{\lambda}{4\sqrt{\pi^3}|x|} \sum_{m_1=0}^{+\infty} \frac{\Gamma(\frac{1}{2} - m_1)}{m_1!} \left( \frac{\lambda |x|^2}{4c^2} \right)^{m_1} \\ & - \frac{\lambda}{\sqrt{(4\pi)^3 t}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{\Gamma(-\frac{1}{2} + m_1 - m_3)}{m_1! m_3! \Gamma(\frac{1}{2} + m_1 - m_3)} (\lambda t)^{m_1} \left( -\frac{|x|^2}{4t} \right)^{m_3}. \end{aligned} \quad (46)$$

Now, we present some plots of (46) for some fixed values of  $\lambda$  and  $(|x|, t) \in [0.1, 4] \times [0.2, 4]$ .

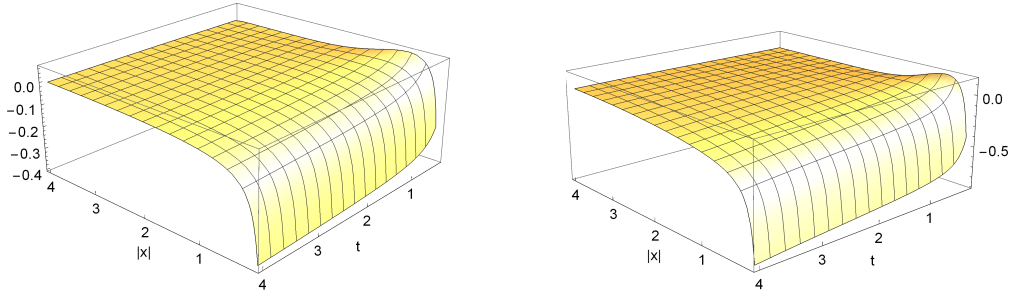


Figure 9: Plots of  $u_\lambda^1$  for  $\lambda = 0.5$  (1st plot) and  $\lambda = 1$  (2nd plot).

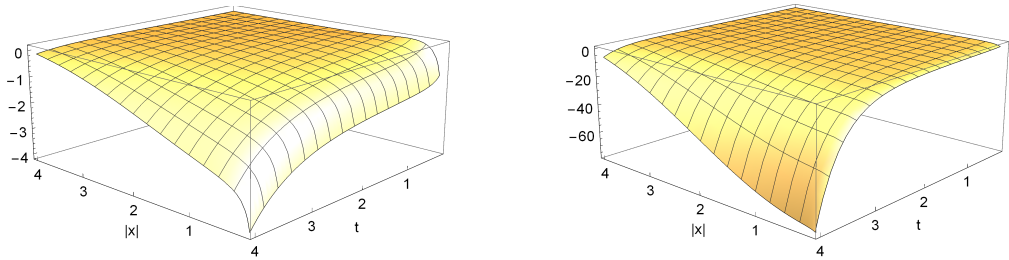


Figure 10: Plots of  $u_\lambda^1$  for  $\lambda = 2$  (1st plot) and  $\lambda = 3$  (2nd plot).

From the plots in Figures 9 and 10 we have similar conclusions as those obtained from the analysis of Figures 7 and 8. The main difference is that the increase of the range of the plot with  $\lambda$  is more pronounced. Moreover, due to the first term of the second series in (46) we have that

$$\lim_{|x| \rightarrow 0^+} u_\lambda^1(x, t) = -\infty,$$

and due to the two double series that appear in (46) we have that

$$\lim_{t \rightarrow 0^+} u_\lambda^1(x, t) = +\infty.$$

## 5 The wave case

In this section we obtain the representation, in terms of double Mellin-Barnes integrals and in terms of double hypergeometric series, of the eigenfunctions of the wave operator  $(\partial_t^2 - c^2 \Delta_x)$  for the cases of odd and even dimensions. In the end of the section we present some plots of the eigenfunctions for some particular values of the dimension  $n$  and the eigenvalue  $\lambda$ . We observe that if we consider  $\lambda = 0$  in the results presented in Subsections 5.1 and 5.2 we recover the results presented in [7].

Considering  $p = 2$  and  $q = 1$ , ( $\beta = 2$ ) in (28) we obtain, after straightforward calculations, the following representation in terms of Mellin-Barnes integrals for the eigenfunctions of the wave operator

$$u_\lambda^2(x, t) = \frac{1}{\pi^{\frac{n-1}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma(\frac{n}{2} - s) \Gamma(w)}{\Gamma(\frac{1}{2} - s - w)} \left(-\frac{\lambda t}{4}\right)^{-w} \left(\frac{c^2 t^2}{|x|^2}\right)^{-s} dw ds \\ - \frac{\lambda t^2}{4\pi^{\frac{n-1}{2}} |x|^n} \frac{1}{(2\pi i)^2} \int_{\mathcal{L}_1} \int_{\mathcal{L}_2} \frac{\Gamma(1 - w - s) \Gamma(\frac{n}{2} - s) \Gamma(w)}{\Gamma(\frac{3}{2} - w - s) \Gamma(2 - w - s)} \left(-\frac{\lambda t}{4}\right)^{-w} \left(\frac{c^2 t^2}{|x|^2}\right)^{-s} dw ds. \quad (47)$$

Moreover, considering  $p = 2$  and  $q = 1$  in (29) we get

$$u_\lambda^2(x, t) \frac{1}{\pi^{\frac{n-1}{2}} |x|^n} \sum_{m_1=0}^{\infty} \frac{1}{m_1!} \left(\frac{\lambda t^2}{4}\right)^{m_1} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \frac{\Gamma(\frac{n}{2} - s)}{\Gamma(\frac{1}{2} - s + m_1)} \left(\frac{c^2 t^2}{|x|^2}\right)^{-s} ds \\ - \frac{\lambda t^2}{4\pi^{\frac{n-1}{2}} |x|^n} \sum_{m_1=0}^{\infty} \frac{1}{m_1!} \left(\frac{\lambda t^2}{4}\right)^{m_1} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \frac{\Gamma(1 + m_1 - s) \Gamma(\frac{n}{2} - s)}{\Gamma(\frac{3}{2} + m_1 - s) \Gamma(2 + m_1 - s)} \left(\frac{c^2 t^2}{|x|^2}\right)^{-s} ds. \quad (48)$$

In order to continue with the application of the Residue theorem, we need to take into account the parity of the space dimension. In the next two subsections we study the behavior of (48) for these two cases and we deduce the correspondent representation in terms of double hypergeometric series.

### 5.1 Series representation for odd dimension

Considering the case of odd  $n$  in (48) we can conclude that the integrals with respect to  $s$  are convergent and  $\mathcal{L}_1 = \mathcal{L}_{+\infty}$  (cf. [16, Thm. 1.1]). The gamma functions  $\Gamma(\frac{n}{2} - s)$  and  $\Gamma(1 + m_1 - s)$  have the following sequence of simple poles, respectively:

- $s = \frac{n}{2} + m_3$ , with  $m_3 \in \mathbb{N}_0$ ;
- $s = 1 + m_1 + m_2$ , with  $m_2 \in \mathbb{N}_0$ .

Hence, applying the Residue Theorem to (48) we obtain

$$u_\lambda^2(x, t) = \frac{1}{\pi^{\frac{n-1}{2}} (ct)^n} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3! \Gamma(\frac{1}{2} - \frac{n}{2} + m_1 - m_3)} \left(\frac{\lambda t^2}{4}\right)^{m_1} \left(-\frac{|x|^2}{4c^2 t}\right)^{m_3} \\ - \frac{\lambda}{4c^2 \pi^{\frac{n-1}{2}} |x|^{n-2}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma(\frac{n}{2} - 1 - m_1 - m_2)}{m_1! m_2! \Gamma(\frac{1}{2} - m_2) \Gamma(1 - m_2)} \left(\frac{\lambda |x|^2}{4c^2}\right)^{m_1} \left(-\frac{|x|^2}{4c^2 t}\right)^{m_2} \\ - \frac{\lambda t^{2-n}}{4c^n \pi^{\frac{n-1}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{\Gamma(1 - \frac{n}{2} + m_1 - m_3)}{m_1! m_3! \Gamma(\frac{3}{2} - \frac{n}{2} + m_1 - m_3) \Gamma(2 - \frac{n}{2} + m_1 - m_3)} \left(\frac{\lambda t^2}{4}\right)^{m_1} \left(-\frac{|x|^2}{4c^2 t}\right)^{m_3}. \quad (49)$$

In a similar way as it was done the case of the diffusion operator (see Section 4.1) it is possible to directly verify that  $u_\lambda^2$  given by (49), with  $n$  odd, is an eigenfunction of  $\partial_t^2 - c^2 \Delta_x$ .

## 5.2 Series representation for the case of even dimension

Here we consider the case of  $n$  even in (48). First, we perform the change of variable  $u = 1 - s + m_1$  in the second term in (48), which leads to

$$u_\lambda^2(x, t) = \frac{1}{\pi^{\frac{n-1}{2}} |x|^n} \sum_{m_1=0}^{+\infty} \frac{1}{m_1!} \left( \frac{\lambda t^2}{4} \right)^{m_1} \frac{1}{2\pi i} \int_{\mathcal{L}_1} \frac{\Gamma(\frac{n}{2} - s)}{\Gamma(\frac{1}{2} + m_1 - s)} \left( \frac{c^2 t^2}{|x|^2} \right)^{-s} ds$$

$$- \frac{\lambda t^2}{4\pi^{\frac{n-1}{2}} |x|^n} \sum_{m_1=0}^{+\infty} \frac{1}{m_1!} \left( \frac{\lambda t^2}{4} \right)^{m_1} \frac{1}{2\pi i} \int_{\mathcal{L}_1^*} \frac{\Gamma(u) \Gamma(\frac{n}{2} - 1 - m_1 + u)}{\Gamma(\frac{1}{2} + u) \Gamma(1 + u)} \left( \frac{c^2 t^2}{|x|^2} \right)^{u-1-m_1} du. \quad (50)$$

Concerning the first integral with respect to  $s$  in (50) it is convergent and  $\mathcal{L}_1 = \mathcal{L}_{-\infty}$  (cf. [16, Thm. 1.1]). The gamma function  $\Gamma(\frac{n}{2} - s)$  has the following sequence of simple poles

- $s = \frac{n}{2} + m_2$ , with  $m_2 \in \mathbb{N}_0$ .

Concerning the second integral with respect to  $s$  in (50) it is convergent and  $\mathcal{L}_1^* = \mathcal{L}_{-\infty}$  (cf. [16, Thm. 1.1]). The gamma function  $\Gamma(u)$  has poles at  $u = -m_3$ , for  $m_3 \in \mathbb{N}_0$ , and the gamma function  $\Gamma(\frac{n}{2} - 1 - m_1 + u)$  has poles at  $u = 1 - \frac{n}{2} + m_1 - m_4$ , for  $m_4 \in \mathbb{N}_0$ . Since  $n$  is even we can have simple and/or double poles in the following two cases:

- for  $m_1 - \frac{n}{2} + 1 < 0$  there are simple poles at  $u = -m_3$ , for  $m_3 = 0, \dots, -m_1 + \frac{n}{2} - 2$  and double poles at  $u = -m_3$ , for  $m_3 \geq -m_1 + \frac{n}{2} - 1$ ;
- for  $m_1 - \frac{n}{2} + 1 \geq 0$  there are simple poles at  $u = 1 - \frac{n}{2} + m_1 - m_4$ , for  $m_4 = 0, \dots, m_1 - \frac{n}{2}$  and double poles at  $u = 1 - \frac{n}{2} + m_1 - m_4$ , for  $m_4 \geq 1 - \frac{n}{2} + m_1$ .

Therefore, applying the Residue Theorem to (50) and after straightforward calculations we arrive to

$$u_\lambda^2(x, t) = \frac{\sqrt{\pi}}{(c^2 \pi t)^{\frac{n}{2}}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{1}{m_1! m_2! \Gamma(\frac{1-n}{2} + m_1 - m_2)} \left( \frac{\lambda t^2}{4} \right)^{m_1} \left( -\frac{|x|^2}{c^2 t^2} \right)^{m_2}$$

$$- \frac{\lambda |x|^{2-n}}{4c^2 \pi^{\frac{n}{2}}} \sum_{m_1=0}^{\frac{n}{2}-2} \sum_{m_3=0}^{\frac{n}{2}-m_1-2} \frac{\Gamma(\frac{n}{2} - 1 - m_1 - m_3)}{m_1! m_3! \Gamma(1 - 2m_3)} \left( \frac{\lambda |x|^2}{4c^2} \right)^{m_1} \left( -\frac{|x|^2}{4c^2 t^2} \right)^{m_3}$$

$$+ \frac{(-1)^{\frac{n}{2}+1} \lambda |x|^{2-n}}{4c^2 \pi^{\frac{n-1}{2}}} \sum_{m_1=0}^{\frac{n}{2}-2} \sum_{m_3=\frac{n}{2}-m_1-1}^{+\infty} \frac{1}{m_1! m_3! \Gamma(2 - \frac{n}{2} + m_1 + m_3) \Gamma(\frac{1}{2} - m_3)} \left( -\frac{\lambda |x|^2}{4c^2} \right)^{m_1} \left( -\frac{|x|^2}{c^2 t^2} \right)^{m_3}$$

$$- \frac{\sqrt{\pi} \lambda t^{2-n}}{4(c^2 \pi)^{\frac{n}{2}}} \sum_{m_1=\frac{n}{2}-1}^{+\infty} \sum_{m_4=0}^{m_1-\frac{n}{2}} \frac{1}{m_1! m_4! (1 - \frac{n}{2} + m_1 - m_4) \Gamma(\frac{3-n}{2} + m_1 - m_4)} \left( \frac{\lambda t^2}{4} \right)^{m_1} \left( -\frac{|x|^2}{c^2 t^2} \right)^{m_4}$$

$$+ \frac{(-1)^{\frac{n}{2}} \lambda}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2}} \sum_{m_1=\frac{n}{2}-1}^{+\infty} \sum_{m_4=0}^{+\infty} \left\{ \frac{\ln\left(\frac{c^2 t^2}{|x|^2}\right) - \psi\left(\frac{1}{2} - m_4\right) - \psi(1 - m_4) + \psi(1 + m_4) + \psi\left(2 - \frac{n}{2} + m_1 + m_4\right)}{m_1! m_4! \Gamma\left(2 - \frac{n}{2} + m_1 + m_4\right) \Gamma\left(\frac{1}{2} - m_4\right) \Gamma(1 - m_4)} \right.$$

$$\left. \times \left( -\frac{\lambda |x|^2}{4c^2} \right)^{m_1} \left( \frac{|x|^2}{c^2 t^2} \right)^{m_4} \right\}. \quad (51)$$

**Remark 5.1** We observe that for some rational values of  $\beta$  we have an indetermination in the series coefficients due to the terms  $\psi(1 - m_4)$  and  $\Gamma(1 - m_4)$  when  $m_4 \geq 1$  in the inner series that appear in the last term of (51). In a similar way as it was done for the diffusion case (see Remark 4.1), these indeterminations can be removed after applying the properties of the gamma and digamma functions presented in (41). In fact, from

(42) the last term in (51) can be rewritten as

$$\begin{aligned} & \frac{(-1)^{\frac{n}{2}} \lambda}{4c^2 \pi^{\frac{n+1}{2}} |x|^{n-2}} \sum_{m_1=\frac{n}{2}-1}^{+\infty} \frac{\ln\left(\frac{c^2 t^2}{|x|^2}\right) - \psi\left(\frac{1}{2}\right) + \psi\left(2 - \frac{n}{2} + m_1\right)}{m_1! \Gamma\left(2 - \frac{n}{2} + m_1\right)} \left(-\frac{\lambda |x|^2}{4c^2}\right)^{m_1} \\ & + \frac{(-1)^{\frac{n}{2}} \lambda}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2}} \sum_{m_1=\frac{n}{2}-1}^{+\infty} \sum_{m_4=1}^{+\infty} \left\{ \frac{1}{m_1! \Gamma\left(\frac{1}{2} - m_4\right) \Gamma\left(2 - \frac{n}{2} + m_1 + m_4\right) m_4} \left(-\frac{\lambda |x|^2}{4c^2}\right)^{m_1} \left(\frac{|x|^2}{t^2}\right)^{m_4} \right\}. \end{aligned} \quad (52)$$

In a similar way as it was done the case of the diffusion operator (see Section 4.1), it possible to directly verify that  $u_\lambda^2$  is an eigenfunction of  $\partial_t - c^2 \Delta$ , when  $n$  is even.

### 5.3 Plots of the eigenfunctions

In this subsection we present some plots of the eigenfunctions of the wave operator.

#### 5.3.1 The case of $n = 1$

Considering  $c = 1$  and  $n = 1$  in (49) we obtain

$$\begin{aligned} u_\lambda^2(x, t) &= \frac{1}{t} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3! \Gamma(m_1 - m_3)} \left(\frac{\lambda t^2}{4}\right)^{m_1} \left(-\frac{|x|^2}{4c^2 t}\right)^{m_3} - \frac{\lambda |x|}{4\sqrt{\pi}} \sum_{m_1=0}^{+\infty} \frac{\Gamma\left(-\frac{1}{2} - m_1\right)}{m_1!} \left(\frac{\lambda |x|^2}{4}\right)^{m_1} \\ &- \frac{\lambda t}{4} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3! \Gamma(1 + m_1 - m_3)} \left(\frac{\lambda t^2}{4}\right)^{m_1} \left(-\frac{|x|^2}{4t}\right)^{m_3}. \end{aligned} \quad (53)$$

Now, we present some plots of (53) for some fixed values of  $t$  and  $\lambda$  and  $x \in [-8, 8]$ .

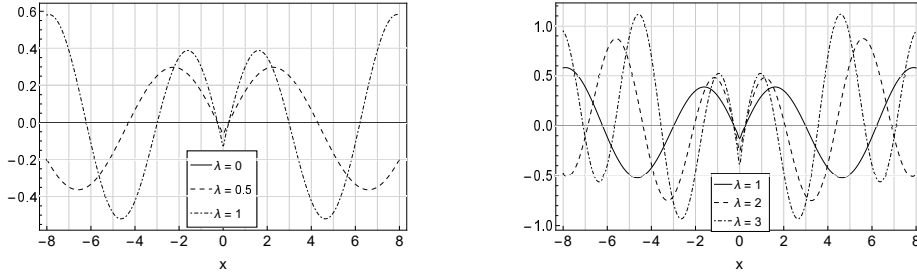


Figure 11: Plots of  $u_\lambda^2(x, t)$  for  $t = 0.5$  and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 0.5$  and  $\lambda = 1, 2, 3$  (2nd plot).

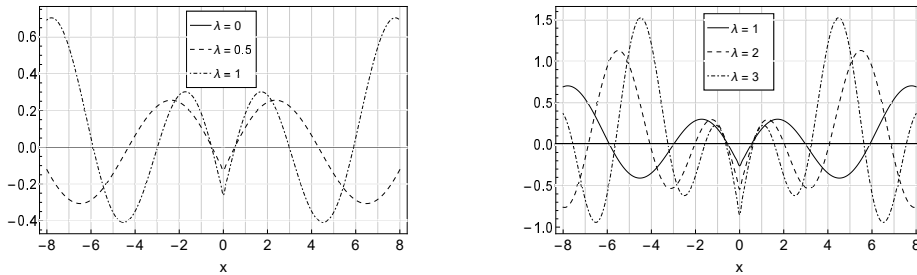


Figure 12: Plots of  $u_\lambda^2(x, t)$  for  $t = 1$  and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 1$  and  $\lambda = 1, 2, 3$  (2nd plot).

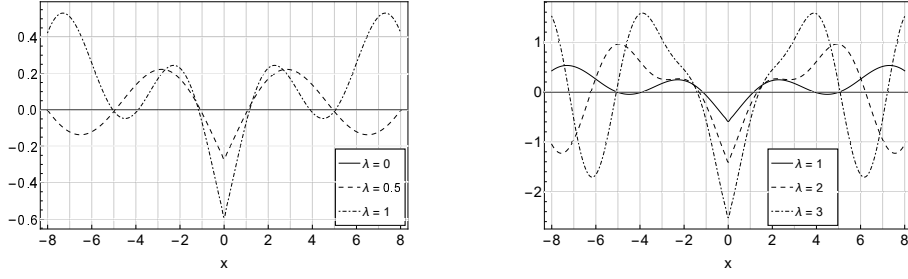


Figure 13: Plots of  $u_\lambda^2(x, t)$  for  $t = 2$  and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 2$  and  $\lambda = 1, 2, 3$  (2nd plot).

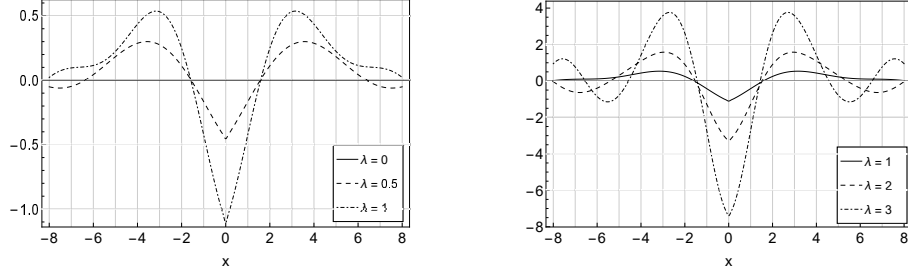


Figure 14: Plots of  $u_\lambda^2(x, t)$  for  $t = 3$  and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 3$  and  $\lambda = 1, 2, 3$  (2nd plot).

From Figures 11, 12, 13, and 14 we can see that the plots are similar to those presented in [7] for the fundamental solution of the wave operator. Each eigenfunction attains several maxima and minima. When  $x = 0$  the local minimum decreases when the time and the parameter  $\lambda$  increase. The range near the origin increases with  $\lambda$ , and for each  $t$  the oscillating behaviour increases with  $\lambda$ . For all values of  $\lambda$  the eigenfunctions are continuous and differentiable in  $\mathbb{R} \times \mathbb{R}^+$ , even when  $x$  and  $t$  tend to zero. In fact, from (53) we have that

$$\begin{aligned} \lim_{x \rightarrow 0} u_\lambda^2(x, t) &= \frac{1}{t} \sum_{m_1=0}^{+\infty} \frac{1}{m_1! (m_1 - 1)!} \left( \frac{\lambda t^2}{4} \right)^{m_1} - \frac{\lambda t}{4} \sum_{m_1=0}^{+\infty} \frac{\Gamma(\frac{1}{2} + m_1)}{(m_1!)^2 \Gamma(\frac{3}{2} + m_1)} \left( \frac{\lambda t^2}{4} \right)^{m_1} \\ &= \frac{\sqrt{\lambda}}{2} I_1(\sqrt{\lambda} t) - \frac{\lambda}{2} {}_1F_2\left(\frac{1}{2}; 1, \frac{3}{2}; \frac{\lambda t^2}{4}\right), \end{aligned} \quad (54)$$

$$\begin{aligned} \lim_{t \rightarrow 0^+} u_\lambda^2(x, t) &= -\frac{\lambda |x|}{4\sqrt{\pi}} \sum_{m_1=0}^{+\infty} \frac{\Gamma(-\frac{1}{2} - m_1)}{m_1!} \left( \frac{\lambda |x|}{4} \right)^{m_1} \\ &= \frac{\sqrt{\lambda}}{2} \sin(\sqrt{\lambda} |x|), \end{aligned} \quad (55)$$

where  $I_\nu(z)$  is the modified Bessel function of the first kind and  ${}_1F_2(a; b, c; z)$  is the Gauss hypergeometric function. It is clear that (54) and (55) are continuous and differentiable functions in  $\mathbb{R}^+ \times \mathbb{R}$ , respectively. In the following figure we present some plots of (53) for some fixed values of  $\lambda$  and  $(x, t) \in [-8, 8] \times [0.2, 4]$ .

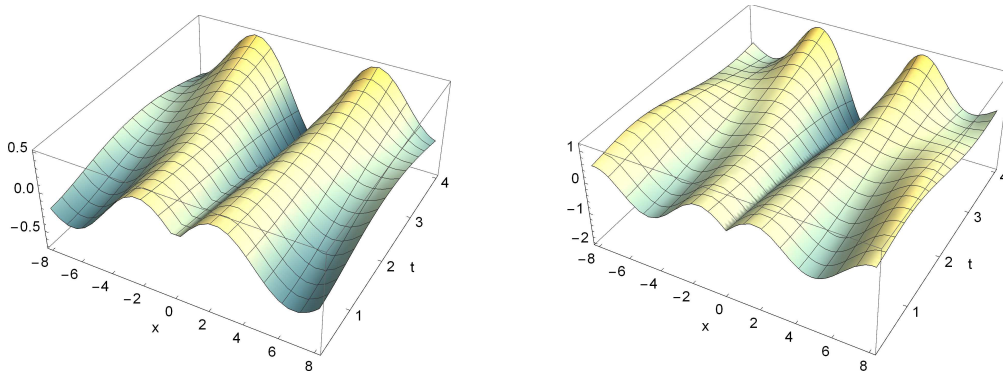


Figure 15: Plots of  $u_\lambda^2$  for  $\lambda = 0.5$  (1st plot) and  $\lambda = 1$  (2nd plot).

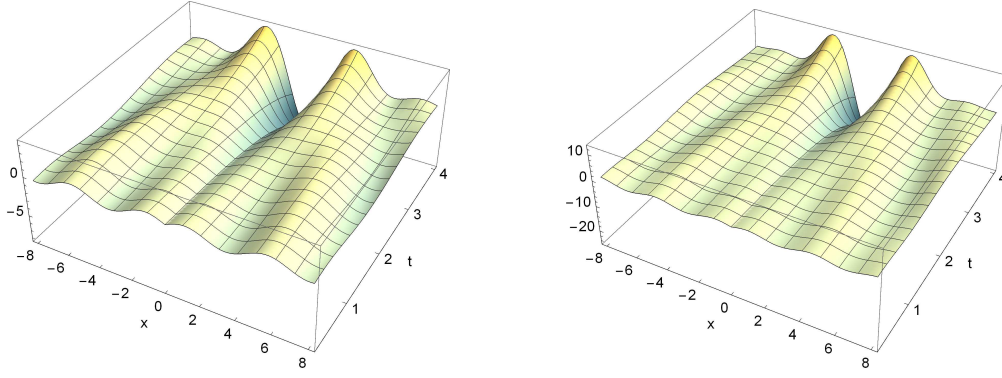


Figure 16: Plots of  $u_\lambda^2$  for  $\lambda = 2$  (1st plot) and  $\lambda = 3$  (2nd plot).

From the plots in Figures 15 and 16 the conclusions are the same as those made in the analysis of the plots in Figures 11, 12, 13, and 14. In these plots it is even more clear that the range near the origin increases with  $\lambda$ . Due to the singularity at  $t = 0$  that appears in (54) we did not plot the functions near  $t = 0$  to see more clearly the behaviour of the eigenfunctions.

### 5.3.2 The case of $n = 3$

Considering  $c = 1$  and  $n = 3$  in (49) we obtain

$$u_\lambda^2(x, t) = \frac{1}{\pi t^3} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3! \Gamma(-1 + m_1 - m_3)} \left(\frac{\lambda t^2}{4}\right)^{m_1} \left(-\frac{|x|^2}{4t}\right)^{m_3} - \frac{\lambda}{4\sqrt{\pi^3}|x|} \sum_{m_1=0}^{+\infty} \frac{\Gamma(\frac{1}{2} - m_1)}{m_1!} \left(\frac{\lambda |x|^2}{4}\right)^{m_1} - \frac{\lambda}{4\pi t} \sum_{m_1=0}^{+\infty} \sum_{m_3=0}^{+\infty} \frac{1}{m_1! m_3! \Gamma(m_1 - m_3) (-\frac{1}{2} + m_1 - m_3)} \left(\frac{\lambda t^2}{4}\right)^{m_1} \left(-\frac{|x|^2}{4t}\right)^{m_3}. \quad (56)$$

Now, we present some plots of (56) for some fixed values of  $\lambda$  and  $(|x|, t) \in [0.1, 6] \times [0.1, 4]$ .

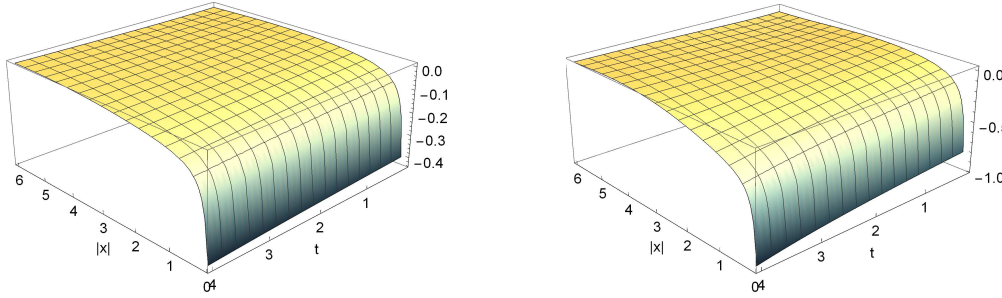


Figure 17: Plots of  $u_\lambda^2$  for  $\lambda = 0.5$  (1st plot) and  $\lambda = 1$  (2nd plot).

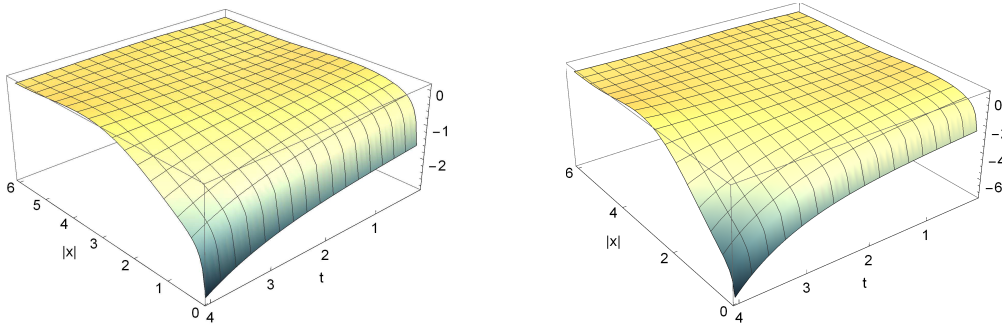


Figure 18: Plots of  $u_\lambda^2$  for  $\lambda = 2$  (1st plot) and  $\lambda = 3$  (2nd plot).



From the plots in Figures 17 and 18 we observe that the decay of  $u_\lambda^2$  is more pronounced in space than in time and the range of the plots increases with  $\lambda$ . Moreover, we have that

$$\lim_{t \rightarrow 0^+} u_\lambda^2(x, t) = -\frac{\lambda}{4\sqrt{\pi^3}|x|} \sum_{m_1=0}^{+\infty} \frac{\Gamma(\frac{1}{2} - m_1)}{m_1!} \left(\frac{\lambda|x|^2}{4}\right)^{m_1} = -\frac{\lambda}{4\pi|x|} \cos(\sqrt{\lambda}|x|),$$

which is a continuous and differentiable function for  $|x| \neq 0$ . From the first term of the second series in (56) we have that

$$\lim_{|x| \rightarrow 0^+} u_\lambda^2(x, t) = -\infty.$$

## 6 The case of $\beta = \frac{3}{2}$ for $n = 1$

In this section we consider the case when  $\beta = \frac{3}{2}$  for  $n = 1$  and  $c = 1$ . Since in this case  $\beta$  is a rational number a natural way to obtain the representation of the eigenfunctions in terms of generalized hypergeometric series for the case of non-coincident sequences of poles is to consider  $p = 3$ ,  $q = 2$ ,  $n = 1$  and  $c = 1$  in (31). However, the obtained expression is very unstable numerically and does not allow to obtain reasonable plots for the eigenfunctions. Therefore, we consider  $\beta = \frac{3}{2}$ ,  $n = 1$ , and  $c = 1$  in (23), which leads to

$$\begin{aligned} u_\lambda^{\frac{3}{2}}(x, t) = & \frac{|x|}{4\sqrt{\pi}t^3} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma(-\frac{1}{2} - m_1 - m_2)}{m_1! m_2! \Gamma(-\frac{1}{2} - \frac{3m_2}{2})} \left(-\frac{\lambda|x|^2}{4}\right)^{m_1} \left(-\frac{|x|^2}{4\sqrt{t^3}}\right)^{m_2} \\ & + \frac{1}{\sqrt{4\pi}\sqrt{t^3}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma(\frac{1}{2} + m_1 - m_2)}{m_1! m_2! \Gamma(\frac{1}{4} + \frac{3}{2}(m_1 - m_2))} (-\lambda\sqrt{t^3})^{m_1} \left(-\frac{|x|^2}{4\sqrt{t^3}}\right)^{m_2} \\ & - \frac{\lambda|x|}{4\sqrt{\pi}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma(-\frac{1}{2} - m_1 - m_2)}{m_1! m_2! \Gamma(1 - \frac{3m_2}{2})} \left(-\frac{\lambda|x|^2}{4}\right)^{m_1} \left(-\frac{|x|^2}{4\sqrt{t^3}}\right)^{m_2} \\ & - \frac{\lambda\sqrt{t^3}}{\sqrt{4\pi}} \sum_{m_1=0}^{+\infty} \sum_{m_2=0}^{+\infty} \frac{\Gamma(\frac{1}{2} + m_1 - m_2)}{m_1! m_2! \Gamma(\frac{7}{4} + \frac{3}{2}(m_1 - m_2))} (-\lambda\sqrt{t^3})^{m_1} \left(-\frac{|x|^2}{4\sqrt{t^3}}\right)^{m_2}. \end{aligned} \quad (57)$$

Now, we present some plots of (57) for some fixed values of  $t$ .

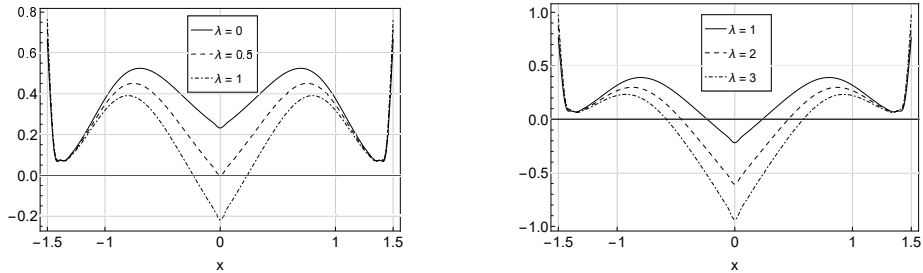


Figure 19: Plots of  $u_\lambda^{\frac{3}{2}}(x, t)$  for  $t = 0.5$ ,  $x \in [-1.5, 1.5]$ , and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 0.5$  and  $\lambda = 1, 2, 3$  (2nd plot).

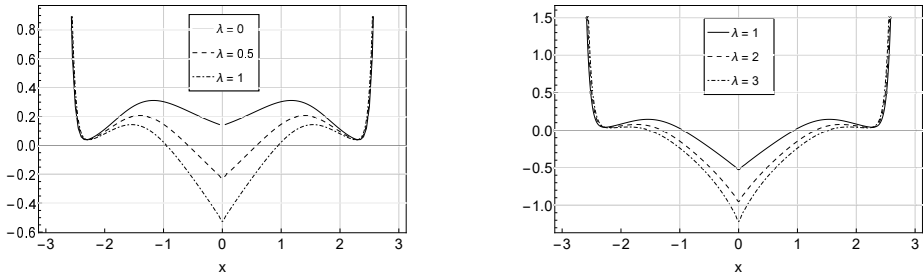


Figure 20: Plots of  $u_\lambda^{\frac{3}{2}}(x, t)$  for  $t = 1$ ,  $x \in [-3, 3]$ , and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 1$  and  $\lambda = 1, 2, 3$  (2nd plot).

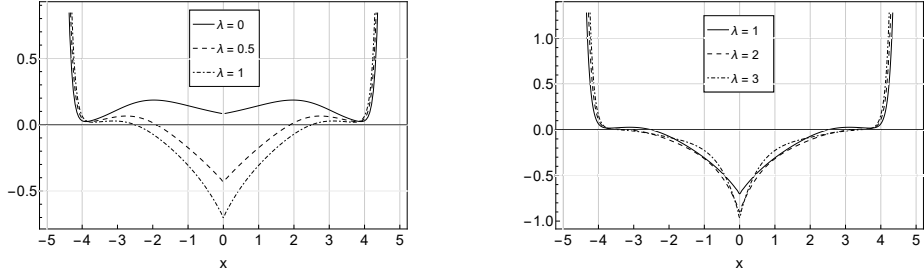


Figure 21: Plots of  $u_{\lambda}^{\frac{3}{2}}(x, t)$  for  $t = 2$ ,  $x \in [-5, 5]$ , and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 2$  and  $\lambda = 1, 2, 3$  (2nd plot).

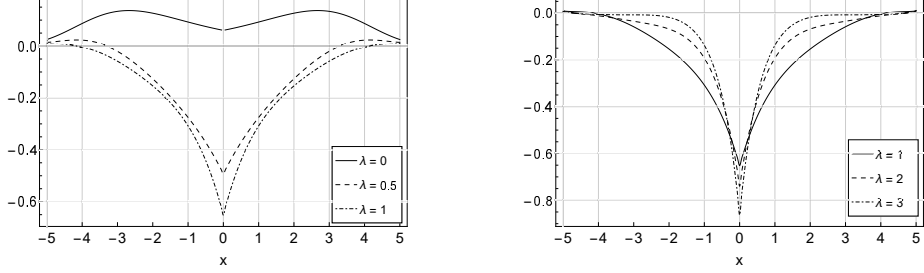


Figure 22: Plots of  $u_{\lambda}^{\frac{3}{2}}(x, t)$  for  $t = 3$ ,  $x \in [-5, 5]$ , and  $\lambda = 0, 0.5, 1$  (1st plot), and  $t = 3$  and  $\lambda = 1, 2, 3$  (2nd plot).

From the plots in Figures 19, 20, 21, and 22, we can see that they have an intermediate behaviour between the plots of the diffusion case (see Figures 1, 2, 3, and 4) and the plots of the wave case (see Figures 11, 12, 13, and 14). The eigenfunction  $u_{\lambda}^{\frac{3}{2}}$  attains two symmetric maxima which decrease with time and the eigenvalue  $\lambda$ , and attains a minimum at  $x = 0$  which decreases when the time and the parameter  $\lambda$  increase. The range near the origin increases with  $\lambda$ , and for all values of  $\lambda$  the eigenfunctions are continuous and differentiable in  $\mathbb{R} \times \mathbb{R}^+$ , even when  $x$  tends to zero. In fact, from (57) we have that

$$\begin{aligned} \lim_{x \rightarrow 0} u_{\lambda}^{\frac{3}{2}}(x, t) &= \frac{1}{\sqrt{4\pi} \sqrt{t^3}} \sum_{m_1=0}^{+\infty} \frac{\Gamma(\frac{1}{2} + m_1)}{m_1! \Gamma(\frac{1}{4} + \frac{3m_1}{2})} (-\lambda \sqrt{t^3})^{m_1} - \frac{\lambda \sqrt[4]{t^3}}{\sqrt{4\pi}} \sum_{m_1=0}^{+\infty} \frac{\Gamma(\frac{1}{2} + m_1)}{m_1! \Gamma(\frac{7}{4} + \frac{3m_1}{2})} (-\lambda \sqrt{t^3})^{m_1} \\ &= \frac{\Gamma(\frac{3}{4})}{2\pi \sqrt{2} \sqrt[4]{3}} \left( {}_1F_3\left(\frac{1}{4}; \frac{1}{12}, \frac{5}{12}, \frac{1}{2}; \frac{\lambda^2 t^3}{27}\right) - 1 \right) - \frac{\lambda \sqrt[4]{t^3} \Gamma(\frac{1}{4})}{3\pi \sqrt{2}} {}_1F_3\left(\frac{3}{4}; \frac{7}{12}, \frac{11}{12}, \frac{3}{2}; \frac{\lambda^2 t^3}{27}\right) \\ &\quad - \frac{2\lambda \sqrt[4]{t^3}}{\Gamma(\frac{3}{4})} \left( {}_2F_4\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}, \frac{7}{12}, \frac{11}{12}, \frac{5}{4}; \frac{\lambda^2 t^3}{27}\right) - 1 \right) + \frac{16\lambda \sqrt[4]{t^9}}{45 \Gamma(\frac{1}{4})} {}_2F_4\left(\frac{3}{4}, \frac{5}{4}; \frac{13}{12}, \frac{17}{12}, \frac{3}{2}, \frac{7}{4}; \frac{\lambda^2 t^3}{27}\right) \end{aligned}$$

which is a continuous and differentiable function in  $\mathbb{R}^+$ . Now, we present some plots of (57) for some fixed values of  $\lambda$  and  $(x, t) \in [-4, 4] \times [0.2, 4]$ .

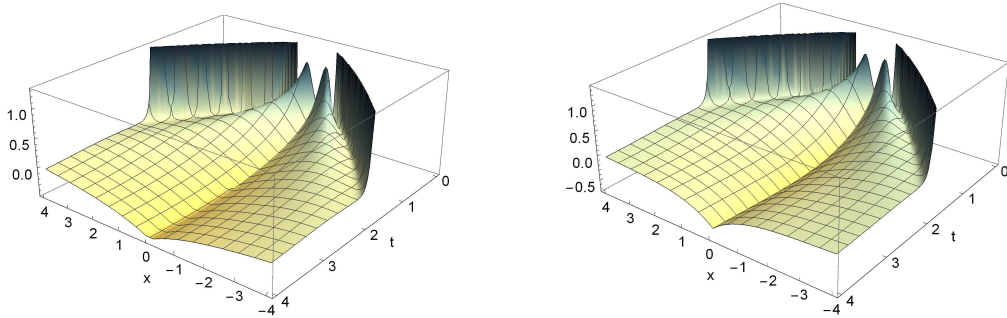


Figure 23: Plots of  $u_{\lambda}^{\frac{3}{2}}$  for  $\lambda = 0.5$  (1st plot) and  $\lambda = 1$  (2nd plot).

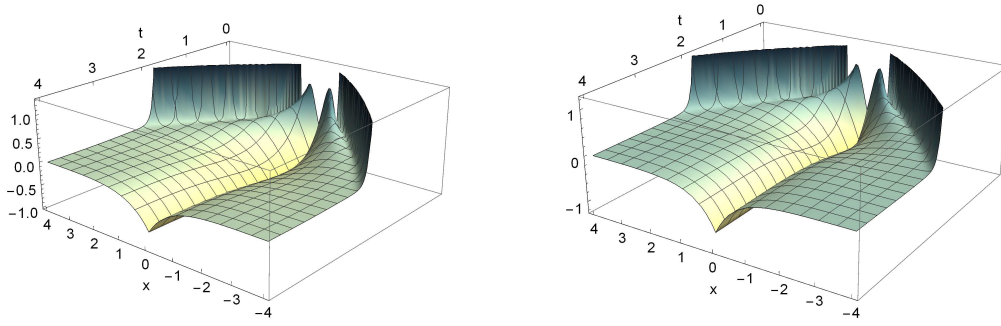


Figure 24: Plots of  $u_{\lambda}^{\frac{3}{2}}$  for  $\lambda = 2$  (1st plot) and  $\lambda = 3$  (2nd plot).

From the plots in Figures 23 and 24 the conclusions are the same as those made in the analysis of the plots in Figures 19, 20, 21, and 22. It is possible to see that when  $|x|$  tends to zero the eigenfunction  $u_{\lambda}^1$  is continuous and differentiable in  $\mathbb{R}^+$ . We remark that

$$\lim_{t \rightarrow 0^+} u_{\lambda}^{\frac{3}{2}}(x, t) = +\infty.$$

## 7 Conclusions

In this work we deduced integral and series representations for the eigenfunctions of the time-fractional diffusion-wave operator of order  $\beta \in ]1, 2[$  in  $\mathbb{R}^n \times \mathbb{R}^+$ . The integral representation is deduced for an arbitrary dimension  $n$ , while the series representation is deduced only for odd  $n$  and an arbitrary rational number  $\beta \in ]1, 2[$ . In this case the obtained series are generalized hypergeometric series of two variables, namely Kampé de Fériet and generalized Lauricella series. The limit cases of the diffusion and wave operators ( $\beta = 1$  and  $\beta = 2$  respectively) and the particular case of  $\beta = \frac{3}{2}$  are considered and studied in detail. Some plots of the eigenfunctions for these particular values of  $\beta$  were presented and discussed. As a future work it is necessary to analyse the asymptotic behaviour of the eigenfunctions at the infinity and to study their scaling properties. Finally, in order to obtain more accurate plots it is necessary to develop stable numerical methods that allow the graphical representation of generalized hypergeometric series.

## Acknowledgements

The work of M. Ferreira, M.M. Rodrigues and N. Vieira was supported by Portuguese funds through CIDMA-Center for Research and Development in Mathematics and Applications, and FCT-Fundação para a Ciência e a Tecnologia, within projects UIDB/04106/2020 and UIDP/04106/2020.

N. Vieira was also supported by FCT via the 2018 FCT program of Stimulus of Scientific Employment - Individual Support (Ref: CEECIND/01131/2018).

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